Harmonic Response of Variable-Structure-Controlled Van der Pol Oscillators

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Abstract —**A variable-structure-controlled (VSC) Van der Pol oscillator is shown to produce an ideal average response of perfect harmonic nature when its motions are made to slide on a circle defined in the interior of its limit cycle. The sinusoidal responses are robust with respect to parameter perturbations and external disturbances.**

I. Introduction

The variable-structure control of dynamic systems undergoing sliding motions on nonlinear manifolds offers a richer variety of design alternatives than those possible with linear hyperplanes. These possibilities are based on the fact that a larger class of static relationships can be synthesized, among the state variables, when nonlinear surface values are used in the switching logic controlling the system. The dynamic behavior of the controlled system is totally determined by the nature of the nonlinear surface, making the controlled system robust with respect to external disturbances and parameter perturbations.

Sliding surface reachability and invariance are essential ingredients of the sliding motion design for variable-structure systems (VSS's). These tasks are accomplished by opportune switchings among feedback laws, which guarantee state trajectories invariably directed towards the sliding surface. In the sliding regime, one of the outstanding characteristics of the controlled motion is that radically new properties are obtained compared with those of the individual structures responsible for its creation.

The reader is referred to several books (Utkin [1], Utkin [2], Itkis [3]) and survey articles (Utkin [4], Utkin [5]) for a complete account of the theory and its many practical applications.

Using the theory of variable-structure control [1], the possibility of creating harmonic limit cycles in controlled Van der Pol oscillators is explored. An ideally sinusoidal response is obtained

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whose amplitude can be varied at will within a certain robustness range where the oscillations are immune to the effects of parameter variations and external perturbations. From two feedback structures yielding nonsinusoidal oscillatory responses, a nearly perfect harmonic behavior is obtained (modulo small chattering dependent upon the switchings velocity).

Undoubtedly, there are simpler ways of creating sinusoidal oscillators. However, the point stressed in this article is the possibility of building a nearly perfect, robust, sinusoidal oscillator out of a Van der Pol system which naturally exhibits a nonsinusoidal periodic response in its limit cycle.

II. VSC Van der Pol Oscillators

A Van der Pol oscillator represents a simple class of variable damped systems. In the region of negative damping, generally occurring when the signals are small, the damping increases the energy level of the response, thus steadily increasing its amplitude. Conversely, as the signals' amplitudes grow larger, the damping becomes positive, thus decreasing the energy of the output signal. As a consequence of this, the motion reaches a stable, or globally attractive, **limit cycle**.

In the past, a good deal of attention was devoted to shaping the response of this oscillator so as to obtain a quasi-sinusoidal response. The proposed schemes were complex and, in many instances, without the benefits of robustness and fast switchings made possible by modern solid-state electronics.

When the time variable t is replaced by $-t$ in the differential equations describing the Van der Pol oscillator, the "reverse time Van der Pol oscillator" is obtained. This system exhibits a globally repulsive limit cycle.

Consider the controlled Van der Pol oscillator represented by the system of differential equations

$$\frac{d}{dt}x_1 = x_2$$
$$\frac{d}{dt}x_2 = 2\xi w\left(1-\mu x_1^2\right)x_2 u - w^2 x_1. \quad (1)$$

For $u=+1$, this system is known to possess an unstable origin and a stable limit cycle bounding a circle of radius $r = 1/\sqrt{\mu}$ (see Fig. 1). A reversal of the state trajectories direction, followed by a $180°$ rotation of the R^2 plane, yields the limit cycle corresponding to $u=-1$. The effect of this control on the state equations is equivalent to still having $u=+1$ and replacing t by $-t$ and x_2 by $-x_2$ in (1). A stable origin and unstable limit cycle are obtained for this value of the input (see Fig. 2).

Consider the switching "surface" (line)

$$S = \left\{(x_1,x_2)\in R^2 : s = x_1^2 + \frac{x_2^2}{w^2} - r^2 = 0 : r < \frac{1}{\sqrt{\mu}}\right\} \quad (2)$$

which represents a circle of radius r in the normalized coordinates $(x_1, x_2/w)$.

Define $\partial/\partial x_1$ and $\partial/\partial x_2$ as the unit vectors in the directions of the global coordinates x_1, x_2, respectively. These vectors span the tangent space of R^2 at each point and constitute a **coordinate frame**. Also, let f and g represent the vector fields defining the integral curves of the Van der Pol oscillator written in control affine form: $\dot{x} = f + gu$:

$$f = x_2\frac{\partial}{\partial x_1} - w^2 x_1\frac{\partial}{\partial x_2}$$
$$g = 2\xi w\left(1-\mu x_1^2\right)x_2\frac{\partial}{\partial x_2}. \quad (3)$$

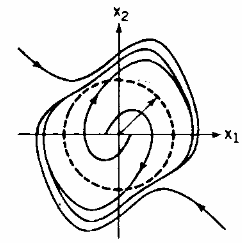

Fig. 1. Van der Pol's stable limit cycle.

