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Sliding-mode control on slow manifolds of DC-to-DC power converters

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In this article, general relationships are established among variable structure control strategies and pulse-width modulated control schemes leading to sliding modes in non-linear systems. The results are applied to the design of a sliding regime for the regulation of DC-to-DC switchmode power converters exhibiting a slow integral manifold due to time-scale separation properties among input and output circuits.

1. Introduction

Using elementary notions from differential geometry (Isidori 1985), a general relationship is established among the sliding modes resulting from a variable structure control (VSC) strategy (Utkin 1977, 1978, 1981, 1983) and those resulting from a pulse-width-modulated (PWM) control scheme for the regulation of non-linear analytic systems.

Under the assumptions of infinitely fast switchings for the VSC scheme and infinite-frequency duty cycles for the PWM approach, the essential similarities and the conceptual differences between these approaches are conveniently exhibited in terms of the average behaviour of the controlled motion.

The average behaviour of the PWM controlled system is obtained from the system model by simply replacing the discrete control input (switch position function) by a smooth analytic function of the state known as the *duty ratio*. The ideal sliding dynamics, obtained from the VSC scheme, is similarly obtained by replacing the switch position function by a smooth input known as the *equivalent control* (Utkin 1978). The equivalent control and the duty ratio coincide when the integral manifold of the average PWM controlled system is taken as a sliding surface for the VSC option. Conversely, the ideal sliding dynamics of the VSC scheme adopts as an integral manifold that of the average PWM controlled system when the equivalent control is made to coincide with the corresponding duty ratio. Integral manifolds of the average PWM controlled system are seen to satisfy the necessary and sufficient conditions for the local existence of a sliding regime. Conversely, sliding manifolds locally qualify as integral manifolds of the average PWM controlled system.

As an example of the proposed theory, a popular DC-to-DC power converter circuit of the boost type is analysed under the assumptions of a constant duty ratio (see Severns and Bloom 1985, Middlebrook and Cuk 1981, Venkataramanan *et al.* 1985, Bilalovich *et al.* 1983). The computation of a local invariant set is considerably simplified by exploiting a time-scale separation property of the average PWM controlled system. This allows the use of the slow manifold as a sliding surface. The time-scale separation property is shown to be physically significant in terms of the relative values of the neutral frequency associated with the input filter with respect

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to the value of the output circuit time constant. The justification for such an assumption stems from desirable non-oscillatory stability properties of the converter response. A sliding-mode approach, based on VSC, is proposed by specifying as a switching surface an affine variety which contains the slow manifold of the average PWM controlled system. This variety does not globally qualify as an integral manifold of the average system and therefore only local stable sliding regimes exist on the switching surface. The region of equivalence of the sliding modes for the VSC and PWM approaches is completely identified.

Section 2 presents a general equivalence among PWM and VSC schemes for non-linear analytic systems, § 3 contains the DC-to-DC converter example and § 4 summarizes some conclusions and suggestions for further research.

2. Background and general results

Consider the non-linear analytic system defined in R^n:

$$\dot{x}(t) = f(x) + ug(x) \tag{2.1}$$

with f, g smooth vector fields defined on an open neighbourhood $\mathbf{X}$ of R^n. The scalar control function u is assumed to take values on the discrete set $\{0, 1\}$.

A common feature of the VSC and PWM control schemes is the discontinuous character of the right-hand side of (2.1). Both schemes coincide in their essential features when their corresponding average behaviours are computed, respectively, under the assumption of infinitely fast switchings and infinite-frequency duty cycles. The rest of the section is devoted to demonstrating this fact.

2.1. *Generalities about sliding modes under VSC*

The sliding-mode control of (2.1) by means of a VSC scheme entitles the use of a feedback control law of the form

$$u = \begin{cases} 1 & \text{for } s(x) > 0 \\ 0 & \text{for } s(x) < 0 \end{cases} \tag{2.2}$$

where the smooth function $s(x)$ determines a 'switching surface' or a 'sliding manifold' in R^n defined as

$$S = \{x \in R^n : s(x) = 0\} \tag{2.3}$$

It is assumed that the gradient of S, denoted by ds, is nowhere zero in $\mathbf{X}$.

We shall henceforth qualify a result or assumption as *local* whenever x is restricted to $\mathbf{X}$, an open neighbourhood of R^n which has non-empty intersection with S.

Let the expressions: $s \to +0$ and $s \to -0$ denote the fact that s approaches zero from positive and negative values respectively. Necessary and sufficient conditions for the local existence of a sliding mode on S are satisfied whenever the switching logic (2.2) and the controlled motion (2.1) are such that

$$\lim_{s \to +0} ds/dt < 0, \quad \lim_{s \to -0} ds/dt > 0 \tag{2.4}$$

Let $\mathbf{L}_h s$ denote the *directional derivative* of the sliding surface coordinate function s in the direction of the vector field h. The equation (2.4) is thus rewritten as

$$\left.\begin{aligned} \lim_{s \to +0} \mathbf{L}_{f+g} s &= \lim_{s \to +0} \langle ds, f+g \rangle < 0 \\ \lim_{s \to -0} \mathbf{L}_f s &= \lim_{s \to -0} \langle ds, f \rangle > 0 \end{aligned}\right\} \tag{2.5}$$

The gradient has the natural interpretation of a linear functional in the tangent space of R^n. However, as a vector, the gradient is assumed to be locally directed from the region where $s < 0$ towards the region where $s > 0$. This fixes the local orientation of S according to the definition of $s(x)$.

