It is then necessary to make a compromise between the frequency resolution, which can be characterized for instance by the noise bandwidth, and the uncertainty in the determination of the amplitude, which is influenced by the number of values in the computed average for each spectral line and the correlation between the values. Formulas (18) and (19) in Harris' paper can be applied and the following expression is obtained:

$$\Delta f \cdot (\Delta A)^2 = \frac{\text{ENBW}\,(1 - \text{OL})\,\text{OC}}{(T - \tau \cdot \text{OL})} \approx \frac{\text{ENBW}\,(1 - \text{OL})\,\text{OC}}{T} \tag{1}$$

where

- Δf noise bandwidth (Hz);
- ENBW normalized noise bandwidth, bins;
- ΔA relative amplitude uncertainty for each spectral line, σ-value;
- OL overlap, 0.5 or 0.75;
- OC overlap correction, i.e., increase in variance due to overlap;
- T total length of input data to be analyzed, seconds.

The overlap correction can be computed from [1, eq. (19)]. If second order terms are disregarded, one obtains:

$$\text{OC}(0.50) \approx 1 + 2C^2(0.50) \tag{2}$$

$$\text{OC}(0.75) \approx 1 + 2C^2(0.75) + 2C^2(0.50) \tag{3}$$

where $C(0.50)$ and $C(0.75)$ are the overlap correlations for 50 percent and 75 percent overlap.

Equation (1) gives a figure of merit for each choice of overlap and window for the application discussed. Table I compares some of the windows from this point of view.

As is to be expected, more overlap will always give better results, but 75 percent overlap seems to be sufficient in all cases investigated; for some windows even 50 percent could be used to save computer time. It is interesting to note that there is practically no difference between the different windows concerning possible amplitude and frequency resolution. The recommendation is, therefore, to use one of the windows with low-sidelobe levels. Nothing is gained by choosing a simpler window.

Reply[1] *by Fredric J. Harris*[2]

The comments made by Blomqvist on scalloping loss representing an easily correctable processing task (by zero filling and performing larger transforms) is indeed correct and in fact is performed in many signal processing tasks. A qualifier here is that the increased size transform be performed in the memory space of a given machine, and the increased size workload is manageable in the processor's time frame. The comment that the figure of merit, $\Delta f(\Delta A)^2$, is essentially a constant for different windows is an interesting property of windows but should not be used as the basis for discarding equivalent noise bandwidth as a criterion for window selection. In fact, I have shown [3] a similar result, that the signal processing gain obtained by averaging overlapped transforms just matches the signal-to-noise ratio penalty incurred by using a given window in the first place. But remember, ENBW is a sensitive measure of main lobe width [slightly greater than the 3-dB width [1]], and as such reflects the ability of the windowed transform to resolve two nearby-similar strength line components. Sidelobe levels, on the other hand, reflect the ability of the windowed transform to resolve two nearby-very dissimilar strength line components. We note that the resolution of a deep notch (as opposed to a strong resonance) in a power spectrum would require simultaneously a narrow main lobe (i.e., small ENBW) and very low-sidelobe levels. Thus the window selection criterion should be to select the window with the narrowest main lobe width for a given sidelobe level. The windows which meet this criterion are the Dolph–Chebyschev, the Kaiser–Bessel and the Blackman–Harris. Of course, if the spectrum being processed exhibits an extremely deep notch, the constant sidelobe level of the Dolph-Chebyshev will prevent it from performing as well as the other two.

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On a Class of Nonlinear Systems with Explicit Solutions

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Abstract—A new class of nonlinear systems with explicit solutions is introduced. The importance of this new class of systems is pointed in terms of their significance in the reachable set computation problem for linear systems with polyhedral constraints on the initial state and prescribed control signals.

I. Introduction

This letter introduces a new class of systems described by nonlinear ordinary vector differential equations with explicit solutions in terms of the initial conditions and the matrices defining the system. This work represents a generalization of a class of systems that naturally appears on the reachable set computation problem for linear systems with set-constrained initial states and prescribed controls [1], [2].

II. Definitions and Main Results

Definition: We say that a function $f(x, u)$ is *locally Lipschitzian* whenever f satisfies the following conditions: 1) f is continuous with respect to x and u, 2) The derivatives $\partial f_i/\partial x_j$ exist and are bounded over any convex, bounded region of the state space, and 3) $u(t)$ is bounded over any finite interval $[t_0, T]$.