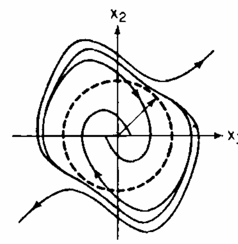

Fig. 2. Van der Pol's unstable limit cycle.

In this notation, the **distribution**, or tangent subspace, to the circle S is characterized by

$$\Delta_s = \text{span}\left\{x_2\frac{\partial}{\partial x_1} - w^2 x_1\frac{\partial}{\partial x_2}\right\} \quad (4)$$

while a normal (gradient or differential) to the switching surface is given by

$$N = w^2 x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}. \quad (5)$$

Under ideal sliding, the following invariance conditions are satisfied:

1) $$s = 0$$

2) $$\frac{ds}{dt} = 0 \quad (6)$$

i.e., in an **average sense**, the system trajectories satisfy the equation of the circle S and they do not abandon S after reaching it. The average, ideal value of the smooth feedback control that maintains the state confined to the switching surface S is known as the **equivalent control** [1], denoted here by u_{EQ}.

Condition 2 in (6) is clearly seen to be equivalent to

$$\frac{ds}{dt} = \frac{\partial s}{\partial x}\dot{x} = \langle ds, f + gu_{EQ}\rangle = 0 \quad (7)$$

where ds denotes the gradient of the surface. This vector, clearly, coincides with N. The kernel of the gradient is the tangent subspace to S. Equation (7) leads then to

$$f + gu_{EQ} \in \text{Ker}\, ds. \quad (8)$$

In other words, under ideal sliding conditions, the controlled vector field $f + gu_{EQ}$ is to belong to the subspace tangent to the surface (also known as the sliding distribution [6]). The equivalent control is then seen to annihilate all components of the controlled vector field which do not belong to this tangent subspace.

From (3) and (4), it follows that the drift vector field f is in the span of the tangent subspace to S and therefore $\langle N, f\rangle = 0$.

By condition (8), the equivalent control must set the components of g to zero. We then have

$$u_{EQ} = 0. \tag{9}$$

Substitution of the equivalent control in the original system equations (1) results in the **ideal sliding dynamics**, governed by

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -w^2 x_1 \tag{10}$$

which are the equations of a harmonic oscillator producing perfect sinusoidal responses.

The variable-structure controller is specified as

$$u = \begin{cases} u^+ & \text{for } s > 0 \\ u^- & \text{for } s < 0. \end{cases} \tag{11}$$

This controller is to locally drive the state trajectories towards the surface S. Once the surface is reached, the variable-structure controller must sustain a sliding motion on the manifold by means of active, persisting switchings, based on the surface coordinate sign, among the limit values (or structures) of the feedback control law.

Necessary and sufficient conditions for local existence of sliding motions on the surface S demand that the resulting controlled vector fields of (1) and (11) point towards S in its immediate vicinity. For this, the following sign conditions have to be satisfied by the projections of the controlled fields onto the vector N, normal to the surface:

$$\lim_{s \to 0^+} \langle N, f + gu^+ \rangle < 0; \ \lim_{s \to 0^-} \langle N, f + gu^- \rangle > 0. \tag{12}$$

Since $\langle N, f \rangle = 0$, it follows from (12), (3), and (5) that

$$2\xi w(1 - \mu x_1^2) x_2^2 u^+ < 0$$

$$2\xi w(1 - \mu x_1^2) x_2^2 u^- > 0. \tag{13}$$

Notice that inside the band $x_1^2 < 1/\mu$, the factor $(1 - \mu x_1^2)$ is always positive for any value of s. The variable-structure controller $u^+ = -1$; $u^- = +1$ guarantees the existence of the sliding motion in the region determined by the intersection of this band and the area covered by the interior of the "reverse time, rotated" limit cycle of Fig. 2. Outside this limit cycle, reachability of the sliding circle cannot be guaranteed. The magnitude restriction on the radius r of the circle given by $r < 1/\sqrt{\mu}$, as specified in (2), represents a measure of robustness for the sinusoidal response of the system, and an upper bound for the achievable wave amplitudes generated by the scheme.

It is also clear that on the limit points of the band, where $x_2 = 0$, the sliding condition (13) cannot be guaranteed. However, it is a fortunate fact that, precisely at these points, the drift vector field f (tangent to the circle at each point) is the only acting field defining the integral curve direction. The sliding motion exists everywhere in the circle except at these two isolated points where the trajectories happen to be tangent to it.