Remark 1

Notice that if a sliding mode locally exists on S for the reversed switching logic: $u = 1$ for $s(x) < 0$ and $u = 0$ for $s(x) > 0$, then the existence conditions (2.5) are obtained with the inequality signs also reversed. If one redefines S in (2.3) by means of a surface coordinate function $s_1(x) = -s(x)$, the assumed inequalities become valid again. Therefore, there is no loss of generality in assuming that the existing sliding regime satisfying (2.5) is obtained from (2.1) and (2.2).

Remark 2

If a sliding motion locally exists on S for the adopted switching logic (2.2), then necessarily $\mathbf{L}_g s = \langle ds, g \rangle < 0$ locally on S. This is immediate from the existence conditions (2.5) and the linearity of the directional derivative operator. The condition $\mathbf{L}_g s < 0$ represents a *transversality condition* of the vector field g with respect to the surface S. The transversality condition is easily seen to be a necessary condition for the existence of a local sliding motion on S with switching logic (2.2). As previously discussed, when a sliding motion exists for a reversed switching logic, the transversality condition adopts the opposite inequality sign.

We shall concentrate our analysis on a region where the transversality condition is assumed to be satisfied. However, in the region where conditions (2.5) fail, the vector field g may be transversal to S and a sliding regime may still locally exist with a reversed switching logic. In general, in the neighbourhood of the intersection of the sliding surface S with the set of points where $\langle ds, g \rangle = 0$ a sliding regime does not exist except in those cases termed *singular* (where $\mathbf{L}_f s$ is also zero). The reader is referred to Utkin (1983) for a survey of existing results about this particular case.

According to (2.2) the state trajectories of the controlled motion of (2.1) are defined everywhere in $\mathbf{X}$ except at the surface of discontinuity S. Several definitions have been proposed to describe the solution of (2.1), (2.2) when a sliding regime exists locally on S. Filippov (1964) defines an average vector field describing the ideal sliding dynamics which generates the ideal corresponding trajectories. This average vector field is obtained by a *geometric average* or *convex combination* of the vector fields defined on each 'side' of S. This average vector field is tangential to the sliding surface at each point of the existence region. Utkin (1978), on the other hand, defines the average controlled trajectories in terms of the response of the system to a smooth control function known as the *equivalent control*. The equivalent control, denoted by $u_{EQ}(x)$ is obtained from the *invariance conditions* (or *ideal sliding conditions* according to Itkis 1976):

$$s = 0, \quad \mathbf{L}_{f + u_{EQ}(x) g} s = 0 \tag{2.6}$$

The equivalent control is thus a smooth control function which locally turns S into an *integral manifold* (invariant set) of (2.1). The ideal sliding dynamics is then governed by

$$\dot{x} = f(x) + u_{EQ}(x) g(x) \tag{2.7}$$

Filippov's and Utkin's definitions of an ideal sliding mode are totally equivalent for the case at hand. In more general situations, typically, when the control function enters non-linearly in the differential equation of the system, the definitions have been shown to yield different results (Utkin 1978). Utkin's definition is adopted below to obtain some of the relevant facts about the sliding mode control scheme by means of VSC.

We denote by Ker ds the *tangent distribution* generated by S and defined as the $n-1$-dimensional subspace of the *tangent space* to $\mathbf{X}$ at each point of S. Ker $ds = \{h : \langle ds, h\rangle = 0\}$. From the definition of the ideal sliding dynamics (2.6) it follows immediately that

$$f + u_{EQ}(x)g \in \text{Ker}\, ds, \quad \text{i.e. } \langle ds, f + u_{EQ}(x)g\rangle = 0 \tag{2.8}$$

In other words, the smooth vector field represented by $f + u_{EQ}(x)g$ will be locally tangent to the sliding surface at each point of existence of the sliding regime. As a consequence of this, the ideal sliding dynamics has as a local integral manifold (invariant set) the sliding surface S.

Lemma 1

The equivalent control, if it locally exists, is unique, i.e. the sliding manifold S uniquely determines the equivalent control for the system (2.1).

Proof

Let $u_{EQ1}(x)$ be also an equivalent control associated with a sliding mode locally created on S. We then have from the definition of equivalent control: $\langle ds, f + u_{EQ}(x)g\rangle = \langle ds, f + u_{EQ1}(x)g\rangle = 0$ and hence $(u_{EQ} - u_{EQ1})\langle ds, g\rangle = 0$. Since by assumption the transversality condition is satisfied, $\langle ds, g\rangle$ is non-zero. It follows that $u_{EQ} = u_{EQ1}$ locally on S. □

Theorem 2

A necessary and sufficient condition for the existence of a sliding mode on S is that the equivalent control satisfies

$$0 < u_{EQ}(x) < 1 \tag{2.9}$$

Proof

Suppose a sliding motion exists on S. Then, from (2.8) the equivalent control is given by

$$u_{EQ}(x) = -\frac{\langle ds, f\rangle}{\langle ds, g\rangle} = -\frac{\mathbf{L}_f s}{\mathbf{L}_g s} \tag{2.10}$$

From (2.5) and the transversality condition, the above quantity is locally positive, i.e. $u_{EQ}(x) > 0$. On the other hand, using (2.5) again one obtains

$$\frac{\mathbf{L}_{f+g} s}{\mathbf{L}_g s} = 1 + \frac{\mathbf{L}_f s}{\mathbf{L}_g s} = 1 - u_{EQ}(x) > 0 \tag{2.11}$$

which implies $u_{EQ} < 1$ locally.