Notice that the above conditions are sufficient conditions for the function f to be Lipschitzian in the ordinary sense [3], [4].

Definition: A scalar function $h(x, u)$ is said to be *m-homogeneous with respect to* x whenever $h(\alpha x, u) = \alpha^m h(x, u)$ for any real number α (m is an integer).

Consider the nonlinear system:

$$\frac{d}{dt}x(t) = A(t)\,x(t) + h(x(t), u(t))\,x(t) \tag{1}$$

with

$$x(t_0) = x_0$$

where $h(x, u)$ is a scalar function m-homogeneous with respect to x. The function h is assumed to have a bounded gradient with respect to x in any convex bounded region of the state space. The control function $u(t)$ is supposed to be piecewise continuous and bounded on the interval $[t_0, T]$ of finite length. $A(t)$ is an $n \times n$ matrix whose entries are continuous functions of time defined on the interval $[t_0, T]$. The initial state x_0 is a known vector.

The following is a theorem which establishes the uniqueness of solutions of system (1).

Theorem 1

Under the above hypothesis, system (1) has a unique solution given by

$$x(t) = \left(1 - m\int_{t_0}^{t} h(\Phi(\sigma, t_0)x_0, u(\sigma))\,d\sigma\right)^{-1/m} \Phi(t, t_0)\,x_0 \tag{2}$$

where $\Phi(t, t_0)$ is the fundamental matrix associated with $A(t)$.

Proof: Uniqueness of solutions is easily established after realizing that the function $f(x, u) = A(t)x + h(x, u)x$ is locally Lipschitzian. The proof that (2) is actually the solution of (1) is done by taking the time derivative of $x(t)$ in (2). For this let

$$q(t) \triangleq \left(1 - m\int_{t_0}^{t} h(\Phi(\sigma, t_0)x_0, u(\sigma))\,d\sigma\right). \tag{3}$$

Manuscript received May 9, 1978; revised September 19, 1978.
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Then, the time derivative of $x(t)$ is given by

$$\frac{d}{dt}x(t) = -\frac{1}{m}q(t)^{-1/m-1}(-mh(\Phi(t,t_0)x_0, u(t))\Phi(t,t_0)x_0 + q(t)^{-1/m}A(t)\,\Phi(t,t_0)x_0.$$

Simplifying and using the m-homogeneous character of $h(x,u)$ we obtain

$$\frac{d}{dt}x(t) = q(t)^{-1/m}\Phi(t,t_0)x_0\, h(q(t)^{-1/m}\Phi(t,t_0)x_0, u(t)) + A(t)\,q(t)^{-1/m}\,\Phi(t,t_0)x_0.$$

Substitution of (2) in the preceding expression yields back the differential equation (1). The proposed solution (2) trivially satisfies the initial conditions. The result is thus established. □

The preceding theorem constitutes an interesting result due to the nonstringent conditions imposed on the nonlinear part of the differential equation. Any finite escape time problems that could arise for certain values of m are easily circumvented by the *explicit* form of the solution and some "monitoring" on $q(t)$. The solution is thus characterized by (2) in the real line, excluding some arbitrarily small neighborhoods around the finite escape times.

Example: Suppose we have a linear system of the form $\dot{x} = A(t)x + B(t)u(t)$ where $u(t)$ is a given piecewise continuous bounded control function. Let the symbols $\langle , \rangle$ stand for inner product of the shown vectors. If the initial state of the system is unknown but bounded by a polyhedron of the form $\{x \in R^n: \langle x, \eta_{i0}\rangle \leq 1; i = 1, 2, \cdots, N \geq n+1\}$ ($N \geq n+1$ is a necessary condition for boundedness) then it is easy to show that at time t, the state $x(t)$ will be bounded by a polyhedron characterized by: $\{x \in R^n: \langle x, \eta_i(t)\rangle \leq 1; i = 1, 2, \cdots, N\}$ where $\eta_i(t)$ is given by the unique solution of

$$\frac{d}{dt}\eta_i(t) = -A'(t)\,\eta_i(t) - \langle \eta_i(t), B(t)u(t)\rangle\, \eta_i(t) \tag{4}$$

with

$$\eta_i(t_0) = \eta_{i0} \quad \text{for all } i.$$

A straightforward application of *Theorem 1* yields for $\eta_i(t)$:

$$\eta_i(t) = \left(1 + \int_{t_0}^{t} < \eta_{i0}, \Phi(t_0,\sigma)B(\sigma)u(\sigma) > d\sigma\right)^{-1} \Phi'(t_0,t)\,\eta_{i0}, \quad i = 1, 2, \cdots, N.$$

