INT. J. SYSTEMS SCI., 1987, VOL. 18, NO. 9, 1673–1689

Variable structure control of non-linear systems

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This article presents a differential geometric approach for the design of variable structure controllers (VSC) leading to sliding regimes (Emelyanov 1967) in non-linear smooth dynamical systems described in R^n. New properties, inherited from the geometry of the sliding manifold, are obtained in the controlled system when sliding conditions prevail. A sliding motion is obtained thanks to opportune switchings among the available structures in the feedback loop. This discontinuous control action is aimed at guaranteeing manifold reachability and invariance.

1. Introduction

More than fifteen years ago the control community witnessed the birth of a sustained interest in the applications of differential geometry concepts to the formulation and solution of a variety of long-standing non-linear control problems. Works by Brockett (1976), Sussman and Jurdjevic (1972), Hermann and Krener (1970), and Isidori *et al.* (1981), are responsible for a wealth of theoretical and practical implications that have nutured the understanding of control theorists about the delicate issues associated with the design of non-linear control systems. The amount of available research and results is impossible to survey in an article of this length. The reader is referred to the excellent surveys by Andreev (1982), Isidori (1982) and a recent book by Isidori (1985), for more detailed information.

The theory of variable structure systems (VSS) and their associated sliding mode behaviour (Emelyanov 1967) has also undergone extensive and detailed studies in the last twenty five years. Scientists from the Soviet Union and, more recently, the United States have contributed with a wide embracing range of applications that ranges from aerospace design problems (Utkin 1968) to gas turbine control (Young *et al.* 1982), robot manipulators (Slotine and Sastry 1983) and hydropower generation (Ershler *et al.* 1974). Survey articles (Utkin 1977, 1983), and several books (Itkis 1976, Utkin 1978, 1981), contain lucid expositions on the state of the art and the potentials for the future of this simple and yet powerful design methodology.

A surface or manifold in the state space represents static relationships among the different state variables describing the behaviour of the system. If these relations are enforced on the dynamic description of the system, the resulting reduced-order dynamics may contain highly desirable features. The idea is then to specify a feedback control action, of variable-structure nature, which guarantees the reachability of the prescribed surface and, once the surface conditions are met, proceeds to maintain the systems motions constrained to this surface. The task is usually accomplished by opportune drastic changes in the structure of the feedback controller which induce velocity vector fields invariably directed towards the 'sliding surface' in its immediately vicinity.

Received 1 May 1986.

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In the linear systems case, the surface is generally a hyperplane designed to bring about asymptotic stability of the controlled dynamics. The non-linear system context is probably richer in so far as invariance of the sliding manifold is by itself a design task.

In this article we examine the problem of desigining VSC leading to sliding regimes in non-linear stationary smooth plants. A differential geometric approach is presented for the formation and characterization of the equivalent control and the ideal sliding dynamics representing the invariant motions of the system on the sliding manifold. We propose local necessary and sufficient conditions for the surface reachability problem in terms of sign conditions on the directional derivatives of the surface coordinate with respect to the resulting vector field of the controlled system. The relation of these conditions to the geometry of the surface and to the equivalent control are both transparent and appealing from this viewpoint.

In § 2, we formulate and solve the problem of designing a sliding regime on a smooth manifold which represents a desirable behaviour condition for the reduced-order system. In this section the interpretation of the equivalent control and conditions for the ideal sliding regime are obtained through a regularization procedure of the system equations.

Section 3 is devoted to some illustrative examples of a simple but important nature. The last section contains conclusions and some suggestions for further research in the area of non-linear variable structure controller (VSC) design.

2. Problem formulation and main results

2.1. *Notation, definitions and main assumptions*

Consider the non-linear dynamic system:

$$(\Sigma): \quad \frac{dx}{dt} = f(x) + g(x)u; \quad x(t_0) = x_0 \tag{2.1}$$

where x is a local coordinate system on a smooth n-dimensional manifold M which we usually take as an open set in R^n. The vectors f and g are local representations of smooth vector fields in M.

In our assertions, a property will be termed 'local' whenever its validity is circumscribed in an open neighbourhood N of x_0.

We assume that, locally, $g(x)$ is a non-zero vector whose span, denoted by $\Delta_G(x)$, defines a smooth one-dimensional distribution on T_xM.

The control function $u: M \to R$ is a real valued discontinuous function of the form:

$$u(x) = \begin{cases} -u^+(x) & \text{for } s(x) > 0 \\ -u^-(x) & \text{for } s(x) < 0 \end{cases} \tag{2.2}$$

where $s: M \to R$ is a smooth function for which the set $S = \{x \in M : s(x) = 0\}$ defines a smooth $n-1$ dimensional manifold (hypersurface or submanifold) in M. S is called the 'sliding manifold'.

The distribution represented by T_xS is, locally, a smooth $n-1$ dimensional distribution of T_xM and coincides with Ker ds. We denote this distribution as $\Delta_S(x)$ and refer to it as the 'sliding distribution'.

We assume that among the control input field $g(x)$ and the vector normal to the sliding surface, $\partial s/\partial x$, the following transversality condition holds true locally:

$$\frac{\partial s}{\partial x} g(x) \neq 0$$

If this condition is satisfied, the input function enjoys a definite influence on the dynamic values of the coordinate surface s.

Proposition 1

$$\frac{\partial s}{\partial x} g(x) \neq 0 \quad \text{if and only if} \quad \Delta_S^{\perp}(x) \cap \Delta_G^{\perp}(x) = 0$$

Proof

To avoid the trivial case, the vectors above are assumed to be non-zero. Let $(\partial s/\partial x) g(x) = 0$, then $\partial s/\partial x \in \text{span}\, \Delta_G^{\perp}(x)$ and since $\partial s/\partial x \in \text{span}\, \Delta_S^{\perp}(x)$ then both distributions have a non-zero intersection. On the other hand, suppose the condition holds: then span $\{\Delta_S(x) + \Delta_G(x)\}$ is rank n. Since the dimensions of both distributions add to n, it follows that their orthogonal complements are linearly independent. □

The specification of a smooth feedback control function which makes S into an invariant manifold is known as the 'equivalent control problem' (Utkin 1977). The equivalent control function describes the motion of Σ on the manifold S in an average sense. This motion is known as the 'ideal sliding dynamics'. The conditions for sliding manifold invariance are seen to be:

$$s = 0; \quad L_{f+gu} s = 0$$

Proposition 2

The equivalent control exists and it is unique if and only if the transversality condition is satisfied.

