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Sliding regimes on slow manifolds of systems with fast actuators

HEBERTT SIRA-RAMIREZ†

In this article the slow manifold of a system with actuator parasitics is used as a sliding surface on which a variable structure controller recovers the qualitative properties of the reduced-order closed-loop system obtained from an ideal actuator-based feedback controller design. Illustrative examples are presented where (*a*) the simplicity of reduced order singular perturbation design method and (*b*) the robustness of variable structure sliding modes, are advantageously combined.

1. Introduction

Recent major publications by Kokotovic *et al.* (1986), and Kokotovic and Khalil (1986), in the area of singular perturbations (SP) as a design and modelling tool for control systems, indicate the growing importance of a broad, active, research field with well-established maturity. This was also evidenced in recent surveys by Kokotovic (1984)—over 250 references on the subject—and Saksena *et al.* (1984). The reader is referred to the above work for details and basic results.

The field of variable structure control (VSC) has also been well-established during the last 25 years. Several specialized books (Utkin 1978, 1981; Itkis 1976), some excellent surveys by Utkin (1977, 1983), and a growing number of archival publications exist for an in-depth study of the area. Except for the publications by Marino (1985), in connection with high-gain and feedback linearizable system models, and Ficola *et al.* (1984), Slotine and Hong (1986), in connection with flexible robot manipulators, no systematic studies have been conducted which explore the inter-breeding of both these important research areas.

It is intuitively clear that a simplified design, based on a reduced-order model which neglects parasitic dynamics, could be advantageously combined with the robust features of a VSC scheme by taking the former feedback controller as the *average, equivalent controller* (Utkin 1978). Generally speaking, the qualitative properties of the reduced-order feedback controller are valid only on the ideal slow manifold of the system.

In this article we obtain general relationships between VSC and SP in terms of the slow manifold (dominating the controlled full-order response) with respect to the ideal slow manifold obtained for the instantaneous actuator system. As a consequence of this, and owing to the fact that a VSC scheme needs only an estimate perfect, non-dynamic actuator) are robustly preserved.

The equivalent control is shown to be identical with the corrective controller scheme (Khorasani and Kokotovic 1986) which annihilates first-order deviations of the slow manifold (dominating the controlled full-order response) with respect to the ideal slow manifold obtained for the instantaneous actuator system. As a consequence of this, and owing to the fact that a VSC scheme needs only an estimate of the equivalent control for feedback gains computation, it is shown that no need

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† Departamento Sistemas de Control, Universidad de Los Andes, Mérida, Venezuela.

arises for adding a correction term on the original smooth reduced-order feedback controller design in order to compute the VSC gains. The necessary VSC gains can be computed solely on the basis of the reduced-order controller.

Section 2 is a brief review of the corrective controller approach to feedback design for systems with fast actuators. The developments by Khorasani and Kokotovic (1986) are closely followed for this purpose. In § 3 a VSC approach is proposed for the robust preservation of the reduced-order design characteristics. Section 4 contains several examples which explore the implications of the proposed method in robust controller design for non-linear systems.

2. Control of systems with fast actuators

A fast actuator dynamic model in standard singular perturbation form is given by (2.1) and (2.2).

$$\dot{x} = f(x, z, \varepsilon), \quad x \in \mathbb{R}^n \tag{2.1}$$

$$\varepsilon \dot{z} = g(x, z, u, \varepsilon), \quad z \in \mathbb{R}, \quad \varepsilon > 0 \tag{2.2}$$

where z is the scalar state of the fast actuator. (The results can be extended to multi-input systems, as will be discussed elsewhere.) The state of the system, x, is an n-dimensional vector and ε is the small perturbation parameter.

