Periodic Sliding Motions

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Abstract—This note introduces a general characterization for the global existence of sliding motions, on compact manifolds, in nonlinear variable structure controlled systems. The results are applied, via some illustrative examples, to the case of periodic sliding motions in the plane.

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I. Introduction

In this note, a general geometric characterization of global sliding regimes [1], [2] is proposed for nonlinear variable structure systems (VSS), defined in R^n, which adopt as sliding surfaces smooth manifolds bounding compact regions of R^n (henceforth, such manifolds are called compact manifolds). For general background about VSS, readers are referred to the excellent tutorial by Zak, De Carlo, and Matthews [3].

Under the influence of the flow map, associated to the vector field defining a nonlinear dynamical system, compact regions of the state space evolve in a rather complicated way. However, at each instant of time, the rate of change of the volume of such evolving region is very simply related; in fact, it is equal to the volume integral, over the evolving region, of the divergence of the vector field. This fundamental theorem constitutes a stronger version of Liouville's theorem for linear systems [4, pp. 198 and 195] (see also [5, p. 69, Lemma 1]). The necessary and sufficient conditions for the existence of a sliding motion on a compact manifold are then translated into set inclusion conditions for the set-valued flow map generated by each possible structure of the controlled vector field. From this alternative, particularly characterization of sliding motions, a simple necessary condition for the global existence of a sliding regime is derived. Such condition involves a difference in sign of the rate of change of the controlled volume for each available feedback structure, i.e., a difference in sign of the volume integral of the divergence of the controlled vector field for each possible feedback structure. The manifold invariance condition, characteristic of smooth responses obtained from formal application of the equivalent control to the original controlled dynamics [1], results in having the equivalent flow map, associated to the ideal sliding dynamics, preserve the volume of the compact region.

Section II contains some background definitions and general results about sliding motions on compact manifolds. As applications of the general results, some illustrative examples of periodic sliding modes in the plane are presented in that section. Section III contains the conclusions of the note.

II. Basic Definitions, Main Results, and Some Applications

A. Background Definitions and Results

Consider a nonlinear dynamical system defined in R^n by

$$dx/dt = X(x, u) \tag{2.1}$$

where u is a scalar, possibly discontinuous, feedback control function and X is a smooth vector field for each given smooth u. The flow, generated by the controlled vector field $X(x, u)$, is defined as the one-parameter group of transformations g_u^t of R^n such that $g_u^t: x(0) \to x(t)$, where $x(t)$ is a solution, at time t, of (2.1) for the given u. The vector field $X(x, u)$ is addressed as the generating field of g_u^t. Let D be a compact subset of R^n, the image at time t of D under the flow of X, denoted by $g_u^t(D)$ or by $D(t)$, is defined as $g_u^t(D) = [x \in R^n: x = g_u^t x_0 \text{ for some } x_0 \in D]$. The divergence of a vector field $F(x)$ is defined as div $F(x)$ = Trace $[\partial F/\partial x]$ $= \Sigma_{i=1n}\, \partial F_i/\partial x_i$. A stronger version of Liouville's theorem [4, p. 198] is constituted by the identity $dV(t)/dt|_{t=\tau} = \int_{D(\tau)} \text{div } X(x, u)$ where $V(t)$ denotes the volume of $g_u^t(D)$ and τ is any instant of time. The proof of this theorem can be found in [5, p. 69]. Throughout this note, it is assumed that the boundary of D, denoted by ∂D, is a smooth $n - 1$ dimensional submanifold of R^n characterized by $\partial D = \{x \in R^n: s(x) = 0\}$ where $s: R^n \to R$ is a smooth function with nonzero gradient vector $\partial s/\partial x$ almost everywhere on ∂D. It is also assumed that ∂D is oriented in such a way that $s(x) < 0$ describes the bounded interior of D, while $s(x) > 0$ is the open unbounded complement of D. The vector field ds will denote the unit outer normal vector field of ∂D, i.e., $\partial s/\partial x = \|\partial s/\partial x\| ds$. The divergence theorem [6, p. 151] establishes that $\int_D \text{div } X(x, u) = \int_{\partial D} \langle ds, X(x, u)\rangle$ where $\langle\cdot, \cdot\rangle$ denotes inner product.