To prove the reverse implication of the theorem, let (2.9) hold true for a smooth control function $u_{EQ}(x)$ which turns S into a local invariant manifold and assume that a sliding motion does not exist locally on S. Then, the inequality $0 < 1 - u_{EQ}(x) < 1$ also holds true. By assumption, the smooth vector field generated by u_{EQ} will locally belong to the distribution associated with S. In this region the following relation will be satisfied:

$$\langle ds, f + u_{EQ}(x)g\rangle = u_{EQ}(x)\langle ds, f+g\rangle + (1 - u_{EQ})\langle ds, f\rangle = 0$$

It follows, necessarily, that the quantities $\langle ds, f+g\rangle$ and $\langle ds, f\rangle$ are opposite in signs on the surface S. The linearity of the inner product implies that $\langle ds, g\rangle$ cannot be zero and thus the transversality condition is locally satisfied since its sign can be arbitrarily established. We then have

$$\langle ds, f+g\rangle|_{s=0} = \lim_{s\to+0} \langle ds, f+g\rangle = \lim_{s\to+0} \dot{s} < 0$$

and also

$$\langle ds, f\rangle|_{s=0} = \lim_{s\to-0} \langle ds, f+0, g\rangle = \lim_{s\to-0} \dot{s} > 0$$

which means that the control law (2.2) locally creates a sliding mode on S. This is a contradiction, and hence the theorem is proved. □

Remark 3

The pair of inequalities (2.9) determines the region or regions of local existence of a sliding mode. The intersection of the open regions defined by each inequality with both the sliding surface candidate S and the region of validity of the transversality condition determine the open set on S where a sliding mode can be achieved by means of the VSC law (2.2) acting on the system (2.1). The existence of a sliding mode in other regions where the transversality condition exhibits a different sign is not precluded provided the opposite switching logic is used.

2.2. *Generalities about PWM control*

In a PWM control option, the scalar control u is switched once within a *duty cycle* of fixed small duration Δ. The instants of time at which the switchings occur are determined by the sample value of the state vector at the beginning of each duty cycle. The fraction of the duty cycle on which the control holds the fixed value, say 1, is known as the *duty ratio* and it is denoted by $D(x(t))$. The duty ratio is usually specified as a smooth function of the state vector x. The duty ratio evidently satisfies $0 < D(x) < 1$. On a typical duty cycle interval, the control input u is defined as (see Fig. 1):

$$u = \begin{cases} 1 & \text{for } t \leqslant \tau \leqslant t + D(x(t))\Delta \\ 0 & \text{for } t + D(x(t))\Delta \leqslant \tau \leqslant t + \Delta \end{cases} \tag{2.12}$$

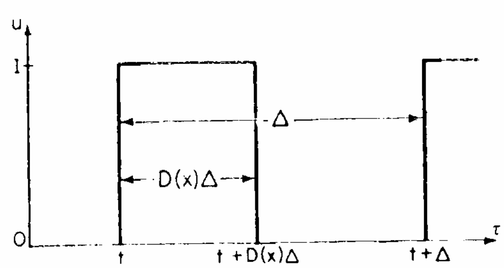

Figure 1. Typical duty cycle in PWM control.

It follows then that, generally,

$$x(t+\Delta) = x(t) + \int_t^{t+D(x(t))\Delta} \{f(x(\tau)) + g(x(\tau))\}\, d\tau + \int_{t+D(x(t))\Delta}^{t+\Delta} f(x(\tau))\, d\tau$$
$$= x(t) + \int_t^{t+\Delta} f(x(\tau))\, d\tau + \int_t^{t+D(x(t))\Delta} g(x(\tau))\, d\tau$$

The ideal average behaviour of the PWM controlled system response is obtained by allowing the duty cycle frequency to tend to infinity with the duty cycle length Δ approaching zero. In the limit the above relation yields

$$\lim_{\Delta\to 0} \frac{x(t+\Delta) - x(t)}{\Delta} = \lim_{\Delta\to 0} \frac{1}{\Delta}\left[\int_t^{t+\Delta} f(x(\tau))\, d\tau + \int_t^{t+D(x(t))\Delta} g(x(\tau))\, d\tau\right]$$
$$= \frac{d}{dt} x = f(x) + D(x)g(x) \tag{2.13}$$

As the duty cycle frequency tends to infinity, the ideal average dynamics of the PWM controlled system is represented by the smooth response of the system (2.1) *to the smooth control function constituted by the duty ratio $D(x)$. The duty ratio replaces the discrete control function u in the same manner as the equivalent control u_{EQ}, of the VSC scheme, replaces u in* (2.1) *to obtain* (2.7).

We refer to (2.13) as the *average PWM controlled system.*

Definition 3

An $n-1$-dimensional manifold Σ is said to be a *local integral manifold*, on an open neighbourhood $\mathbf{X}$ of R^n, of the average PWM controlled system (2.13) if for some $0 < D(x) < 1$, there exists a smooth function $\sigma: R^n \to R$ such that $\Sigma = \{x \in R^n : \sigma(x) = 0\}$, whose gradient $d\sigma$ is nowhere zero in $\mathbf{X}$, and such that locally on Σ

$$\langle d\sigma, f(x) + D(x)g(x)\rangle = 0 \tag{2.14}$$

From the above definition it follows that the duty ratio satisfies

$$D(x) = -\frac{\langle d\sigma, f\rangle}{\langle d\sigma, g\rangle} = -\frac{\mathbf{L}_f\sigma}{\mathbf{L}_g\sigma} \tag{2.15}$$

Notice that $\langle d\sigma, g\rangle = 0$ makes $D(x)$ unbounded unless $\langle d\sigma, f\rangle$ is itself zero, in which case $f + ug$ trivially belongs to the tangent distribution $\operatorname{Ker} d\sigma$ and thus the trajectories are locally constrained to Σ for any arbitrary control function u. We therefore assume that locally on Σ the quantity $\langle d\sigma, g\rangle \neq 0$. From this assumption, it follows that, without loss of generality, we may consider $\langle d\sigma, g\rangle$ as a negative quantity in the region of interest. To see this, suppose $\langle d\sigma, g\rangle > 0$ and consider the following alternative definition of the integral manifold $\Sigma = \{x : \sigma_1(x) = -\sigma(x) = 0\}$. On the region of interest we now have $\langle d\sigma, g\rangle = \langle -d\sigma_1, g\rangle = -\langle d\sigma_1, g\rangle > 0$, i.e. $\langle d\sigma_1, g\rangle < 0$. From the above assumption and the definition of the duty ratio, it follows that in order to have $0 < D(x) < 1$, necessarily $\langle d\sigma, f\rangle > 0$ locally on Σ.