A feedback control law $u = u(x, \varepsilon)$ is sought, under which action the system is known to possess an *invariant manifold*, referred to as the *ε-slow manifold*, specified in (2.3):

$$M_\varepsilon : z = h(x, u(x, \varepsilon), \varepsilon) \tag{2.3}$$

The manifold M_ε is obtained as the solution of the partial differential equation (2.4), arising from the insertion of (2.3) into (2.2):

$$\varepsilon \left[\frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \frac{\partial u}{\partial x} \right] f(x, h(x, u(x, \varepsilon), \varepsilon), \varepsilon) - g(x, h(x, u(x, \varepsilon), \varepsilon), u(x, \varepsilon), \varepsilon) = 0 \tag{2.4}$$

A sufficient condition for the existence of the ε- slow manifold is that the jacobian matrix $(\partial g/\partial z)|_{\varepsilon=0}$ be non-singular for all x in the range of interest. When the fast transients associated with (2.1) and (2.2) are asymptotically stable towards M_ε, the *ε-slow subsystem* evolves on M_ε according to the dynamics described by (2.5):

$$\dot{x} = f(x, h(x, u(x, \varepsilon), \varepsilon), \varepsilon) \tag{2.5}$$

The feedback control is usually designed on the basis of two simplifying assumptions and an additional corrective procedure. The assumptions are as follows.

(*a*) The actuator exhibits instantaneous response, i.e. no dynamics is associated with the actuator operation.

(*b*) A feedback controller $u^r(x)$ exists which stabilizes the reduced-order dynamics on a uniquely defined slow manifold, i.e. the reduced-order controlled plant evolves, in a stable fashion, according to (2.6),

$$\dot{x} = f(x, h^r(x, u^r(x)), 0) =: f^r(x, h^r(x)) \tag{2.6}$$

where $h^r(x, u^r(x)) =: h^r(x)$ is the *unique solution*, $z = h^r(x)$, of $g(x, z, u^r(x), 0) = 0$, i.e. one has:

$$g(x, h^r(x), w(x), 0) = 0 \quad \text{iff} \quad w(x) = u^r(x) \tag{2.7}$$

Here $z = h^r(x)$ will be called the *ideal slow manifold*, denoted by M_0, while (2.6) will be referred to as the *ideal actuator system*.

When $u^r(x)$ is used in the full-order system (2.1) and (2.2), with $\varepsilon \neq 0$, the asymptotic behaviour of the controlled dynamics takes place in (2.3) computed now as shown in (2.8).

$$M_\varepsilon : z = h(x, u^r(x), \varepsilon) \neq h^r(x) \tag{2.8}$$

and the evolution of the resulting slow trajectories is governed by (2.5) with

$$h(x, u, \varepsilon) = h(x, u^r(x), \varepsilon)$$

Unless $g(x, z, u, \varepsilon)$ is independent of ε, M_0 does not coincide, in general, with M_ε. For this reason, an additional corrective controller design is needed.

In Khorasani and Kokotovic (1986) a procedure is proposed for compensating the deviation of the ε-slow manifold M_ε from the slow manifold M_0 when the control law derived for the ideal actuator system is used. A control scheme is obtained in which a correction term is added to such an ideal control law. This controller is of the form given by (2.9),

$$u(x, \varepsilon) = u^r(x) + \varepsilon u^c(x) + \sigma(\varepsilon) \tag{2.9}$$

with $\sigma(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$ and $u^c(x)$ is computed in such a manner that the first-order term in the expansion series for the deviation of the slow manifold M_ε, from M_0, namely $\varepsilon(dh/d\varepsilon)$, is made zero. This corrective term for the controller is given by:

$$u^c(x) = -\left(\frac{\partial g}{\partial u}\right)^{-1} \left\{ \left(\frac{\partial g}{\partial z}\right)^{-1} \left[\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\frac{\partial u^r}{\partial x} \right] f^r(x, h^r(x)) + \frac{\partial g}{\partial \varepsilon} \right\} \tag{2.10}$$

The following definition characterizes the existence of locally non-intersecting control-dependent slow manifolds.

Definition 2.1

The scalar function $g(x, z, u, \varepsilon)$ is said to satisfy a *local* (*global*) *foliation property* about the manifold M_0, uniformly for all ε in the interval $[0, \varepsilon^*]$ for some small ε^*, whenever, the following inequalities are verified,

$$g(x, h^r(x)u^+(x), \varepsilon) < g(x, h^r(x), u(x, \varepsilon), \varepsilon) < g(x, h^r(x), u^-(x), \varepsilon) \tag{2.11}$$

for any choice of controls u^+, u, u^- that locally (globally) satisfy one of the inequalities below:

$$u^+(x) > u(x, \varepsilon) > u^-(x) \tag{2.12}$$

$$u^+(x) < u(x, \varepsilon) < u^+(x) \tag{2.13}$$

An alternative to (2.12) or (2.13) is:

$$\min\,(u^+(x), u^-(x)) < u(x, \varepsilon) < \max\,(u^+(x), u^-(x)) \tag{2.14}$$

The following assumption will be proven to be crucial for the existence of a local (global) sliding motion on the slow manifold M_0.