Proof

The proof is seen immediately from the fact that:

$$\frac{ds}{dt} = \frac{\partial s}{\partial x} f(x) + \frac{\partial s}{\partial x} g(x) u$$

and the fact that the invariance of S demands for satisfaction the conditions:

$$s = 0; \quad L_{f+gu} s = \frac{ds}{dt} = 0$$ □

When the feedback control action (2.2) is capable of driving and maintaining the state of the system on the manifold S, we say that a 'sliding regime' is created by the VSC.

2.2. *Problem formulation*

It is desired to specify a smooth $n - 1$ dimensional sliding manifold S and a VSC of the form (2.2) so that the integral curves of Σ (locally) approach S and remain constrained to this surface thanks to the active switchings of the feedback controller. The average invariant motions in S are deemed desirable in the following optional senses.

(1) An asymptotically stable motion is obtained on the manifold which converges towards a pre-specified equilibrium point located in S.

(2) A sustained oscillatory motion of the desirable characteristics is obtained as a result of satisfying the manifold conditions.

Notice that the motions on S are totally determined by the geometry of the manifold. Therefore, new features which are not present in any of the closed-loop available structures may emerge in the controlled system as characteristics borrowed from the shape and nature of the sliding surface.

We shall start our developments by assuming that a sliding manifold S has been given which reflects one of the above stated idealized desirable behaviours for the reduced-order system. It will become clear at some point that the ideal sliding dynamics is the outcome of a feedback control design problem on the reduced system. The feedback control function will be interpreted as a static relationship among the original full system state variables which in turn explicitly defines the required sliding manifold. However, these issues become transparent only if the system is placed in regular form.

Definition 1

We define Σ to be in regular form (Luk'yanov and Utkin 1981), whenever a local coordinate system exists in which (2.1) can be expressed as:

$$\left.\begin{aligned} \frac{d}{dt}\mathbf{x}_1 &= f_1(\mathbf{x}_1, x_n) \\ \frac{d}{dt}x_n &= f_n(\mathbf{x}_1, x_n) + g_n(\mathbf{x}_1, x_n)u \end{aligned}\right\} \tag{2.3}$$

with $\mathbf{x}_1$ an $n-1$ dimensional vector containing all but the last coordinate function of the vector x, while x_n is the last scalar coordinate function component of x.

A system of the form (2.1) can always be written in regular form after a suitable diffeomorphic state coordinate transformation, or, equivalently, after a coordinate adaptation procedure (Isidori 1985). This is true from the fact that the one-dimensional distribution $\Delta_G(x)$ is always involutive. Our assumption about its nonzero character, in a local sense, guarantees the constancy of its one-dimensional span. The Frobenius theorem (Isidori 1985) establishes, under these assumptions, complete integrability of $\Delta_G(x)$. From here it is easy to see that regularity follows simply from the definition of integrability.

Notice that from the constant dimensionality assumption on $\Delta_G(x)$ it follows that $g_n(\mathbf{x}_1, x_n) \neq 0$ in N.

We assume the $n-1$ dimensional manifold S in M can be expressed as

$$S = \{x \in M : s = x_n + m(\mathbf{x}_1) = 0\} \tag{2.4}$$

In the above class of sliding manifolds, the transversality condition of Proposition 1 is trivially equivalent to the non-vanishing of the input vector field component $g_n(\mathbf{x}_1, x_n)$.

2.3. *Ideal sliding dynamics and the equivalent control*

In order to be able to relate the effects of the control input functions on the reachability of S, as well as to be able to have an assessment of the 'equivalent control' and its associated ideal sliding dynamics, we replace the x_n coordinate function by the 'surface' coordinate function s and let $x_n = s - m(\mathbf{x}_1)$. We then obtain:

$$\left.\begin{aligned} \frac{d}{dt}\mathbf{x}_1 &= \hat{f}_1(\mathbf{x}_1, s) \\ \frac{d}{dt}s &= \hat{f}_n(\mathbf{x}_1, s) + \hat{g}_n(\mathbf{x}_1, s)u \end{aligned}\right\} \tag{2.5}$$

where

$$\left.\begin{aligned} \hat{f}_1(\mathbf{x}_1, s) &= f_1(\mathbf{x}_1, s - m(\mathbf{x}_1)) \\ \hat{f}_n(\mathbf{x}_1, s) &= f_n(\mathbf{x}_1, s - m(\mathbf{x}_1)) + \frac{\partial s}{\partial \mathbf{x}_1} f_1(\mathbf{x}_1, s - m(\mathbf{x}_1)) \\ &= f_n(\mathbf{x}_1, s - m(\mathbf{x}_1)) + \frac{\partial m}{\partial \mathbf{x}_1} f_1(\mathbf{x}_1, s - m(\mathbf{x}_1)) \\ \hat{g}_n(\mathbf{x}_1, s) &= g_n(\mathbf{x}_1, s - m(\mathbf{x}_1)) \end{aligned}\right\} \tag{2.6}$$

We let:

$$\hat{f} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_n \end{bmatrix} \quad \text{and} \quad \hat{g} = \begin{bmatrix} 0 \\ \hat{g}_n \end{bmatrix}$$

Geometrically, this new change of coordinates amounts to considering $\Delta_S(x)$ and $\Delta_G(x)$ as projection subspaces of the tangent space $T_x M$. Thus, the velocity vector field defining s is the sum of the projections of the involved vector fields onto $\Delta_G(x)$ along $\Delta_S(x)$. On the other hand, $f_1(\mathbf{x}_1, s - m(\mathbf{x}_1))$ is the projection on $\Delta_S(x)$ of the component $f_1(\mathbf{x}_1, x_n)$ of the drift field along $\Delta_G(x)$ (Fig. 1 illustrates the geometry of this decomposition).