Lemma 4

If Σ is a locally integral manifold for (2.13), then Σ is also a local integral manifold for $\dot{x} = f(x) + D_1(x)g(x)$ if and only if $D(x) = D_1(x)$ in the region of interest.

Proof

Sufficiency is obvious. To prove necessity suppose $D_1(x) \neq D(x)$ locally on Σ, but assume that Σ is a local integral manifold for both controlled systems. It follows from Definition 3 that on $\Sigma: \langle d\sigma, f + D(x)g \rangle = \langle d\sigma, f + D_1(x)g \rangle = 0$. From this equality it is easy to see that $\langle d\sigma, (D(x) - D_1(x))g \rangle = (D(x) - D_1(x)) \langle d\sigma, g \rangle = 0$. Since by hypothesis $\langle d\sigma, g \rangle \neq 0$ then necessarily $D(x) = D_1(x)$ locally on Σ. This is a contradiction and the lemma is established. □

Theorem 5

A necessary and sufficient condition for Σ to be a local integral manifold of (2.13) is that locally on Σ

$$\langle d\sigma, f + g \rangle < 0, \quad \langle d\sigma, f \rangle > 0 \tag{2.16}$$

Proof

Let Σ be a local integral manifold for (2.13), then $f + D(x)g$ satisfies (2.14) and from the definition of duty ratio,

$$0 < -\frac{\langle d\sigma, f \rangle}{\langle d\sigma, g \rangle} = D(x) = -\frac{\mathbf{L}_f \sigma}{\mathbf{L}_g \sigma} < 1 \tag{2.17}$$

Using the hypothesis that $\langle d\sigma, g \rangle < 0$, it follows from the right-hand side of (2.17) that $-\langle d\sigma, f \rangle > \langle d\sigma, g \rangle$ and therefore $\langle d\sigma, f + g \rangle < 0$. On the other hand, using the first inequality of (2.17), it follows that $-\langle d\sigma, f \rangle < 0$, i.e. $\langle d\sigma, f \rangle > 0$.

To prove sufficiency, suppose (2.16) holds true locally on Σ. Then, there exists positive smooth functions $a(x)$ and $b(x)$ such that on the region of interest

$$a(x)\langle d\sigma, f + g \rangle + b(x)\langle d\sigma, f \rangle = 0 \tag{2.18}$$

It then follows that rearranging the above expression $\langle d\sigma, f + [a(x)/(a(x) + b(x))]g \rangle = 0$, i.e. there exists a smooth control function $0 < D(x) = a(x)/(a(x) + b(x)) < 1$ such that, locally on Σ, $\langle d\sigma, f + D(x)g \rangle = 0$. In other words, in the region of interest, Σ is a local integral manifold of (2.13). □

In a similar form to the VSC case, (2.16) determines the regions of existence of a local integral manifold for the average PWM controlled system. Note that the definition of duty ratio could have been established in the opposite manner by ascribing it to the portion of the duty cycle in which the control u is zero, i.e. this control value occurs at the beginning of each cycle. Local portions of the integral manifolds may exist in the regions where both conditions (2.16) are violated with this alternative use of the switching sequence. The situation is totally parallel to that encountered in the VSC case. Further equivalences are established in the following section.

2.3. *Ideal equivalence of VS and PWM control*

In a VSC scheme, the switching surface S is necessarily a local integral manifold for the ideal sliding dynamical system, controlled by $u_{EQ}(x)$. In a PWM control scheme, the average system (2.13) may possess a local integral manifold corresponding to the duty ratio $D(x)$. The following theorem demonstrates that local integral manifolds of the average PWM controlled system, if they exist, qualify as a local sliding surface.

When a sliding regime is created on such portions of the manifold, the corresponding equivalent control coincides with the duty ratio. In two-dimensional systems this is particularly interesting since any state trajectory of the average PWM controlled system, which happens to be globally, or only locally, inside the region determined by (2.16) also qualifies, respectively, as a global, or local, integral manifold itself. Hence, a global or local sliding motion can be treated along these one-dimensional surfaces. This last fact will be exploited in § 3.

Theorem 6

If there exists a smooth function $\sigma : R^n \to R$ such that $\Sigma = \{x \in R^n : \sigma(x) = 0\}$ is an $n-1$-dimensional local integral manifold (invariant set) for the average PWM controlled system (2.13) then a sliding mode exists, locally on Σ, in the same region where it qualifies as a local integral manifold. Moreover, the equivalent control corresponding to such a sliding motion coincides, locally in the same region, with the duty cycle.