Theorem 2.1: Let $X(x, u)$ be the generating field of g_u^t. Then, g_u^t is locally a contraction on D (i.e., for an arbitrarily small $\epsilon > 0$, $D \supset g_u^t(D)$ for all $0 < t < \epsilon$) if and only if for all $x \in \partial D$, $\langle ds, X(x, u)\rangle < 0$. Similarly, g_t^u is locally an expansion on D (i.e., for an arbitrarily small $\epsilon > 0$ $g^t u(D) \supset D$ for all $0 < t < \epsilon$) if and only if for all $x \in \partial D$ $\langle ds, X(x, u)\rangle > 0$.

Proof: Only the contraction part in the theorem will be proved. The expansion part follows by similar arguments.

Let g_u^t be locally a contraction on D, then for each $x \in \partial D$, the vector $g_u^\epsilon(x) - x$ points inwards to the interior of D. Hence, the inner product $\langle ds, g_u^\epsilon(x) - x\rangle < 0$, for an arbitrarily small ϵ. Substituting $g_u^\epsilon(x)$ by its series expansion about x, $g_u^\epsilon(x) = x + \epsilon X(x, u) + h.o.t.$, one finds $\epsilon\langle ds, X(x, u)\rangle + o(\epsilon^2) < 0$, which holds true for any arbitrarily small; positive ϵ if and only if $\langle ds, X(x, u)\rangle < 0$. To prove sufficiency, let $\langle ds, X(x, u)\rangle < 0$ for all $x \in \partial D$, but suppose that $g_u^t(D)$ is not entirely contained in D, i.e., g_u^t is not locally a contraction (see Fig. 1). Then there exists at least one open region of ∂D which has a nonempty intersection with $g_u^t(D)$. Take any x in $g_u^t(D) \cap \partial D$. For a sufficiently small $\epsilon > 0$, the vector $g_u^\epsilon(x) - x$ points outwards of D. Hence, $\langle ds, g_u^\epsilon(x) - x\rangle > 0$. Using again, $g_u^\epsilon(x) = x - \epsilon X(x, u) + h.o.t.$, in the inner product one concludes that $\epsilon \langle ds, X(x, u)\rangle + O(\epsilon^2) > 0$, i.e., $\langle ds, X(x, u)\rangle > 0$ on an open region of ∂D. This is a contradiction. □

Corollary 2.2: Let g_u^t be locally a contraction (expansion) on D. Then, $dV/dt|_{t=0} < 0$ (>0), i.e., $\int_{\partial D} \langle ds, X(x, u)\rangle = \int_D \text{div } X(x, u) < 0$ (>0).

Proof: Immediate from Theorem 2.1 and the divergence theorem. □

B. Conditions for the Existence of Sliding Modes on Compact Manifolds

A variable structure control law, with discontinuity surface ∂D, is a specification of a feedback control policy $u(x)$, on (2.1), according to

$$u(x) = \begin{cases} u^+(x) & \text{for } s(x) > 0 \\ u^-(x) & \text{for } s(x) < 0 \end{cases} \quad u^+ \neq u^- \tag{2.2}$$

where one may assume, without loss of generality, that pointwise in x, $u^+(x) < u^-(x)$. A global sliding regime (i.e., one existing everywhere except, possibly, on a set of measure zero) is said to exist on ∂D if and only if at every point $x \in \partial D$, the variable structure control law (2.2), acting on (2.1), is such that

$$\lim_{s \to +0} L_{X(x, u^+)} s < 0 \Leftrightarrow \lim_{s \to +0} \langle ds, X(x, u^+)\rangle < 0$$
$$\lim_{s \to -0} L_{X(x, u^-)} s > 0 \Leftrightarrow \lim_{s \to -0} \langle ds, X(x, u^-)\rangle > 0 \tag{2.3}$$

where L_X denotes the Lie derivative (directional derivative) of the scalar function s with respect to the controlled vector field X.