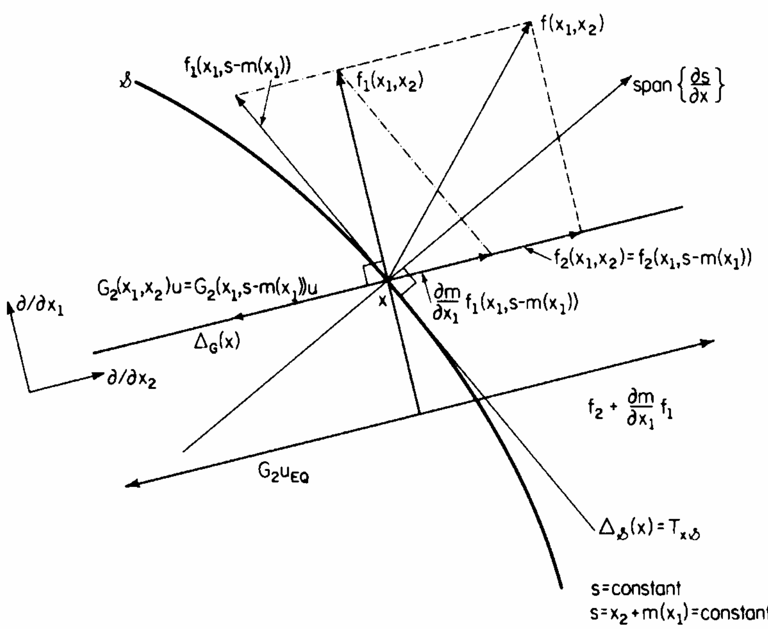


Figure 1. Geometry of ideal sliding motion.

Clearly, the sliding manifold S will be locally invariant if and only if the following conditions are satisfied:

$$\left.\begin{array}{l}(1)\ s=0 \\ (2)\ L_{\hat{f}+\hat{g}u}s=0\end{array}\right\} \tag{2.7}$$

Ideal sliding motion corresponds to having null components of the directional derivative of the surface coordinate s with respect to the velocity vector field defining the integral curves of the system. The integral curves are therefore locally constrained to the sliding manifold. The equivalent control role is to annihilate the components of $\hat{f}$ along directions not corresponding with $\Delta_S(x)$, i.e.

$$\langle ds, \hat{f} - \hat{g}u_{EQ} \rangle = 0$$

from where

$$u_{EQ} = \frac{\langle ds, \hat{f} \rangle}{\langle ds, \hat{g} \rangle}$$

where use has been made of the negative-feedback convention adopted in (2.2) by letting $u = -u_{EQ}(\mathbf{x}_1)$ in (2.7). Using (2.6) and the fact that

$$\langle ds, h \rangle = \frac{\partial s}{\partial x} h = \frac{\partial m}{\partial \mathbf{x}_1} h$$

for any h, we obtain

$$u_{EQ}(\mathbf{x}_1) = \frac{\left[f_n(\mathbf{x}_1, -m(\mathbf{x}_1)) + \dfrac{\partial m}{\partial \mathbf{x}_1} f_1(\mathbf{x}_1, -m(\mathbf{x}_1)) \right]}{g_n(\mathbf{x}_1, -m(\mathbf{x}_1))} \tag{2.8}$$

The existence and uniqueness of the equivalent control is a crucial factor in the determination of the necessary gains to achieve a sliding motion. The case where the input vector field component $g_n(\mathbf{x}_1, -m(\mathbf{x}_1)) = 0$ is known as a 'singular case'. Under this condition, the sliding motions can be lost by excursions of the velocity vector fields along directions transverse to the sliding distribution. In some special cases, however, a sliding regime may still exist, i.e. when the drift vector field happens to be contained by the sliding distribution in the singularity region or at the isolated singular point (see Sira-Ramirez 1986 and Example 2).

Condition (2) of (2.7) is also seen to be equivalent to:

$$\hat{f} + \hat{g}u \in \operatorname{Ker} ds = \Delta_S \tag{2.9}$$

The ideal sliding dynamics and the equivalent control are then completely specified by requiring that, locally, the velocity vector field defining the integral curves of the system belong to the tangent subspace of the sliding manifold in M.

The reduced-order ideal sliding dynamics is governed by

$$\frac{d}{dt}\mathbf{x}_1 = \hat{f}_1(\mathbf{x}_1, 0) = f_1(\mathbf{x}_1, -m(\mathbf{x}_1)) \tag{2.10}$$

which may be viewed as

$$\frac{d}{dt}\mathbf{x}_1 = f_1(\mathbf{x}_1, v); \quad v = -m(\mathbf{x}_1) \tag{2.11}$$

Since, according to (2.4), the function $m(\mathbf{x}_1)$ completely defines the sliding manifold S, it is seen that the sliding manifold specification problem for a required ideal sliding motion is equivalent to the specification of an appropriate smooth feedback control law of the form $v = -m(\mathbf{x}_1)$ on the reduced-order equations (2.11).

Several approaches have been proposed in the literature of variable structure systems in connection with the problem of specification of a desirable sliding surface where the sliding motions are to take place. Among these, the most important are: optimal control, parametric optimization, and pole placement in the linear case (see Utkin and Young 1978, for details on the available techniques in the linear system case).

In the non-linear system case, several options are available as design objectives. If stabilization of the system is the main objective, then a Lyapunov approach may be adopted. In this case, a positive function $V(\mathbf{x}_1)$ is to be proposed such that

$$\frac{d}{dt}V(\mathbf{x}_1) = \frac{\partial V}{\partial \mathbf{x}_1} f_1(\mathbf{x}_1, v) < 0 \quad \text{for some} \quad v = -m(\mathbf{x}_1)$$

the literature on this topic is extensive and well known.

A second possibility is to obtain an invariance property of the sliding manifold without necessarily imposing an asymptotically stable behaviour on the system. In particular, a robust stable limit cycle may be highly desirable in some cases. The variable-structure approach allows for this possibility, as the examples in § 3 show. In the stabilization case, the ideal asymptotically-stable dynamics define the sliding manifold; in the limit-cycle case, the sliding surface specification (circle, sphere, etc.) determines the corresponding dynamics.