Proof

Since Σ is an integral manifold for the average PWM controlled system (2.13), then Theorem 5 applies and (2.16) holds true. It follows that locally on Σ:

$$\langle d\sigma, f+g \rangle = \lim_{\sigma \to +0} \langle d\sigma, f+g \rangle = \lim_{\sigma \to +0} \sigma < 0$$

$$\langle d\sigma, f \rangle = \lim_{\sigma \to -0} \langle d\sigma, f \rangle = \lim_{\sigma \to -0} \ > 0$$

i.e. the variable structure control law:

$$u = \begin{cases} 1, & \text{for } \sigma(x) > 0 \\ 0, & \text{for } \sigma(x) < 0 \end{cases} \tag{2.19}$$

applied on system (2.1) creates a sliding mode locally on Σ.

To prove the second part of the theorem, note that if a sliding mode exists on Σ then, necessarily, the transversality condition $\langle d\sigma, g \rangle = \mathbf{L}_g \sigma < 0$ is satisfied. By definition, the corresponding equivalent control satisfies the invariance condition on Σ:

$$\langle d\sigma, f + u_{\text{EQ}}(x) g \rangle = 0 \tag{2.20}$$

Subtracting then the expression (2.20) from (2.14), which represents the local integral manifold condition for Σ, one obtains after letting $s = \sigma$,

$$\langle d\sigma, (u_{\text{EQ}}(x) - D(x)) g \rangle = (u_{\text{EQ}}(x) - D(x)) \langle d\sigma, g \rangle = 0$$

and by virtue of the assumption on $\langle d\sigma, g \rangle$ being non-zero, it follows that $u_{\text{EQ}}(x) = D(x)$ locally on Σ. □

The converse theorem completes the equivalence between VSC and PWM control schemes in their respective idealized features.

Theorem 7

If a local sliding motion exists on the $n-1$-dimensional smooth manifold $S = \{x \in R^n : s(x) = 0\}$ then, locally, S qualifies as an integral manifold for an average

PWM controlled system. Moreover, the duty ratio corresponding to this average system coincides, locally in the same region, with the equivalent control.

Proof

Suppose a sliding motion locally exists on S, then (2.5) holds true locally on S and hence the conditions for Theorem 5 are easily seen to be also valid. Hence, S qualifies as a local integral manifold of the average PWM controlled system. Notice, furthermore, that from Theorem 2, $0 < u_{EQ}(x) < 1$ is satisfied in the region of interest. The equivalent control also turns S into a local integral manifold in the region where the inequalities (2.16) are also valid. By virtue of the uniqueness of the duty ratio, the equivalent control necessarily coincides with the duty ratio as smooth functions of the state vector. □

Theorems 6 and 7 allow us to draw the following conclusion:

The necessary and sufficient condition for the local existence of a sliding mode on an $n-1$-dimensional manifold S is that it also locally qualifies as an integral manifold for an average PWM controlled system in the region of existence of the sliding mode. In this region of equivalence, the equivalent control and the duty ratio totally coincide.

It should be pointed out that the previous result can also be derived from Filippov's definition of average trajectory on a surface of discontinuity. This approach leads to the above result in a more direct fashion and it will be presented elsewhere.

It is, generally speaking, very difficult to compute an integral manifold for a system of non-linear differential equations. However, for the class of linear time-invariant systems exhibiting a two-time-scale separation property, known as singularly perturbed systems (Kokotovic *et al.* 1986), affine varieties containing the *slow manifold* of the system can be explicitly computed with considerable simplicity. Generally speaking, hypersurfaces or affine varieties containing slow manifolds are not, themselves, global integral manifolds.

3. Sliding-mode control of DC-to-DC switchmode power converters

3.1. *Generalities*

Linear networks on which intentional switchings are conveniently carried out to achieve electrical energy transfer among dynamic storage elements or, more commonly, to achieve steady state regulation of the output variables, constitute a special class of variable structure systems. The control action is represented by a voluntary change in the network topology, associated with the operation of a switch (transistor, diode or combination thereof).

Switchmode controlled networks have important applications in DC-to-DC power converters, and other related areas (switched capacitor circuits). They constitute a vast developing field of knowledge with on-going applications in modern power electronics and robotic control. It is not possible to survey, in a paper of this nature, the vast amount of contributions available in this exciting field of applied research. For this reason, the reader is referred to the book by Severns and Bloom (1985) and the multivolume series of Middlebrook and Cuk (1981). For closely related results, the reader is referred to Venkataramanan *et al.* (1985) and Bilalovich *et al.* (1983).

3.2. *Sliding motions on the slow manifold of the boost converter*

Consider the boost converter of Fig. 2, where the state variables are defined as: $x_1 = I\sqrt{L}$, $x_2 = V\sqrt{C}$, and parameters; $b = E/\sqrt{L}$, $w_0 = 1/\sqrt{LC}$, $w_1 = 1/RC$. The control 'input' is again represented by a discrete-valued variable $u \in \{0, 1\}$ representing the ideal switch position.

$$\left.\begin{aligned} \frac{d}{dt}x_1 &= -w_0x_2 + uw_0x_2 + b \\ \frac{d}{dt}x_2 &= w_0x_1 - w_1x_2 - uw_0x_1 \end{aligned}\right\} \tag{3.1}$$

The system of differential equations describing the converter corresponds to a bilinear system due to the multiplicative character of the control function u with repsect to the state vector. However, the average response of the system to a *constant duty ratio* PWM control scheme is governed by a *linear system* in which u is replaced by a value $0 < \mu < 1$, as demonstrated in § 2.

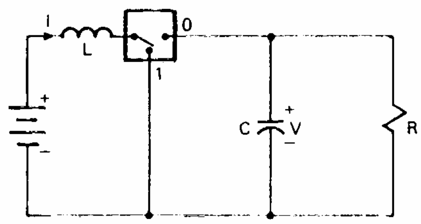

Figure 2. Boost converter.