Assumption 2.1

There exists an ε^* for which the scalar function $g(x, z, u, \varepsilon)$, defining the actuator state z, satisfies, uniformly in $[0, \varepsilon^*]$ a local (global) foliation property about the ideal slow manifold M_0.

Remark 2.1

This assumption simply means that the functions $g(x, z, u, \varepsilon)$ for $u = u^+, u, u^-$, evaluated along the manifold $M_0 : z = h^r(x)$ are bounded away from each other, for a certain range of values of ε. Since the solutions of $g(x, z, u, 0) = 0$ represent the slow manifolds for each controller u, these manifolds do not intersect, at least locally, along the ideal slow manifold M_0. In other words, the family of slow manifolds, for a convenient parametrization of the control inputs, produces an n-dimensional, local, *regular foliation* of the state space (x, z).

Remark 2.2

If $g(x, z, u, \varepsilon)$ depends linearly or affinely on u, i.e.

$$g(x, z, u, \varepsilon) = g_0(x, z, \varepsilon) + g_l(x, z, \varepsilon)u$$

then the foliation property is trivially satisfied globally on the corresponding M_0.

3. VSC of systems with fast actuators

In this section a VSC control approach is proposed, which induces a sliding mode on the ideal slow manifold M_0. The sliding surface is expressed as follows:

$$M_0 \equiv S := \{(x, z) \in \mathbb{R}^{n+1} : s = z - h^r(x) = 0\} \tag{3.1}$$

It should be noticed that the surface coordinate s is defined similarly to the 'fast variable' of time scale decomposition schemes associated with the SP technique. The main difference is that (3.1) is defined in terms of the ideal slow manifold, characterized by $h^r(x)$, rather than in terms of the ε-slow manifold corresponding to $u^r(x)$, characterized by $h(x, \varepsilon)$.

The design objective in a VSC approach is to specify a feedback controller of the form of (3.2),

$$u(x) = \begin{cases} u^+(x) & \text{for } s > 0 \\ u^-(x) & \text{for } s < 0 \end{cases} \tag{3.2}$$

such that the following properties are verified.

(*a*) The switching surface S is made reachable, even if in a local sense.

(*b*) By means of opportune, active switchings between the available feedback paths, represented by (3.2), the resulting motions are forced to evolve along the switching surface S.

The state evolution takes place in the immediate vicinity of S in a fast chattering fashion, caused by the 'overshoot and correct' control action (3.2). The second control objective is guaranteed if and only if the controlled state trajectories, associated with each one of the feedback control paths in (3.2), satisfy the *sliding mode existence conditions* (Utkin 1978) defined by:

$$\lim_{s \to 0^+} \dot{s} > 0, \quad \lim_{s \to 0^-} \dot{s} < 0 \tag{3.3}$$

The differential equation governing the evolution of the sliding surface coordinate s is obtained by the differentiation of the surface coordinate function s in (3.1) and by the use of (2.1) and (2.2). This leads to

$$\begin{aligned}\dot{s} &= \dot{z} - \frac{dh^r(x)}{dt}\\ &= \frac{1}{\varepsilon}g(x, z, u, \varepsilon) - \left[\frac{\partial h^r}{\partial x} + \frac{\partial h^r}{\partial u^r}\frac{\partial u^r}{\partial x}\right]f(x, z, \varepsilon)\\ &= \frac{1}{\varepsilon}g(x, s + h^r(x), u, \varepsilon) - \left[\frac{\partial h^r}{\partial x} + \frac{\partial h^r}{\partial u^r}\frac{\partial u^r}{\partial x}\right]f(x, s + h^r(x), \varepsilon) \qquad (3.4)\end{aligned}$$

The switching surface S is made quasi-invariant with respect to the controlled state trajectories (sliding trajectories) and the *average dynamic* behaviour is obtained as the result of imposing on (3.4) the *invariance conditions* (Itkis 1976):

$$s = 0, \quad \dot{s} = 0 \qquad (3.5)$$

The use of the invariance conditions (3.5) on (3.4) leads to the definition of the *equivalent control*. This idealized control represents the smooth feedback control action that, under the assumption of no perturbations, time delays, etc., would generate invariant trajectories on the sliding manifold. Although use of this control is avoided due to its lack of robustness to plant and parameter perturbations, it serves as a reference control 'level' on which the variable structure feedback gains are specified. The VSC scheme, however, will 'emulate', on the average, the action of the equivalent control.