2.4. *Local reachability conditions*

Once the ideal sliding dynamics and the existence of the equivalent control have been established, the design process is usually turned into the issue of specifying the VSC gains that guarantee sliding manifold reachability. Locally, this task is accomplished if and only if the resulting velocity vector field of the controlled system is made to point towards the manifold in its immediate vicinity. In other words, the directional derivative of the scalar function representing the surface with respect to the velocity vector field $f + gu$ must have different signs on each side of the surface. In the limit, these conditions must hold and we have the following proposition.

Proposition 4

The necessary and sufficient condition for a sliding regime to exist is that

$$\lim_{s \to 0^+} L_{f+gu}s < 0 \quad \text{and} \quad \lim_{s \to 0^-} L_{f+gu}s > 0 \tag{2.12}$$

The proof of this fact is obvious from the geometric considerations already furnished above.

The meaning of the above conditions is geometrically clear, i.e. if the trajectories approach S from negative values of s the control function has to produce a positive rate of approach to the surface in order to hit it. If, on the contrary, the trajectories approach S by positive values of s, the control function should make s decrease and therefore produce a negative rate of growth to s. If these two conditions are locally enforced, the sliding manifold is reached from any point in its immediate vicinity.

Notice that

$$L_{\hat{f}+\hat{g}u}s = \frac{\partial s}{\partial \mathbf{x}_1}\hat{f}_1(\mathbf{x}_1, s) + \hat{f}_n(\mathbf{x}_1, s) + \hat{g}_n(\mathbf{x}_1, s)u$$

$$= \frac{\partial m}{\partial \mathbf{x}_1}f_1(\mathbf{x}_1, s - m(\mathbf{x}_1)) + f_1(\mathbf{x}_1, s - m(\mathbf{x}_1)) + g_n(\mathbf{x}_1, s - m(\mathbf{x}_1))u$$

Using conditions (2.12) we immediately obtain

$$\frac{\partial m}{\partial \mathbf{x}_1}f_1(\mathbf{x}_1, -m(\mathbf{x}_1)) + f_n(\mathbf{x}_1, -m(\mathbf{x}_1)) - g_n(\mathbf{x}_1, -m(\mathbf{x}_1))u^+(\mathbf{x}_1) < 0$$

i.e.

$$u^+(\mathbf{x}_1) > \frac{\frac{\partial m}{\partial \mathbf{x}_1}f_1(\mathbf{x}_1, -m(\mathbf{x}_1)) + f_n(\mathbf{x}_1, -m(\mathbf{x}_1))}{g_n(\mathbf{x}_1, -m(\mathbf{x}_1))} = u_{EQ}(\mathbf{x}_1) \tag{2.13}$$

Similarly, from the second condition of (2.12):

$$\frac{\partial m}{\partial \mathbf{x}_1}f_1(\mathbf{x}_1, -m(\mathbf{x}_1)) + f_n(\mathbf{x}_1, -m(\mathbf{x}_1)) - g_n(\mathbf{x}_1, -m(\mathbf{x}_1))u^-(\mathbf{x}_1) > 0$$

i.e.

$$u^-(\mathbf{x}_1) < \frac{\frac{\partial m}{\partial \mathbf{x}_1}f_1(\mathbf{x}_1, -m(\mathbf{x}_1)) + f_n(\mathbf{x}_1, -m(\mathbf{x}_1))}{g_n(\mathbf{x}_1, -m(\mathbf{x}_1))} = u_{EQ}(\mathbf{x}_1) \tag{2.14}$$

The variable structure feedback gains are responsible for the transversality condition that makes the trajectory reach the surface. The stipulation is highly dependent upon the value of the equivalent control and takes its value as a reference level in order to produce a convenient 'tilting' of the ideal sliding vector field produced by u_{EQ} such that surface reachability is guaranteed from any 'side' of the sliding manifold.

3. Examples

Example 1: *Sliding motions on a circle*

Consider the system in R^2:

$$\dot{x}_1 = -x_2 - x_1(x_1^2 + x_2^2 - u); \quad \dot{x}_2 = x_1 - x_2(x_1^2 + x_2^2 - u)$$

For $u = r^2 =$ constant, the system exhibits an asymptotically stable limit cycle represented by a circle of radius r centred at the origin of coordinates. We show that a VSC makes this limit cycle reachable in a finite time.

The vector fields $f(x)$, $g(x)$ and the distribution $\Delta_G(x)$ are given by

$$f(x_1, x_2) = (-x_2 - x_1(x_1^2 + x_2^2))\frac{\partial}{\partial x_1} + (x_1 - x_2(x_1^2 + x_2^2))\frac{\partial}{\partial x_2}$$

$$g = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}; \quad \Delta_G(x) = \text{span}\left\{x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right\}$$

A transformation to regular form is accomplished by describing the system in polar coordinates: $\rho = (x_1^2 + x_2^2)^{1/2}$; $\theta = \tan^{-1}(x_2/x_1)$. Then

$$\frac{d}{dt}\theta = 1 \quad \text{and} \quad \frac{d}{dt}\rho = -\rho(\rho^2 - u)$$

describe the system. The transformed vector fields are now

$$\hat{f} = \frac{\partial}{\partial \theta} + \rho^3 \frac{\partial}{\partial \rho}; \quad \hat{g} = -\rho \frac{\partial}{\partial \rho}$$

Let $s = \rho - r$ be the surface coordinate equation with $r > 0$. A sliding surface candidate is obtained when $s = 0$, represented in this case by a circle of radius r. Then, $\Delta_S = \text{span}\ \{\partial/\partial\theta\}$. The transversality condition is clearly satisfied.

In terms of 'sliding and surface coordinates' we have

$$\frac{d}{dt}\theta = 1 = \frac{(\rho - s)}{r}$$

$$\frac{d}{dt}s = -(s + r)[(s + r)^2 - u]$$

The ideal sliding conditions require that $s = 0$ and $\partial/\partial\theta - r[r^2 - u]\,\partial/\partial\rho \in \text{span}\ \{\partial/\partial\theta\}$ which is only possible if $r^2 - u = 0$, i.e. $u_{\text{EQ}} = r^2$, and the ideal sliding dynamics take place on $\rho = r$. It follows from the reachability conditions that the controller

$$u = \begin{cases} \alpha > r^2 & \text{for } s < 0 \\ \beta < r^2 & \text{for } s > 0 \end{cases}$$

achieves sliding motions on $s = 0$. In the original coordinates, the control law is exactly the same except that the value of s above must be substituted by $(x_1^2 + x_2^2 - r)^{1/2}$, which is not in the standard form we adopted in (2.4). Since the trajectories reach the sliding manifold transversally, the limit cycle is substituted by a sliding circle, which is reachable in a finite time (see Figs. 2(*a*), (*b*) and (*c*)).