The average PWM controlled system is then

$$\left.\begin{aligned} \frac{d}{dt}x_1 &= -(1-\mu)w_0x_2 + b \\ \frac{d}{dt}x_2 &= (1-\mu)w_0x_1 - w_1x_2 \end{aligned}\right\} \tag{3.2}$$

Equilibrium points

These are given by

$$x_{1,\mathrm{ss}} = \frac{w_1 b}{(1-\mu)^2 w_0^2}, \quad x_{2,\mathrm{ss}} = \frac{b}{(1-\mu)w_0} \tag{3.3}$$

Elimination of the parameter μ demonstrates that the family of equilibrium points for constant duty ratios describes a locus represented by a parabola:

$$x_{1,\mathrm{ss}} = \frac{w_1}{b} x_{2,\mathrm{ss}}^2 \tag{3.4}$$

DC gain

The DC gain is obtained as the quotient of the steady state values of the output voltage and the input voltage:

$$G(\mu) = \frac{V_{\mathrm{ss}}}{E} = \frac{x_{2,\mathrm{ss}}}{\sqrt{C}E} = \frac{1}{(1-\mu)} \tag{3.5}$$

As the duty ratio μ is increased from 0 to 1, the DC gain increases from 1 to infinity. This indicates the 'step up' character of the boost converter (in more realistic models, including parasitic resistances, the efficiency of the circuit never reaches infinity).

Stability of equilibrium points

The characteristic equation, associated with the average model describing the response of the circuit to a constant duty ratio, is given by

$$\lambda^2 + w_1 \lambda + (1-\mu)^2 w_0^2 = 0 \tag{3.6}$$

The roots of this characteristic equation always have negative real parts and hence, the average trajectories are asymptotically stable towards the equilibrium point. The damping coefficient ζ of the system is simply $\zeta = w_1/[2w_0(1-\mu)]$, while the natural frequency of the average system coincides with $(1-\mu)w_0$. The value of the damping coefficient ξ determines the nature of the average response. Thus, for values of $\xi < 1$ the response is oscillatory, while for values of $\xi \gg 1$ one of the modes in the response is overdamped, while the other represents a fast transient. The damping coefficient is also a measure of the ratio of the time constant of the output circuit $w_1 = 1/RC$ to the natural frequency w_0 of the LC input circuit.

Eigenspaces and affine varieties containing the slow manifold of the controlled system

If it is assumed that the component values of the boost converter satisfy $\xi \gg 1$, a time-scale separation is obtained in the nature of the response specified by the eigenvalues of the controlled system. A fast eigenvalue $\lambda_1(\mu)$ and a slow eigenvalue $\lambda_2(\mu)$ are obtained from this assumption. This has an important bearing in the non-oscillatory nature of the converter and it is deemed as a desirable design property. The eigenvalue $\lambda_1(\mu)$ is assumed to be large while $\lambda_2(\mu)$ is assumed to be small.

The eigenspace associated to each eigenvalue is given by either one of the following equivalent expressions:

$$x_2 = \frac{(1-\mu)w_0}{w_1 + \lambda(\mu)} x_1; \quad x_2 = -\frac{\lambda(\mu)}{(1-\mu)w_0} x_1 \tag{3.7}$$

where substitution of $\lambda(\mu)$ by $\lambda_1(\mu)$ or $\lambda_2(\mu)$ identifies, respectively, the fast or slow eigenspace. In spite of their equivalence for $\mu \in (0, 1)$, the first expression better describes the slow eigenspace (eigenline) for $\mu = 1$, while either of them can be used when $\mu = 0$.

A unique feature of linear time-invariant systems with time-scale separation properties is that the slow manifolds are contained in affine varieties obtained by rigid translation, to the equilibrium point, of the slow eigenspace. However, generally, *such varieties are not themselves global integral manifolds.* This will be demonstrated in the particular example under consideration.

Instead of introducing a singular perturbation parameter ε, the above feature will be used in the computation of the affine varieties containing the slow manifolds of the system for the extreme controls $u = 0$, $u = 1$ and for the variety corresponding to the slow manifold of the average controlled system with constant duty ratio μ.

For $u = 0$, the equilibrium point is located at

$$x_{2,ss} = \frac{b}{w_0}; \quad x_{1,ss} = \frac{w_1 b}{w_0^2} \tag{3.8}$$

and the affine variety containing the slow manifolds of the different state trajectories is constituted by the straight line that contains the equilibrium point and has slope determined by the slow eigenline direction. Using the slopes of the slow eigenlines found in expression (3.7), with $\mu = 0$, the following equivalent definitions are obtained for such an affine variety:

$$S_0 = \left\{ x \in R^2 : x_2 = -\frac{\lambda_2(0)}{w_0} x_1 + \frac{b}{w_0} \left(1 + \lambda_2(0) \frac{w_1}{w_0^2}\right) \right\} \tag{3.9}$$

$$S_0 = \left\{ x \in R^2 : x_2 = \frac{w_0}{w_1 + \lambda_2(0)} x_1 + \frac{b}{w_0} \left(-\frac{\lambda_2(0)}{w_1 + \lambda_2(0)} \right) \right\} \tag{3.10}$$

where $\lambda_2(0)$ is the slow eigenvalue obtained from (3.6) with $\mu = 0$. Figure 3 shows the linear variety S_0 and the state trajectories corresponding to different initial conditions. In this case S_0 is also a global integral manifold for the system (3.1) with $u = 0$, as is easily verified.