Sliding motions on a manifold S appear as a 'brute force' imposition of invariance on the state response due to drastic control action of a discontinuous nature. However, the existence of an equivalent control is a necessary condition for the creation of a sliding regime on S. In other words, a candidate manifold S will not allow the existence of a sliding motion unless it is invariant for some smooth control action. It is the suprising merit of the VSC to force the controlled system to such invariant behaviour without *a priori* knowledge of the corresponding equivalent control. As a matter of fact, its precise value can be totally ignored for design purposes once the existence of a sliding motion has been concluded by other means.

By use of (3.5) in (3.4), the following defining relation is obtained for the equivalent control:

$$g(x, h^r(x), u_{EQ}(x, \varepsilon), \varepsilon) = \varepsilon\left[\frac{\partial h^r}{\partial x} + \frac{\partial h^r}{\partial u^r}\frac{\partial u^r}{\partial x}\right]f(x, h^r(x), \varepsilon) \qquad (3.6)$$

A convenient decomposition for the equivalent control will be imposed:

$$u_{EQ}(x, \varepsilon) = u^0_{EQ}(x) + \varepsilon u^1_{EQ}(x) + \text{h.o.t}\ (\varepsilon) \qquad (3.7)$$

Expanding both terms of the equality (3.6) as a power series in ε, and equating the first-order terms of the expansion in both sides of the resulting equation, one obtains the algebraic equation shown,

$$g(x, h^r(x), u^0_{EQ}(x)) = 0 \qquad (3.8)$$

which by virtue of the uniqueness assumption in (2.7) implies that:

$$u^0_{EQ}(x) = u^r(x) \qquad (3.9)$$

By equating the second-order terms in the series expansion of both sides of (3.6) one also obtains:

$$\frac{\partial g}{\partial z}\frac{dh^r}{d\varepsilon}+\frac{\partial g}{\partial u}u^1_{\mathrm{EQ}}+\frac{\partial g}{\partial \varepsilon}=\left(\frac{\partial h^r}{\partial x}+\frac{\partial h^r}{\partial u^r}\frac{\partial u^r}{\partial x}\right)f^r(x,h^r(x)) \tag{3.10}$$

Since on M_0 the total variation of g with respect to state space coordinates is zero, the following relation holds true:

$$\frac{\partial g}{\partial x}+\frac{\partial g}{\partial u}\frac{\partial u^r}{\partial x}+\frac{\partial g}{\partial z}\left(\frac{\partial h^r}{\partial x}+\frac{\partial h^r}{\partial u^r}\frac{\partial u^r}{\partial x}\right)=0 \tag{3.11}$$

By use of the identity (3.11) in (3.10) one finds:

$$\frac{\partial g}{\partial z}\frac{dh^r}{d\varepsilon}+\frac{\partial g}{\partial u}u^1_{\mathrm{EQ}}+\frac{\partial g}{\partial \varepsilon}=-\left(\frac{\partial g}{\partial z}\right)^{-1}\left(\frac{\partial g}{\partial x}+\frac{\partial g}{\partial u}\frac{\partial u^r}{\partial x}\right)f^r(x,h^r(x)) \tag{3.12}$$

For $s=0$, $h^r(x)$ does not depend on ε, so that $dh^r/d\varepsilon=0$, and u^1_{eq} is defined precisely by the same expression yielding the corrective controller $u^c(x)$, in (2.10).

It follows that the average controller 'emulated' by the VSC approach coincides, up to first-order in ε, with the corrective controller (2.10). The average properties of the variable structure controlled system are then confirmed to capture the properties of the reduced-order closed-loop design. The equivalent control is precisely the smooth control that would be required to make the full order system evolve on M_0. Its precise determination would be interesting but unessential to the method, due to the fact that if necessary, the zeroth-order approximation alone, represented by $u^r(x)$, is sufficient for the creation of a sliding motion when one is known to exist.