Example 2: Harmonic Van der Pol Oscillator

Consider the structure-controlled Van der Pol equation

$$\dot{x}_1 = x_2; \quad \dot{x}_2 = 2\zeta\omega_0(1 - \mu x_1^2)x_2 u - \omega_0^2 x_1$$

When $u = +1$, a stable limit cycle (unstable origin) exists surrounding the origin of coordinates. With $u = -1$, an unstable limit cycle (stable origin) is obtained. This limit cycle is the mirror image on the x_2 axis of the preceding limit cycle (see Figs. 3 and 4).

In this case we have:

$$f = x_2 \frac{\partial}{\partial x_1} - \omega_0^2 x_1 \frac{\partial}{\partial x_2}$$

$$g = 2\zeta\omega_0(1 - \mu x_1^2)x_2 \frac{\partial}{\partial x_2}$$

The distribution $\Delta_G(x)$ is given by the span $\{(1 - \mu x_1^2)x_2(\partial/\partial x_2)\}$. We take as the sliding surface a circle of radius r in the normalized coordinates $(x_1, x_2\omega_0)$ (an ellipse in the unnormalized coordinates (x_1, x_2)). The tangent space to the sliding manifold,

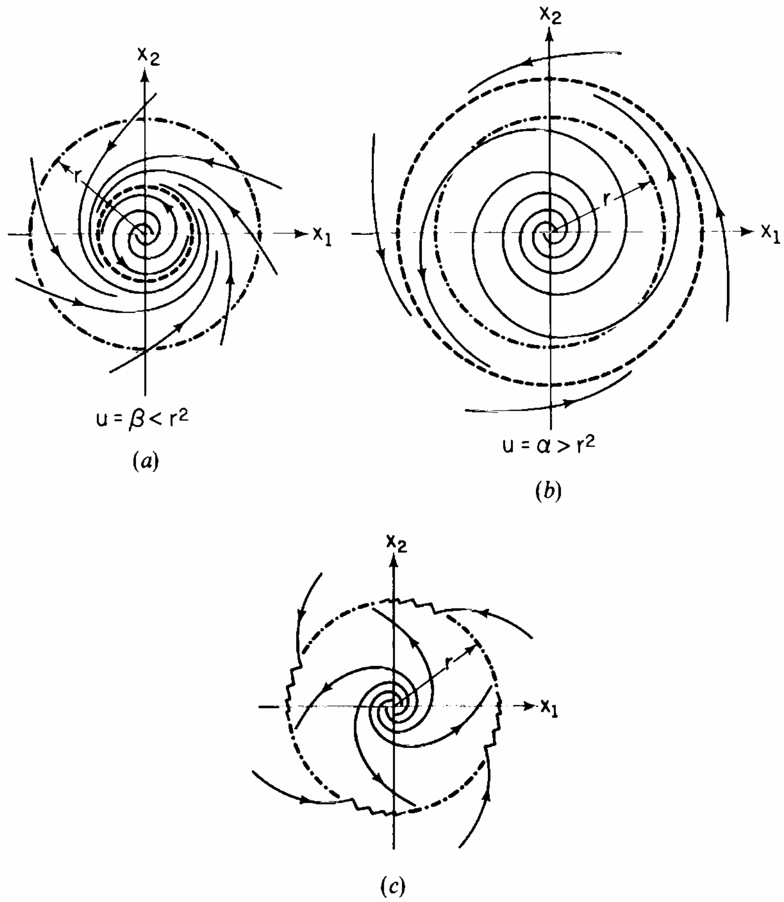


Figure 2. Finite time reachability of limit cycle via variable structure control.

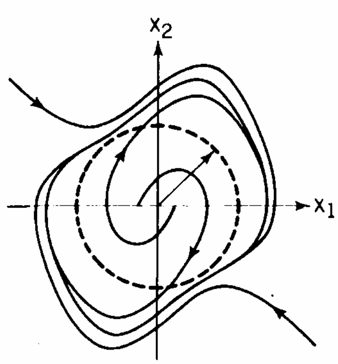


Figure 3. Van der Pol's stable limit cycle.

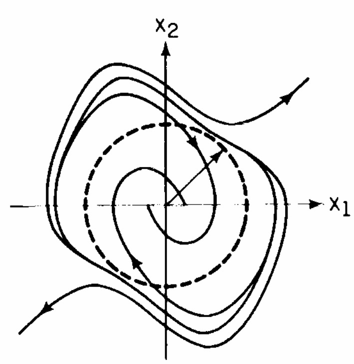


Figure 4. Van der Pol's unstable limit cycle.

or sliding distribution, is $\Delta_S(x) = \text{span}\,\{x_2(\partial/\partial x_1) - \omega_0 x_1(\partial/\partial x_2)\}$. Since in this case $f \in \Delta_S(x)$, the equivalent control must anihilate the components of the input field along $\Delta_G(x)$. It follows immediately that $u_{EQ} = 0$. The ideal sliding is governed by

$$\frac{d}{dt}x_1 = \omega_0(r^2 - x_1^2)^{1/2} \quad \text{for} \quad x_2 > 0$$

$$\frac{d}{dt}x_1 = -\omega_0(r^2 - x_1^2)^{1/2} \quad \text{for} \quad x_2 < 0$$

whose solution is $x_1 = r\cos(w_0 t)$, which, joined to $x_2 = -rw_0 \sin(w_0 t)$, renders the equation of a circle of radius r in the normalized coordinates. Note that although $g(x) = 0$ on $x_2 = 0$, the sliding condition is not violated at this point thanks to the fact that $f(x)$ is a tangent to the circle at these points. The variable structure gains which locally guarantee reachability are given by the conditions:

$$\lim_{s\to 0^+} L_{f+gu}s = 2\zeta\omega_0(1 - \mu x_1^2)x_2 u^+ < 0$$

$$\lim_{s\to 0^-} L_{f+gu}s = 2\zeta\omega_0(1 - \mu x_1^2)x_2 u^- > 0$$

The above conditions imply that the switching logic is simply: $u = -1$ for $s > 0$, and $u = +1$ for $s < 0$. It is also clear from above that the sign of the term $(1 - \mu x_1^2)$ must remain unchanged for the reachability conditions to hold true. This in turn implies that the sliding circle must satisfy the constraint: $|x_1| < 1/\sqrt{\mu}$, i.e. the radius of the sliding circle is constrained to $r < 1/\sqrt{\mu}$.