Notice that upon sliding surface synthesis by means of appropriate hardware, the variable-structure scheme requires only one bit of information in order to decide control switching, namely, sliding surface sign (rather than its actual value). Maintaining the state trajectories on the sliding surface requires only an accurate sign detector and a fast relay.

Fig. 3 shows the phase trajectories of the variable-structure-controlled system, while Fig. 4(a) and (b) depicts some typical time responses. These graphs were obtained using $w = 1$; $\mu = 1$; $\xi = 0.5$. The *Matrix*$_x$ package was used for the computer simulations.

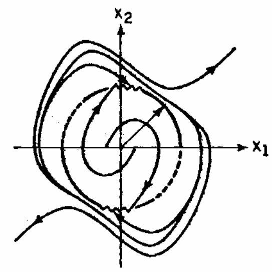

Fig. 3. Sliding harmonic motions for a VS-controlled Van der Pol oscillator.

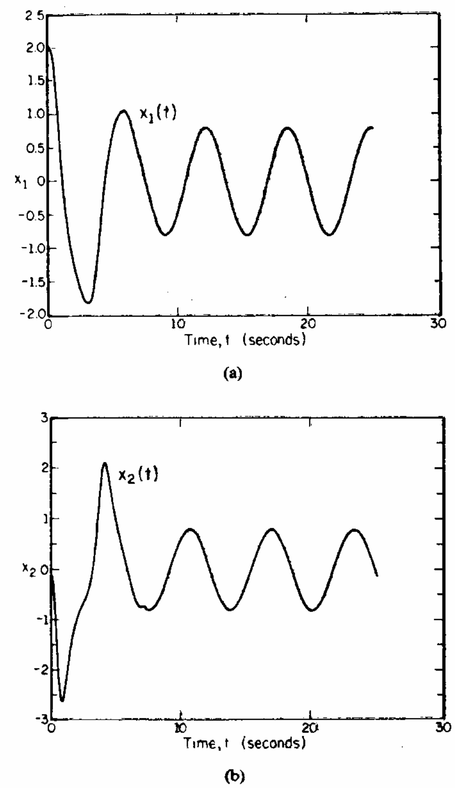

Fig. 4. Time response of variable-structure-controlled Van der Pol oscillator.

III. Conclusions and Suggestions for Further Research

The theory of variable-structure systems was used to generate sinusoidal oscillations on a nonlinear system described by the Van der Pol differential equations. Such response is obtained by using a circle, in the phase space, as a sliding surface. Robust sliding conditions are obtained almost everywhere on the sliding circle. The use of nonlinear sliding surfaces offers a richer variety of reduced order motions. For instance, limit cycles, which are impossible to obtain using linear sliding surfaces, constitute just one of the possibilities when using nonlinear switching manifolds.

Some elementary notions of differential geometry were shown to be intuitively useful in the determination of essential features regarding the sliding-mode design process (see Sira-Ramirez [6]).

The variable-structure-control approach for control systems design is gaining more and more popularity thanks to the wide area of possible applications (power systems control, aerospace design problems, robot and manipulator control, switch-mode power conversion, etc.). The inherent robustness and simplicity involved in its underlying "overshoot and correct" philosophy make it an attractive control scheme demanding little on-line information.

Coupled Van der Pol oscillators were recently used for coordinated biped locomotion schemes in Katoh and Mori [7]. The system describing this motion also enjoys sliding regimes on a toruslike manifold in R^4. This fact suggests the use of VSC for the induction of robust, stable, biped locomotion.

References

[1] V. I. Utkin, *Sliding Modes and Its Applications to Variable Structure Systems*. Moscow: MIR, 1978.

[2] V. I. Utkin, *Sliding Modes in Problems of Optimization and Control* (in Russian). Moscow: Nauka, 1981.

[3] U. Itkis, *Control Systems of Variable Structure*. New York: Wiley, 1976.

[4] V. I. Utkin, "Variable structure systems with sliding modes," *IEEE Trans. Automat. Contr.*, vol. AC-22, pp. 212–222, 1977.

[5] V. I. Utkin, "Variable structure systems: Present and future," *Automat. Remote Contr.*, vol. 44, pp. 1105–1120, 1983.

[6] H. Sira-Ramirez, "A differential geometric approach for the design of variable structure systems," in *Proc. Conf. Information Sciences and Syst.* (Princeton University), Mar. 1986.

[7] R. Katoh and M. Mori, "Control method of biped locomotion giving asymptotic stability trajectory," *Automatica*, vol. 20, pp. 405–414, 1984.