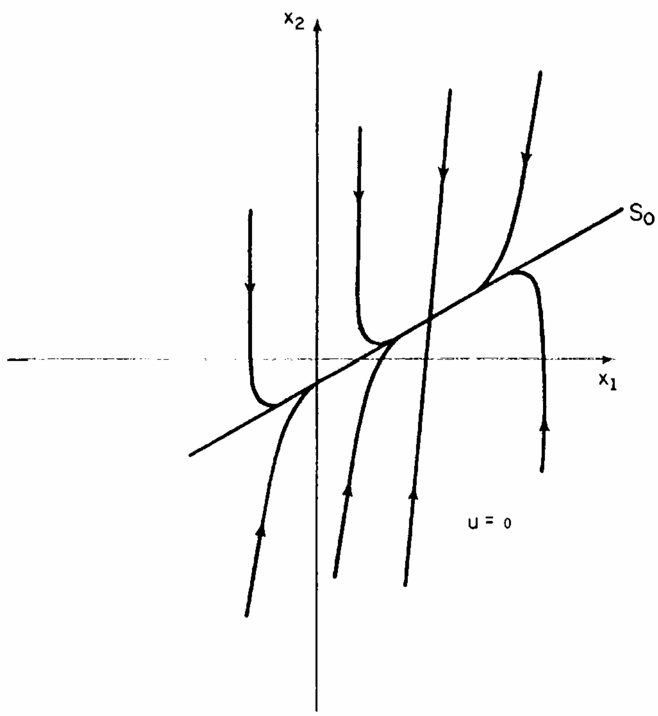

Figure 3. Slow manifold for $u = 0$.

For $u = 1$, an equilibrium manifold exists only for the output voltage variable x_2, represented by the x_1 axis. Using the first expression in (3.7), the affine variety containing the slow manifold is represented by the x_1 axis, i.e.

$$S_1 = \{x \in R^2 : x_2 = 0\} \tag{3.11}$$

The family of fast (conservative) manifolds is parametrized by the value of x_1 in the initial condition of the trajectory (see Fig. 4). The system is then unstable since,

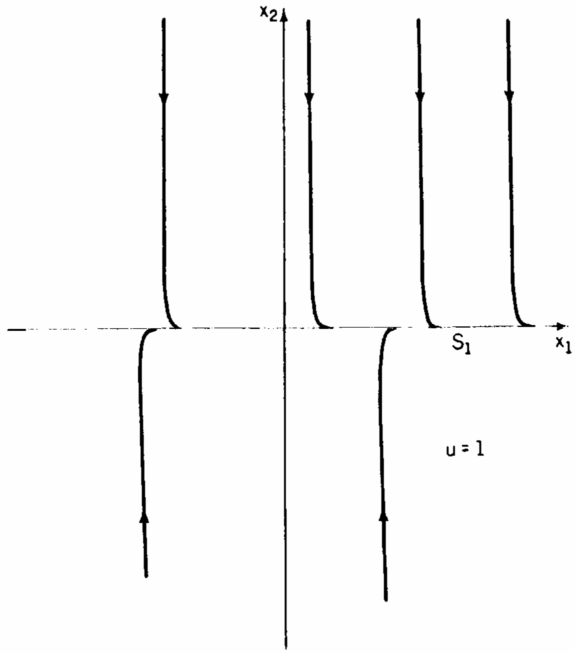

Figure 4. Slow manifold for $u = 1$.

physically, the switch position at $u = 1$ corresponds to having an ideal (non-resistive) inductor connected in parallel to a constant voltage source. In this case, again, S_1 is a global integral manifold for the system (3.1) with $u = 1$.

When the system is controlled by a switching law with constant duty ratio $\mu \in (0, 1)$, the average PWM controlled system is described by (3.2). The affine variety containing the slow manifold of the different trajectories of (3.2) is given by any of the following equivalent expressions:

$$S_\mu = \left\{ x \in R^2 : \sigma = x_2 - \frac{(1-\mu)w_0}{w_1 + \lambda_2(\mu)} x_1 - \frac{b}{(1-\mu)w_0} \frac{\lambda_2(\mu)}{\lambda_2(\mu) + w_1} = 0 \right\} \quad (3.12)$$

$$S_\mu = \left\{ x \in R^2 : \sigma = x_2 + \frac{\lambda_2(\mu)}{(1-\mu)w_0} x_1 - \frac{b}{(1-\mu)w_0} \left[1 + \frac{w_1 \lambda_2(\mu)}{(1-\mu)^2 w_0^2} \right] = 0 \right\} \quad (3.13)$$

Contrary to the previous cases, the affine variety S_μ, shown in Fig. 5, *is not a global integral manifold* for system (3.2) with μ restricted to the interval (0, 1). As shown in the previous section, the region of the state space where a local integral manifold exists for system (3.2) is given explicitly by the existence conditions (2.16).

The intervening fields and the gradient covector for the linear variety are given by

$$f(x) = (b - w_0 x_2) \frac{\partial}{\partial x_1} + (w_0 w_1 - w_1 x_2) \frac{\partial}{\partial x_2}$$

$$g(x) = w_0 x_2 \frac{\partial}{\partial x_1} - w_0 x_1 \frac{\partial}{\partial x_2}$$