Figure 5 shows that a sinusoidal response is obtained by switching among two Van der Pol systems. The sliding motion is robust inside the region covered by the 'reverse time' Van der Pol oscillator limit cycle.

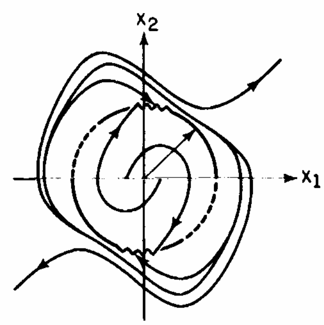


Figure 5. Sliding harmonic motions for a VSC controlled Van der Pol oscillator.

Example 3: *sliding modes on the torus*

Consider the system of equations

$$\frac{dx_1}{dt} = x_2; \quad \frac{dx_2}{dt} = -w_1^2 x_1; \quad \frac{dx_3}{dt} = ux_4; \quad \frac{dx_4}{dt} = -uw_2^2 x_3$$

in R^4. With $u = \text{constant}$, this system evolves on the direct product manifold represented by two two-dimensional circles in R^4. The motion is thus representable as confined to a torus in R^3. This surface in turn can be diffeomorphically represented in the plane R^2 by specifying two angular coordinates θ_1 (longitude) and θ_2 (latitude) modulo 2π. If additionally, a 'pasting' of points in the square: $0 \leqslant \theta_1 \leqslant 2\pi$; $0 \leqslant \theta_2 \leqslant 2\pi$ is exercised by identifying the points $(\theta_1, 0)$ and $(\theta_2, 2\pi)$ as well as the pair of points $(0, \theta_2)$ and $(2\pi, \theta_2)$, the motions can then be analysed on this 'mapping' of the torus. The differential equations involved are now simply

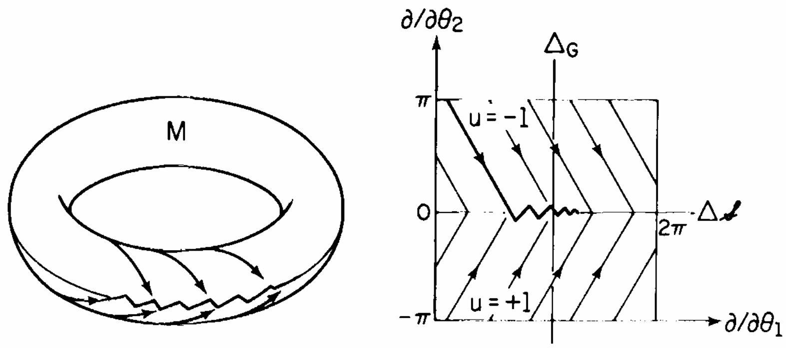
$$\frac{d}{dt}\theta_1 = w_1; \quad \frac{d}{dt}\theta_2 = uw_2$$

where θ_1 is the angle measured from x_2 towards x_1 in the plane x_1, x_2 and similarly θ_2 is measured from x_4 towards x_3 in the plane x_3, x_4. When $u = +1$ we have 'inwards forward winding' and with $u = -1$ we obtain 'outwards forward winding' of the torus surface in R^3. Also, $w_2/w_1 > 1$ provides us with 'fast winding' while $w_2/w_1 < 1$ represents 'slow winding'. Let us assume the former case in our example.

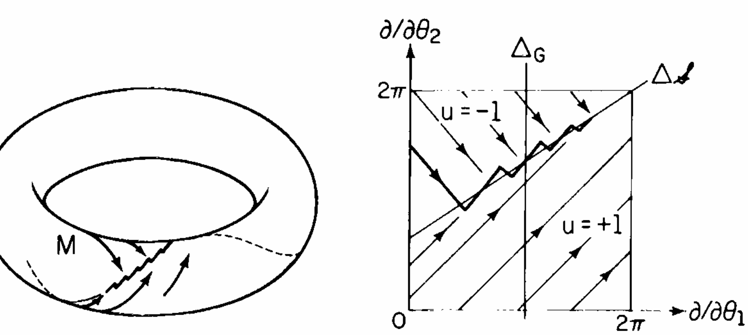
In this case $f = w_1(\partial/\partial\theta_1)$, $g = w_2(\partial/\partial\theta_2)$; $\Delta_g = \text{span}\,\{\partial/\partial\theta_1\}$. Let $s = \theta_2$. The condition $s = 0$ describes the zero-latitude greater circle of the torus. The tangent space to this submanifold is $\Delta_S = \text{span}\,\{\partial/\partial\theta\}$, then, $\theta_1 = w_1$; $\dot{s} = uw_2$. The ideal sliding requires $w_1(\partial/\partial\theta_1) + uw_2(\partial/\partial\theta_2) \in \text{span}\,\{\partial/\partial\theta_1\}$, i.e. $uw_2 = 0$; $u_{EQ} = 0$. Thus the VSC:

$$u = \begin{cases} +1 & \text{for } \theta_2 < 0 \\ -1 & \text{for } \theta_2 > 0 \end{cases}$$

produces a sliding motion on $\theta_1 = 0$, as desired (see Fig. 6(*a*)).



(*a*) Longitudinal sliding on the torus.



(*b*) Sliding motion along a winding line of the torus.

Figure 6. Sliding motions on the torus.