It is notable that when the vector function $f(x,z,\varepsilon)=f(x,z)$ is independent of ε the right-hand side of (3.6) has only one non-identically zero term in its power series expansion. For systems in which the defining function, $g(x,z,\varepsilon)$, of the actuator state is independent of ε, this fact results in having the equivalent control *exactly* as the sum of only two terms: the nominal control u^r, and a correction term which is first-order in ε—see the examples. This exact truncation property is *not* generally true of the corrective feedback controller since h would depend on ε on the left-hand side of (3.6).

Slow manifold sliding mode conditions

Necessary and sufficient conditions for the existence of a sliding regime on S, defined by (3.1), are given by (3.3). Using these conditions on (3.4), with a controller of the form (3.2), one obtains:

$$g(x,h^r(x),u^+(x),\varepsilon)<\varepsilon\left[\frac{\partial h^r}{\partial x}+\frac{\partial h^r}{\partial u^r}\frac{\partial u^r}{\partial x}\right]f(x,h^r(x),\varepsilon) \tag{3.13}$$

$$g(x,h^r(x),u^-(x),\varepsilon)>\varepsilon\left[\frac{\partial h^r}{\partial x}+\frac{\partial h^r}{\partial u^r}\frac{\partial u^r}{\partial x}\right]f(x,h^r(x),\varepsilon) \tag{3.14}$$

By applying (3.6) to the above expressions, the sliding conditions are found to be equivalent to the following inequalities:

$$g(x,h^r(x),u^+(x),\varepsilon)<g(x,h^r(x),u_{\mathrm{EQ}}(x,\varepsilon),\varepsilon) \tag{3.15}$$

$$g(x,h^r(x),u^-(x),\varepsilon)>g(x,h^r(x),u_{\mathrm{EQ}}(x,\varepsilon),\varepsilon) \tag{3.16}$$

By virtue of the foliation property (2.11), it follows that, point-wise in x, the following relation is satisfied:

$$\min (u^+(x), u^-(x)) < u_{EQ}(x, \varepsilon) < \max (u^+(x), u^-(x)) \tag{3.17}$$

The preceding considerations lead to the following theorem.

Theorem 3.1

A sliding regime exists locally (globally) along M_0 if and only if $g(x, z, u, \varepsilon)$ satisfies the local (global) foliation property on M_0 for $u = u_{EQ}(x, \varepsilon)$.

Remark 3.1

The globality or non-globality of the above results is referred to the existence of a sliding motion *along* the ideal slow manifold M_0. Globality issues regarding the nature of the *set of initial states* from where a trajectory can be brought to S for the creation of a sliding motion are not addressed here, and constitute an open area for research (see Example 4.7 where a sliding motion exists globally along the ideal slow manifold, but only states circumscribed to a certain non-convex region of $\mathbb{R}^2$ can actually be brought towards the sliding surface).

A variable structure controller, satisfying the sliding conditions, can be prescribed on the basis of the ideal reduced-order controller $u^r(x)$. This controller takes the form:

$$u = K|u^r(x)| \operatorname{sign}(s), \quad |K| > 1 \tag{3.18}$$

An appropriate choice of K—positive or negative, depending on the attractive or repulsive character of the slow manifolds for each extreme value of $|u^r(x)| \operatorname{sign}(s)$—results in a sliding motion on S, i.e. on M_0.

4. Examples

Example 4.1: *Stable sliding regimes among repulsive slow manifolds*

In this example it is shown that the slow manifold, and the corresponding 'leaves' of the ideal controller-induced foliation, do not have to be attractive in order to obtain a stable sliding mode controlled system.

The non-linear system to be considered (Saberi and Khalil 1985) is described by (4.1) and (4.2):

$$\dot{x} = xz^3 \tag{4.1}$$

$$\varepsilon \dot{z} = z + u \tag{4.2}$$

For $\varepsilon = 0$ the slow, control-dependent manifold is given by $z = -u$, and the reduced-order system is shown below:

$$\dot{x} = -xu^3 \tag{4.3}$$

An ideal reduced-order feedback controller of the following form,

$$u^r(x) = x^{4/3} \tag{4.4}$$

results in a *globally asymptotically stable* controlled system $\dot{x} = -x^5$. However, use of such a controller on the system model with a non-ideal actuator (4.1) results in *unstable dynamics*:

$$\dot{x} = xz^3 \tag{4.5}$$

$$\varepsilon \dot{z} = z + x^{4/3} \tag{4.6}$$

The properties of the controlled system on the slow manifold $M_0 : z + x^{4/3} = 0$ are therefore preferable. A VSC for which the motions of the system are locally forced to evolve on the stabilizing manifold is designed for the following sliding surface,

$$S = M_0 = \{(x, z) : s = z + x^{4/3} = 0\} \tag{4.7}$$

and the corresponding VS controller,

$$u = ku^r(x) = kx^{4/3}, \quad k = \{k^+, k^-\} \tag{4.8}$$

with $k^+ > 1$, $k^- < 1$.

The linearity of the equation defining the actuator state z indicates that the foliation property is trivially satisfied. Samples of the regular parametrization of the ideal slow manifolds, induced by the controller gain in (4.8), are shown in Fig. 1.

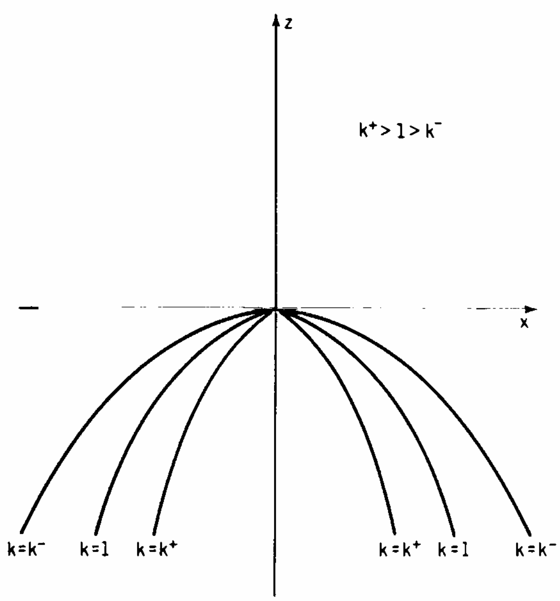

Figure 1. Controller gain-induced foliation.

From (4.7), (4.1) and (4.2) it follows that the surface coordinate s satisfies the following differential equation:

$$\dot{s} = \frac{1}{\varepsilon}(s - x^{4/3}) + \frac{u}{\varepsilon} + \frac{4}{3}x^{4/3}(s - x^{4/3})^3 \tag{4.9}$$

The invariance conditions $s = 0$, $\dot{s} = 0$ on (4.9) define the equivalent control given below,

$$u_{EQ} = x^{4/3} + \tfrac{4}{3}\varepsilon x^{16/3} \tag{4.10}$$

which after substitution in (4.1) and (4.2), along with the invariance conditions, results in the following *stable* average sliding dynamics:

$$\dot{x} = xz^3 = -x^5 \tag{4.11}$$

$$\dot{z} = \tfrac{4}{3}x^{16/3} \tag{4.12}$$

Enforcing the sliding mode conditions (3.3) on (4.8) and (4.9), the variable structure gain k is synthesized according to the appropriate switching logic:

$$k = \begin{cases} k^+ > 1 & \text{for } s > 0 \\ k^- < 1 & \text{for } s < 0 \end{cases} \tag{4.13}$$

Figure 2 shows the local character of the initial states for which a sliding mode exists on the chosen surface.

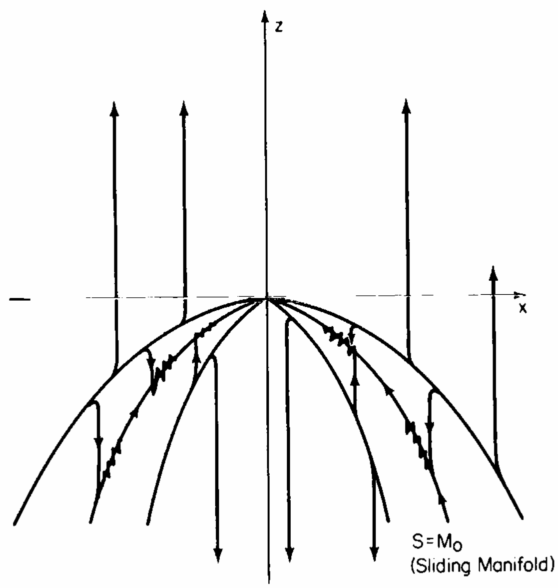

Figure 2. Stable sliding motion on repulsive slow manifold.