Theorem 2.3: A sliding motion globally exists on ∂D if and only if, g_{u+}^t is a local contraction on D and g_{u-}^t is a local expansion on D, i.e., given a sufficiently small positive ϵ, for all $0 < t < \epsilon$

$$D \supset g_{u^+}^t(D) \text{ and } g_{u^-}^t(D) \supset D. \tag{2.4}$$

Proof: Suppose a sliding regime globally exists on ∂D, then condition (2.3) holds true. From Theorem 2.1 the set-inclusions (2.4) are also true. Suppose now (2.4) holds true. Then, using the results of Theorem 2.1, one obtains on ∂D, $\langle ds, X(x, u^+(x))\rangle|_{x \in \partial D} = \lim_{s \to +0} \langle ds, X(x, u^+(x))\rangle < 0$. On the other hand, $\langle ds, X(x, u^-(x))\rangle|_x \in \partial D = \lim_{s \to -0} \langle ds, X(x, u - (x))\rangle > 0$. Hence, conditions (2.3) hold true and a sliding motion globally exists on ∂D. □

Corollary 2.4: If a sliding regime globally exists on ∂D, then

$$\int_D \text{div } X(x, u^+(x)) < 0 \text{ and } \int_D \text{div } X(x, u^-(x)) > 0. \tag{2.5}$$

Proof: Suppose a sliding regime globally exists on ∂D, then from (2.3), for all $x \in \partial D$, $\langle ds, X(x, u^+(x))\rangle < 0$ and $\langle ds, X(x, u^-(x))\rangle > 0$ hold valid. Taking the surface integral, over ∂D, of the inner products and using the divergence theorem on each case, conditions (2.5) follow. □

C. Characterization of the Ideal Sliding Dynamics and the Equivalent Control

Let $s(x) = 0$ be a smooth manifold in R^n. The manifold $s(x) = 0$ is said to be a global integral manifold for the controlled system (2.1) [8, p. 266]

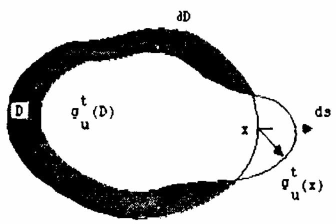

Fig. 1. Proof by contradiction of Theorem 2.1.

if for certain smooth control function $u(x)$, the state trajectories that start anywhere on $s(x) = 0$ remain on $s(x) = 0$ for all time, i.e., for each $x \in \partial D$, $g_u^t(x) \in \partial D$ for all $t > 0$.

Theorem 2.5: The compact manifold ∂D is an integral manifold of (2.1), for a given smooth u, if and only if $g_u^t(\partial D) = \partial D$ for all $t > 0$.

Proof: By definition of integral manifold, for each $x \in \partial D$ and all t, $g_u^t(x) \in \partial D$, i.e., $\partial D \supset g_u^t(\partial D)$ for all t. Suppose now that $g_u^t(\partial D)$ does not properly contain ∂D for some t, then there exist open sets in ∂D which have an empty intersection with $g_u^t(\partial D)$. Taking any x on such an open set, one concludes that $g_u^t(x) \notin \partial D$, i.e., ∂D is not a global integral manifold for (2.1). This is a contradiction. Hence, $g_u^t(\partial D) \supset \partial D$ for all t. From the double inclusion just shown, it follows that $g_u^t(\partial D) = \partial D$. Sufficiency is obvious. □

If a sliding motion exists on ∂D, then the mathematical description of the average, or ideal sliding mode, response is obtained by means of the equivalent control associated to the sliding regime. This is defined as a smooth state feedback control function, denoted by $u^{EQ}(x)$, for which the sliding manifold ∂D globally becomes an integral manifold of (2.1) [9]. The ideal sliding dynamics is then obtained by formally substituting u by $u^{EQ}(x)$ in (2.1), i.e., $dx/dt = X(x, u^{EQ}(x))$, for all $x \in \partial D$.

Theorem 2.6: For all $t > 0$

$$g_u^{tEQ}(\partial D) = \partial D. \tag{2.6}$$

Proof: Immediate by definition of the equivalent control and Theorem 2.5. □

The ideal sliding trajectories are thus tangent to ∂D, and hence characterized by

$$L_{X(x,u^{EQ}(x))}s = 0 \text{ on } s = 0 \text{ i.e., } \langle ds, X(x, u^{EQ}(x))\rangle|_{s=0} = 0. \tag{2.7}$$