A sliding motion can also be created around a closed or otherwise dense winding line of the torus. In this case $\theta_2 = K\theta_1$ for $0 < K < 1$ (deceleration of winding frequency). Then, $s = \theta_2 - K\theta_1$ and $\theta_1 = w_1$; $\dot{s} = -Kw_1 + uw_2$,

$$\Delta_S = \text{span}\left\{\frac{\partial}{\partial\theta_1} + K\frac{\partial}{\partial\theta_2}\right\} \quad \text{and} \quad \Delta_g = \text{span}\left\{\frac{\partial}{\partial\theta_2}\right\}$$

The ideal sliding conditions require $f + gu \in \Delta_S$. The ideal sliding takes place under the condition:

$$w_1\left(\frac{\partial}{\partial\theta_1} + K\frac{\partial}{\partial\theta_2}\right) + (-Kw_1 + uw_2)\frac{\partial}{\partial\theta_2} \in \text{span}\left\{\frac{\partial}{\partial\theta_1} + K\frac{\partial}{\partial\theta_2}\right\}$$

It follows that $w_2 u = Kw_1$; then $u_{\text{EQ}} = K(w_1/w_2) < 1$, thus obtaining a deceleration of the winding motion, which is now described by: $\dot{\theta}_1 = w_1$; $\dot{\theta}_2 = Kw_1$. Reachability is then accomplished by the VSC:

$$u = \begin{cases} +1 & \text{for } \theta_2 - K\theta_1 < 0 \\ -1 & \text{for } \theta_2 - K\theta_1 > 0 \end{cases}$$

The sliding motion for this part of the example is shown in Fig. 6(*b*).

The following proposition is an easy consequence of well-known results about differential equations describing winding lines of the torus (see Arnold 1985).

Proposition 3

If K is a rational number, then the sliding motion occurs in a closed winding line (i.e. open subsets of the torus are not reachable). Otherwise, the sliding submanifold is dense in the torus and, ideally, every point of the torus is made reachable through the proposed VSC.

Example 4: *sliding motions on the sphere*

Consider the system of differential equations:

$$\frac{d}{dt}\xi = \frac{1}{2}\frac{[\xi - u(2\eta - \xi)]}{1 + \xi^2 + \eta^2}; \quad \frac{d}{dt}\eta = \frac{1}{2}\frac{[2\xi + \eta + u\eta]}{1 + \xi^2 + \eta^2}$$

representing the equatorial stereographic description of a dynamic system evolving on the sphere. With $u = -1$, the north pole of the sphere is a saddle point for the unstable trajectories that emerge and die at the south pole (see Fig. 7). For $u = +1$, the south pole of the sphere is a stable equilibrium point of the spiral trajectories arising from the unstable north pole. This is illustrated in Fig. 8.

It is easy to show that there exists a meridian line (obtained as the intersection of the sphere with a plane containing both poles) on which an asymptotically stable sliding motion can be created. This motion leads all trajectories to the north pole, which now acts as a global attractor (see Fig. 9). In stereographic coordinates, this meridian line is represented as $S = \{(\xi, \eta) : s = \eta + K\xi = 0;\ 1 < K < \infty\}$. In this case we have

$$\frac{ds}{dt} = \frac{1}{2}\frac{[s + 2\xi + u(s - 2Ks + 2K^2\xi]}{1 + \xi^2 + (s - K\xi)^2} = \frac{1}{2}\frac{M(1 + 2M)s - 2M^2\eta + u(Ms - 2\eta)}{2M[1 + \eta^2 + M^2(s - \eta)^2]}$$

where $MK = 1$.

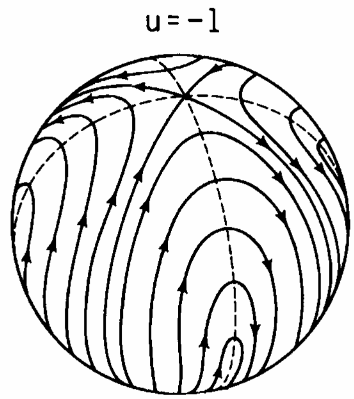


Figure 7. Trajectories on the sphere for $u = -1$.

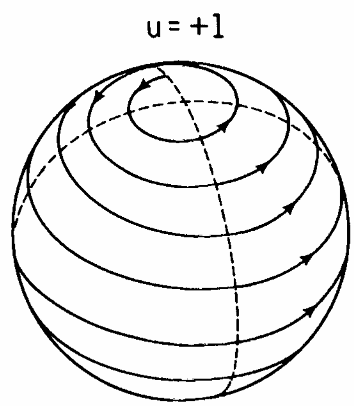


Figure 8. Trajectories on the sphere for $u = +1$.

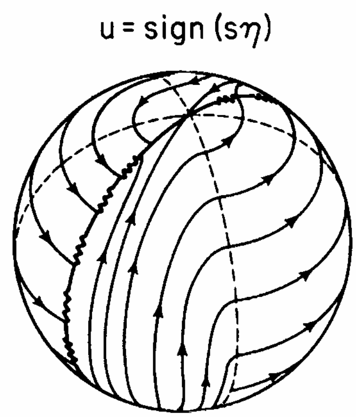


Figure 9. Sliding motions along a meridian line on the sphere.

From above, it follows that the equivalent control is simply

$$u_{EQ} = -\frac{1}{K^2}$$

and the equivalent dynamics is given by either

$$\frac{d}{dt}\xi = \frac{1}{2}\frac{[K^2 - 2K - 1]}{1 + (1 + K^2)\xi^2}\xi$$

or

$$\frac{d}{dt}\eta = \frac{1}{2}\frac{[K^2 - 2K - 1]}{K^2 + (1 + K^2)\eta^2}\eta$$

which, in either case, represents an asymptotically stable system for any value of $K \in (-0{\cdot}41, 2{\cdot}4142)$, i.e. our design may use any value of $K \in (1, 2{\cdot}4142)$.

The swtiching logic is synthesized as

$$u = \begin{cases} +1 & \text{for } s\eta > 0 \\ -1 & \text{for } s\eta < 0 \end{cases}$$

Above, we obtained the sliding regime in a rather direct fashion. We may however proceed to state the analytic aspects of this problem, following the developments of the paper. These are summarized below.