Example 4.2: *Non-linear magnetic suspension system*

In this example, whose model is due to Khorasani and Kokotovic (1986), it is shown that the desirable properties of a linear stabilizing controller, derived for the system with ideal actuator, are retained by means of a VSC. No need arises for altering the inherent simplicity of the reduced-order feedback control law, since its value is directly used to synthesize the required variable structure feedback gains according to the scheme proposed in (3.18).

A non-linear dynamic model of a magnetic suspension system is given by (4.14)–(4.16),

$$\dot{x}_1 = x_2 \tag{4.14}$$

$$\dot{x}_2 = 1 - x_2 - \left(\frac{1+z}{x_1}\right)^2 \tag{4.15}$$

$$\varepsilon\dot{z} = -z + u \tag{4.16}$$

where x_1 denotes the position of the suspended magnet, x_2 represents its velocity, z is the solenoid current, and u is the applied input voltage. The solenoid inductance is assumed to be very small, and is represented by the perturbation parameter ε.

If the voltage-current dynamics is ignored ($\varepsilon = 0$) then the slow manifold is an identity,

$$z = u \tag{4.17}$$

which yields the following reduced-order controlled system:

$$\dot{x}_1 = x_2 \tag{4.18}$$

$$\dot{x}_2 = 1 - x_2 - \left(\frac{1+u}{x_1}\right)^2 \tag{4.19}$$

A stabilizing controller for this reduced-order system is given by the feedback law shown below,

$$u^r = 2(x_1 - 1) + x_2 \tag{4.20}$$

and the evolution of the reduced-order controlled system takes place on the corresponding ideal slow manifold:

$$M_0 \equiv S = \{(x_1, x_2, z) : s = z - 2x_1 - x_2 + 2 = 0\} \tag{4.21}$$

Figure 3 shows a computer-simulated response of the full-order system to the ideal feedback control (4.20).

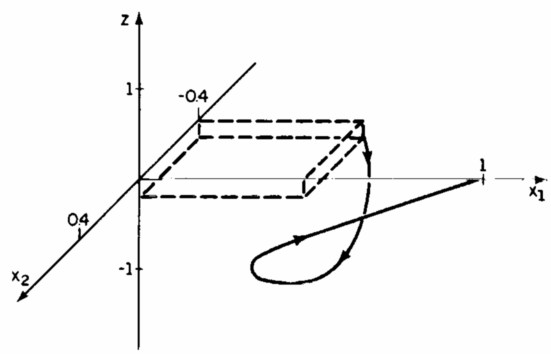

Figure 3. State-space response to reduced-order controller design.

When the control law (4.20) is used in the full-order system, a small non-linear correction term has to be added to (4.20) in order to retain the performance characteristics of the reduced-order design, and to guarantee approximate invariance of the reduced-order manifold. The corrective controller is given by Khorasani and Kokotovic (1986):

$$u = 2(x_1 - 1) + x_2 + \varepsilon\left[x_2 + 1 - \left(\frac{2x_1 + x_2 - 1}{x_1}\right)^2\right] + \sigma(\varepsilon) \tag{4.22}$$

A VSC approach which takes the slow manifold (4.21) as the switching surface yields, from (4.21) and (4.14)–(4.16), the following equations for the surface co-ordinate evolution:

$$\dot{s} = -\frac{1}{\varepsilon}[s + 2(x_1 - 1) + x_2] + \frac{u}{\varepsilon} - \left[x_2 + 1 + \left(\frac{s + 2x_1 + x_2 - 1}{x_1}\right)^2\right] \tag{4.23}$$

The foliation property of the term defining the actuator state is guaranteed from the linearity of the corresponding differential equation in (4.16).

The invariance conditions ($s = 0$, $\dot{s} = 0$), enforced on (4.23), yield the equivalent control shown,

$$u_{EQ} = 2(x_1 - 1) + x_2 + \varepsilon\left[x_2 + 1 - \left(\frac{2x_1 + x_2 - 1}{x_1}\right)^2\right] \tag{4.24}$$

which coincides with the corrective controller (4.22) up to first-order in the perturbation parameter.