Corollary 2.7: If an equivalent control globally exists on ∂D, then

$$dV/dt|_{t=\tau} = \int_{D(\tau)} \text{div } X(x, u^{EQ}(x)) = 0 \tag{2.8}$$

holds true, i.e., the volume of D remains constant under g_u^{tEQ} for all $t > 0$. Moreover, an equivalent control u^{EQ} can be found as a solution of

$$\sum_{i=1}^{n} \{\partial X_i(x, u^{EQ})/\partial x_i + [\partial X_i(x, u^{EQ})/\partial u^{EQ}](\partial u^{EQ}/\partial x_i)\} = 0. \tag{2.9}$$

Proof: By definition of equivalent control, for all $x \in \partial D$, $\langle ds, X(x, u^{EQ}(x))\rangle = 0$ at all times. From the divergence theorem, the stronger version of the Liousville theorem, Theorem 2.5, and the fact that the boundary of $g_u^t(D)$ equals the image of the boundary of D, under g_u^t, i.e., $\partial[g_u^t(D)] = g_u^t(\partial D)$, it follows that for any $\tau \geq 0$

$$dV/dt|_{t=\tau} = \int_{g_u^{\tau}EQ(D)} \text{div } X(x, u^{EQ}(x)) = \int_{\partial[g_u^{\tau}EQ(D)]} \langle ds, X(x, u^{EQ}(x))\rangle$$

$$= \int_{g_u^{\tau}EQ(\partial D)} \langle ds, X(x, u^{EQ}(x))\rangle = \int_{\partial D} \langle ds, X(x, u^{EQ}(x))\rangle = 0.$$

$$\square \tag{2.10}$$

A sufficient condition to have (2.10) valid is that the subintegral quantity becomes zero, i.e., div $X(x, u^{EQ}(x)) = 0$. This condition leads

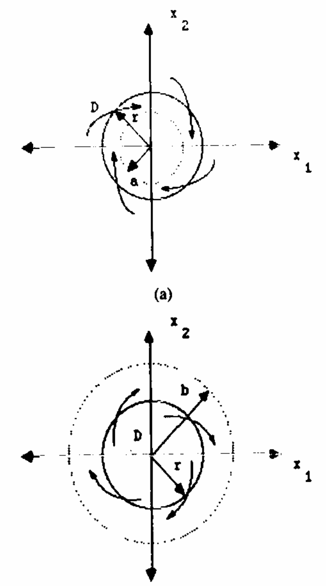

Fig. 2. (a) A local contradiction on D. (b) A local expansion on D.

to the first-order quasi-linear partial differential equation (2.9) from where an $u^{EQ}(x)$ may be found. □

Remarks 2.8: The mere existence of a smooth feedback control $u^{EQ}(x)$ turning ∂D into an integral manifold of (2.1), constitutes only a necessary, but not sufficient, condition for the existence of a sliding regime on ∂D.

Remarks 2.9: In general, for controlled vector fields of the form $X(x, u)$, (2.6), or (2.7), do not uniquely define the equivalent control (this topic is considered at length in [1, p. 64–66]) except in some special cases (typically, when the controlled vector field is of the linear-in-the-control form $X(x, u) = f(x) + u\, g(x)$, provided the transversality condition $\langle ds, g\rangle \neq 0$ is satisfied [7]).

Example 2.10: Consider a disk D of radius r in R^2. The flow g_u^t generated by the vector field $[x_2 - x_1(x_1^2 + x_2^2 - u)]\partial/\partial x_1 + [-x_1 - x_2(x_1^2 + x_2^2 - u)]\partial/\partial x_2$, with $u = a^2 = \text{constant} < r^2$, is locally a contraction on D. On the other hand, g_u^t is locally an expansion on D for $u = b^2 = \text{constant} > r^2$ [see Figs. 2(a) and (b)]. A global sliding motion exists on the circumference ∂D when the switching logic $u = u^+(x) = a^2 < r^2$ for $x_1^2 + x_2^2 - r^2 > 0$ and $u = u^-(x) = b^2 > r^2$ for $x_1^2 + x_2^2 - r^2 < 0$ is used (see Fig. 3). The necessary conditions (2.5) are easily verified. Indeed, $\int_D \text{div } X(x, a^2) = \int_D [-4(x_1^2 + x_2^2) + 2_a^2] = -2\pi r^2[r^2 - a^2] < 0$. On the other hand, $\int_D \text{div } X(x, b^2) = \int_D [-4(x_1^2 + x_2^2) + 2_b^2] = -2\pi r^2[r^2 - b^2] > 0$. An equivalent control may be obtained, using (2.9), as a solution of div $x(x, u^{EQ}(x)) = -4(x_1^2 + x_2^2) + 2u^{EQ} + x_1\partial u^{EQ}/\partial x_1 + x_2\partial u^{EQ}/\partial x_2 = 0$. It is easily verified that $u^{EQ}(x) = x_1^2 + x_2^2$ is such a solution, hence, $u^{EQ}(x)|_{x\in\partial D} = x_1^2 + x_2^2 = r^2$. The ideal sliding is thus governed, on ∂D, by the vector field $x_2\partial/\partial x_1 - x_1\partial/\partial x_2$, the dynamics of an ideal oscillator. □