We propose a state coordinate transformation of the form $z_1 = T_1(\xi, \eta)$; $z_2 = T_2(\xi, \eta)$. The system is in regular form whenever

$$\frac{\partial T_1}{\partial \xi}(\xi - 2\eta) + \frac{\partial T_1}{\partial \eta}\eta = 0$$

The solution to this partial differential equation is given by

$$z_1 = \eta^2 \exp(\xi/\eta)$$

If we further let

$$z_2 = \xi^2 \exp(\xi/\eta)$$

The inverse transformation is easily seen to be

$$\xi = \sqrt{z_2}\exp[-\tfrac{1}{2}(z_2/z_1)^{1/2}]; \quad \eta = \sqrt{z_1}\exp[-\tfrac{1}{2}(z_2/z_1)^{1/2}]$$

The differential equation for z_1 and z_2 is then obtained as

$$\frac{d}{dt}z_1 = \frac{z_1 - z_2 + 2(z_1 z_2)^{1/2}\exp[-(z_2/z_1)^{1/2}]}{1 + (z_1 + z_2)\exp[-(z_1/z_2)^{1/2}]}$$

$$\frac{d}{dt}z_2 = \frac{z_2}{1 + (z_1 + z_2)\exp[-(z_2/z_1)^{1/2}]}\left(1 - \frac{z_2}{z_1}\right) - 2\frac{(z_1 z_2)^{1/2}}{1 + (z_1 + z_2)\exp[-(z_2/z_1)^{1/2}]}u$$

The sliding meridian, in the new coordinates, is represented by the equation

$$\sqrt{z_2} = -\frac{\sqrt{z_1}}{K}$$

The system has now been reduced to regular form, and one may proceed as in the previous examples.

4. Conclusions and suggestions for further research

In this article, we have explored the possibilities of addressing the problem of inducing sliding regimes in non-linear systems governed by variable structure feedback controllers using notions from differential geometry. It was shown that all the notions and concepts dealt with in the design of variable structure controllers can naturally be placed in terms of differential geometric objects. As a result, the design issue is provided with the advantages of an intuitive understanding of the main steps by which sliding regimes are created in variable structure systems. The approach is general enough to be directly applied to the case of non-linear smooth systems that are linear in the control and evolve naturally on differentiable manifolds embedded in R^n.

The interpretation, in geometric terms, of the key ingredients to be considered in the design problem, namely specification of the sliding submanifold in terms of desirable invariant behaviour, the notion of equivalent control and finally, the local reachability conditions for the existence of a sliding regime, result in a convenient, simple and sufficiently general methodology for the attack of this class of problem. The equivalent control is seen to play an essential role in the specification of the variable structure feedback gains. This idealized feedback control provides a reference level on which to assess the necessary feedback variable structure gains level leading to a control action that achieves sliding manifold reachability.

Sliding motions in non-linear systems possess a much richer variety of possibilities for inducing desirable motions than their linear counterparts do. For instance, in non-linear systems, asymptotic stability of the controlled system is just one of the options: limit cycles of a preassigned nature, usually impossible in linear structures, may be highly desirable in certain applications (biped locomotion being a typical example). One of the advantages of using the VSS approach in non-linear plants is also the possibility of obtaining a speeding up of the time response towards a limit cycle without destroying the nature of the sustained oscillatory response. In general, radical new properties of the controlled system can be obtained which are absent from any of the intervening structures.

A number of interesting applications may come out as a result of using VSS theory on non-linear dynamic systems. Aerospace applications of control theory usually deal with systems naturally described on differentiable manifolds (tumbling satellites, etc). The VSS approach has been of interest for the stabilization of linearized power systems plants. The possibility of working directly on the non-linear descriptions and inducing stabilizing non-linear sliding regimes is totally open for research and contributions.

Finally, the design of asymptotic observers for non-linear systems with variable structure gains deserves attention, specifically in connection with the surface co-ordinate estimation. This problem is of paramount importance in the design of the logic portion of the VSC regulator.

Acknowledgment

This work was supported in part by the Joint Services Electronics Program under Contract N00014-84-C-0149.

References

Andreev, Y. N., 1982, *Automn Remote Control*, **43**, 1231.

ARNOLD, V. I., 1985, *Ordinary Differential Equations*, fourth edition (Cambridge, Mass.: MIT Press).

BROCKETT, R. W., 1976, *Proc. Inst. elect. electron. Engrs*, **64**, 61.

EMELYANOV, S. V., 1967, *Variable Structure Controls* (Moscow: Nauka), in Russian, *also* (Munich: Oldenburg Verlag), in German.

ERSHLER, J., ROUBELLAT, F., and VERNHES, J. P., 1974, *Automatica*, **10**, 37.

HERMANN, R., and KRENER, A., 1970, *I.E.E.E. Trans. autom. Control*, **22**, 728.

ISIDORI, A., 1982, *Lecture Notes in Control and Informations Sciences* (New York: Springer-Verlag); 1985, *Lecture Notes in Control and Information Sciences*, Vol. 72 (New York: Springer-Verlag).

ISIDORI, A., KRENER, A. J., GORI-GIORI, C., and MONACO, S., 1981, *I.E.E.E. Trans. autom. Control*, **26**, 331.

ITKIS, U., 1976, *Control Systems of Variable Structure* (New York: Wiley).

LUK'YANOV, A. G., and UTKIN, V. I., 1981, *Automn Remote Control*, **42**, 5.

SLOTINE, J. J., and SASTRY, S. S., 1983, *Int. J. Control*, **38**, 465.

SIRA-RAMIREZ, H., 1986, *Conf. on Information Sciences and Systems*, Princeton University, Princeton.

SUSSMAN, H. J., and JURDJEVIC, V. J., 1972, *J. diff. Eqns*, **12**, 95.

UTKIN, V. I., (editor), 1968, *Variable Structure Systems and Their Application to Flight Automation* (Moscow: Nauka); 1977, *I.E.E.E. Trans. autom. Control*, **22**, 212; 1978, *Sliding Modes and Their Applications to Variable Structure Systems* (Moscow: MIR); 1981, *Sliding Modes in Problems of Optimization and Control* (Moscow: Nauka), in Russian; 1983, *Automn Remote Control*, **44**, 1105.

UTKIN, V. I., and YOUNG, K. K. D., 1978, *Automn Remote Control*, **10**, 72.

YOUNG, K. K. D., and KWATNY, H. G., 1982, *Automatica*, **18**, 384.