The sliding mode conditions (3.3), enforced on (4.23), for a controller in the form of the switching law,

$$u = \begin{cases} u^+ & \text{for } s > 0 \\ u^- & \text{for } s < 0 \end{cases} \tag{4.25}$$

result in the existence conditions:

$$\frac{1}{\varepsilon}[u^+ - u_{EQ}] < 0 \tag{4.26}$$

$$\frac{1}{\varepsilon}[u^- - u_{EQ}] > 0 \tag{4.27}$$

It should be noted that only an estimate of u_{EQ}, and a convenient constant gain multiplying this estimated value, is needed for synthesizing the variable structure feedback controller gains u^+, u^- which produce a sliding motion on S. It is easy to see that for a suitable constant $K > 1$ the controller given below,

$$u = -K|\hat{u}_{EQ}| \operatorname{sign}(s) \tag{4.28}$$

with the following equivalent control estimate,

$$\hat{u}_{EQ} = 2(x_1 - 1) + x_2 \tag{4.29}$$

guarantees satisfaction of the conditions (4.26) and (4.27) for the existence of a sliding regime on S. Figure 4 depicts the full-order system response to the VSC law (4.28). The results are strikingly similar to those of the ideal system response in Fig. 3. The latter evolves on the ε-slow manifold, and those corresponding to the VSC (4.28) evolve on the ideal slow manifold M_0.

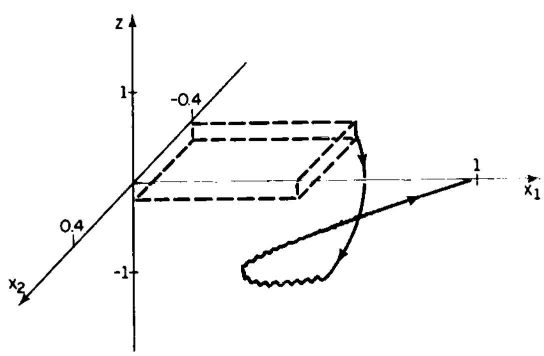

Figure 4. Stable sliding motion on slow manifold.

5. Conclusions and suggestions for further research

In this article, the general relationships between the theory of variable structure systems and the slow manifold approach to singular perturbations in systems with fast dynamic actuators have been explored. The ideal slow manifold serves as a natural sliding surface if and only if a foliation property is satisfied by the control-dependent slow manifolds. The ideal slow manifold must be an intermediate manifold existing between the slow manifolds associated with the extreme control laws that determine the VS gains.

The sliding manifold was assumed to be stable, while the 'leaves' of the foliation, including the sliding manifold, did not have to be necessarily attractive for a stable sliding regime to exist. The equivalent control was shown to be identical to the corrective controller design which guarantees approximate invariance of the reduced-order slow manifold. This was interpreted as imposing, on the average, the reduced-order feedback design, with all its advantageous characteristics, on the full-order system dynamics. The required variable feedback gains were computed using the ideal feedback controller as an estimate of the equivalent control, without further approximations. Several illustrative examples were furnished.

The choice of the slow manifold as a sliding surface was not only natural but it also resulted in a convenient approximation to the problem of finding a sliding surface for which the equivalent control was known. The integral manifold of the full-order system excited by the reduced-order feedback control can be found by solving a quasi-linear partial differential equation. The properties of the state trajectories on this manifold do not necessarily coincide with those of the reduced-order controlled system on the ideal slow manifold. If one wishes to retain these properties by means of a VSC approach, the ideal slow manifold should be taken as a sliding surface. The neglected dynamics causes the corresponding equivalent control to be slightly different from the reduced-order controller. Fortunately, this first-order approximation to the equivalent control is sufficient to synthesize a switching feedback law that guarantees reachability of the sliding surface.

More general results can be obtained by considering non-linear singularly perturbed system models with fast parasitic dynamics in the plant rather than in the actuator alone. The foliation property still represents a necessary and sufficiency condition for the existence of a sliding mode. This avenue is currently being explored for the variable structure feedback linearization scheme for flexible spacecraft attitude manoeuvres (see Dwyer and Sira-Ramirez 1988 for the rigid body case) and the control of DC-to-DC switch-mode power converters (Sira-Ramirez 1987).

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