The preceding result follows easily from well-known properties of fundamental matrices of adjoint systems.

An interplay among vector and matrix differential systems frequently occurs. For this reason and because of their importance in related optimization problems, we extend the previous theorem to systems of the form

$$\frac{d}{dt}\Sigma(t) = A(t)\,\Sigma(t) + \Sigma(t)\,A'(t) + h(\Sigma(t), Q(t))\,\Sigma(t) \tag{5}$$

with

$$\Sigma(t_0) = \Sigma_0.$$

The function h is scalar m-homogeneous with respect to Σ. Under the appropriate hypothesis of continuity and boundedness of the *matrix gradient* of h (see [5]), it is easy to show that the following theorem holds true:

Theorem 2

System (5) has a unique solution given by

$$\Sigma(t) = \left(1 - m\int_{t_0}^{t} h(\Phi(\sigma,t_0)\Sigma_0\Phi'(\sigma,t_0), \quad Q(\sigma))d\sigma\right)^{-1/m} \cdot \Phi(t,t_0)\,\Sigma_0\Phi'(t,t_0)$$

Proof: This theorem is an immediate extension of *Theorem 1* to the matrix case. □

III. Conclusions

This paper has introduced a new class of systems described by nonlinear vector (and also matrix) differential equations with explicit solutions. Uniqueness and form of solutions were established under rather relaxed conditions regarding the nonlinearity of the system. The class was shown to have importance in the study of set-theoretic issues regarding the evolution of the initial state uncertainty. The vector results were shown to be extended easily to the matrix case.

Further research is needed in specific areas such as stability, controllability, and optimization problems associated with this class of nonlinear systems. The advantage offered by their explicit solution can be exploited to obtain some results in the above mentioned areas.

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Simulation of a Discrete PLL with Variable Parameters

LUIS F. ROCHA

***Abstract*—The discrete phase lock loop (DPLL) proposed by G. S. Gill and S. C. Gupta [1] is simulated with a programmable calculator (TI-58). Many interesting properties of its behavior are easily shown such as the pulling-up time, the effects of varying the coefficients of the filter, and the effect of phase and frequency steps on the output.**

If the constant K_1 of the filter is initially made high, and its value decreases with time, it is possible to obtain a DPLL with large pull-in range and which is very insensitive to input noise.

Introduction

The discrete phase lock loop (DPLL) [1]–[3] has recently received a good deal of attention due to its simplicity and easy implementation.

The system is based on a variable period clock (see Fig. 1) controlled by a sampler through an A/D converter and digital filter. The sampling period $T(j)$ is changed every cycle according to:

$$T(j) = T - c(j-1) \tag{1}$$

where T is a constant, the free running period of the clock, and $c(j-1)$ is the output of the digital filter.

For a second-order filter the difference equation is given by

$$c(k) = K_1 e(k) + K_2 \sum_{i=1}^{\infty} e(k-1) \tag{2}$$

where $e(k)$ is the sampled input at instant k, and K_1 and K_2 are the filter parameters.

In order to simulate this system with an input of one (or more) sinusoids plus white noise using a programmable calculator (TI-58), we have to define also the time elapsed:

$$t = t(0) + \sum_{j=0}^{\infty} T(j) \tag{3}$$

where $T(j)$ are the delay sampling periods and $t(0)$ is the initial time.

The program first finds the input value at instant $T(0)$ for a sinusoid of amplitude one and frequency ω_1, plus a second sinusoid of amplitude 0.5 and frequency ω_2, plus a noise value with deviation σ. Then it

Manuscript received July 24, 1978.

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0018-9219/79/0300-0440$00.75 © 1979 IEEE