Example 2.11: Consider a DC to DC power converter of the Boost type, shown in Fig. 4 [7].

$$dx_1/dt = b - w_0x_2 + uw_0x_2 = X_1(x, u)$$

$$dx_2/dt = w_0x_1 - w_1x_2 - uw_0x_1 = X_2(x, u) \tag{2.11}$$

where $x_1 = \sqrt{L}I$, $x_2 = \sqrt{C}V$, $b = E/\sqrt{L}$, $w_0 = 1/\sqrt{LC}$, $w_1 = 1/RC$ and u denotes the switch position function, acting as a control input, which takes values in the discrete set $U = \{0, 1\}$. It is desired to know whether or not, using a suitable switching policy, harmonic motions are possible for the boost converter responses (DC to AC conversion), i.e., D

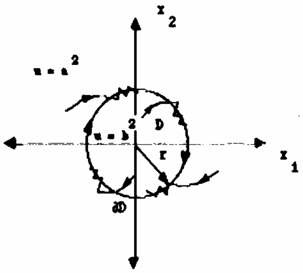

Fig. 3. Periodic sliding motions in R^2.

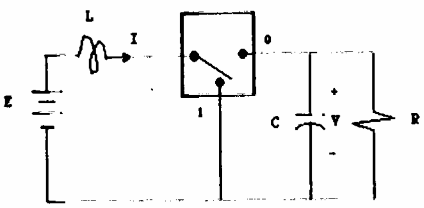

Fig. 4. Boost converter.

is to be taken as a disk of radius r, centered at the origin. The boundary ∂D is then the bounding circumference. An evaluation of the necessary conditions (2.5) leads to

$$\int_D \text{div } X(x, 1) = -\pi r^2 w_1 < 0, \text{ and } \int_D \text{div } X(x, 0) = -\pi r^2 w_1 < 0 \quad (2.12)$$

which readily reveals that a global sliding motion does not exist on ∂D for the available control inputs in the discrete set U. As a matter of fact, a sliding motion does not exist on any nontrivial circumference in R^2. The invariance condition (2.8) is easily seen to merely be a necessary, but not sufficient, condition for the existence of a sliding regime. Indeed, consider the problem of finding an $u^{EQ}(x)$ which renders smooth oscillatory responses of harmonic nature on (2.11). Using (2.9) one immediately obtains $x_2[\partial u^{EQ}/\partial x_1] - x_1[\partial u^{EQ}/\partial x_2] = w_1/w_0$. This equation has as a solution, $u^{EQ}(x_1, x_2) = [w_1/w_0] \tan^{-1}[x_1 x_2^{-1}]$. A smooth feedback control thus exists which satisfies the volume invariance condition (2.8). However, a sliding regime does not globally exist on ∂D. The bound $u^{EQ}(x)$ is not truly associated to a global sliding motion.

III. Conclusions

A general geometric characterization for the global existence of sliding regimes, on compact manifolds, in nonlinear variable structure feedback systems has been given. The characterization involves a set-theoretic inclusion condition to be satisfied by the control-dependent flow map acting on the compact region contained by the sliding manifold. A sign condition is derived on the volume integral of the divergence of the generating controlled vector field. The condition is a necessary, but not sufficient, condition for the existence of a sliding regime. The manifold invariance conditions, or ideal sliding conditions, are characterized in terms of volume-preserving evolution of the flow map associated with the ideal sliding dynamics. An application of the general results to periodic sliding motions in R^2 was illustrated via some simple examples. A generalization of the obtained results to the case of noncompact manifolds is by no means trivial and constitutes an area for further research.

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