$$d\sigma = -\frac{(1-\mu)w_0}{w_1 + \lambda_2(\mu)} dx_1 + dx_2$$

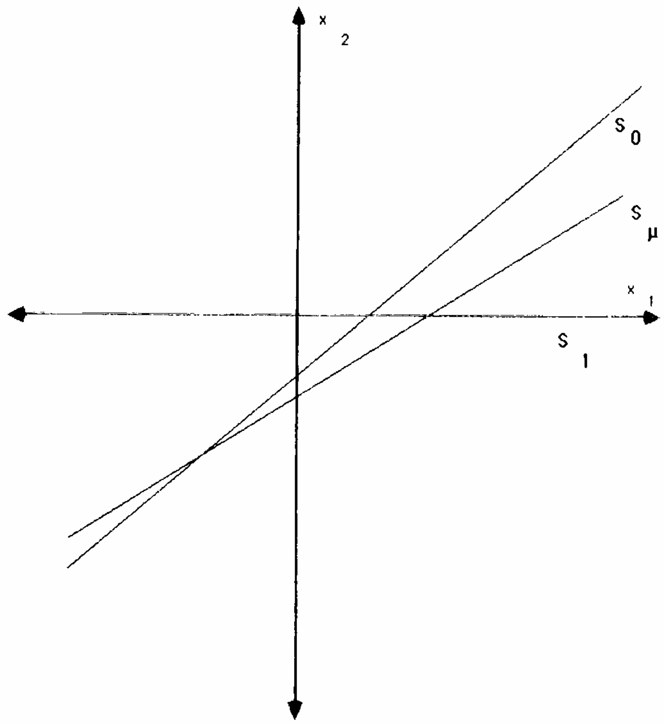

Figure 5. Affine varieties containing slow manifolds.

Particularizing the existence conditions (2.16) of Theorem 5 for the dynamical system (3.1), represented by the above vector fields, leads to the following region of existence of a local integral manifold for the average PWM controlled system (3.2):

$$x_2 > \frac{\lambda_2(\mu)b}{(1-\mu)w_0 w_1} \tag{3.14}$$

$$\left[\frac{(1-\mu)w_0^2}{w_1+\lambda_2(\mu)} - w_1\right]x_2 + w_0 x_1 - \frac{b(1-\mu)w_0}{w_1+\lambda_2(\mu)} > 0 \tag{3.15}$$

The condition $\langle d\sigma, g\rangle < 0$ is represented by an open hemispace given by the inequality constraint

$$x_2 > -\frac{w_1+\lambda_2(\mu)}{(1-\mu)w_0}x_1 \tag{3.16}$$

As was justified in the previous section we restrict our attention to the region where the constraints (3.14)–(3.16) are valid. This unbounded subset of R^2 is shown in Fig. 6. After some tedious but straightforward calculations, it can be verified that the three lines bounding the region of interest have a common intersection point P_1 on the linear variety S_μ. The coordinates of this intersection point P_1 are given by

$$\left.\begin{aligned} x_1 &= -\left[\frac{b\lambda_2(\mu)}{(1-\mu)^2 w_0^2} + \frac{b}{w_1}\right] \\ x_2 &= \frac{\lambda_2(\mu)b}{(1-\mu)w_0 w_1} \end{aligned}\right\} \tag{3.17}$$

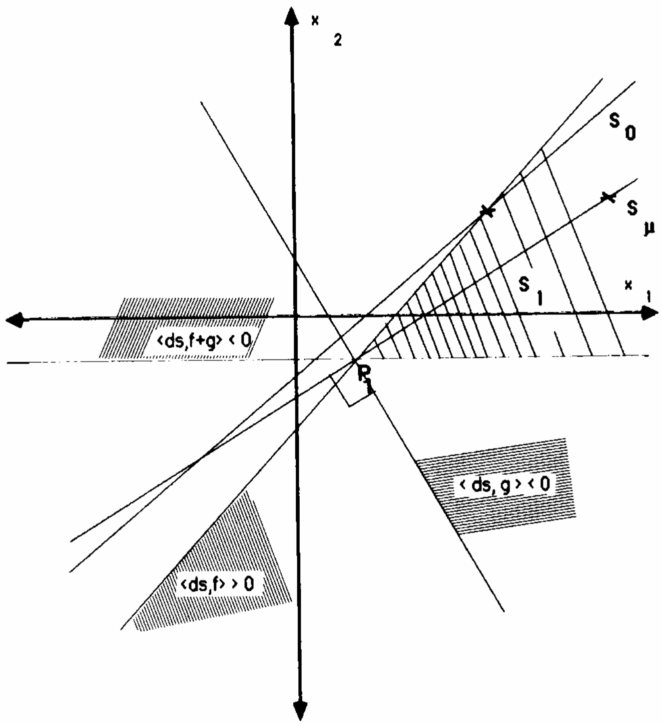

Figure 6. Region of existence for average PWM integral manifold.

It is evident that the local integral manifold is constituted by the unbounded portion of S_μ *to the right* of this point P_1.

Example

Figure 7 depicts simulated state trajectories in a boost converter, for $u = 0$, $u = 1$ and a PWM control scheme with constant duty ratio $\mu = D = 0{\cdot}5$. The component values of the circuit are $L = 150\ \mu$H, $C = 72\ \eta$F, $R = 10\Omega$ and $E = 400$ V.

3.3. *A variable structure control approach*

The preceding developments identified a portion of S_μ where a local integral manifold exists for the average PWM controlled trajectories. This portion of the integral manifold may be utilized to synthesize an equivalent sliding mode by means of variable structure feedback control. The simplicity of the VSC is then used advantageously in combination with the stable, non-oscillatory behaviour of the average dynamics on the slow manifold while maintaining a complete equivalence of the ideal sliding dynamics of the VSC scheme with the average behaviour of the PWM control alternative.

Next, S_μ is taken as the switching surface S for a VSC scheme creating a local sliding regime leading to stable equilibrium. Notice that the sliding mode will be local due to the fact that S_μ does not globally qualify as an integral manifold for (3.2) and hence Theorem 5 only applies to that portion of S_μ which does exhibit such a property.

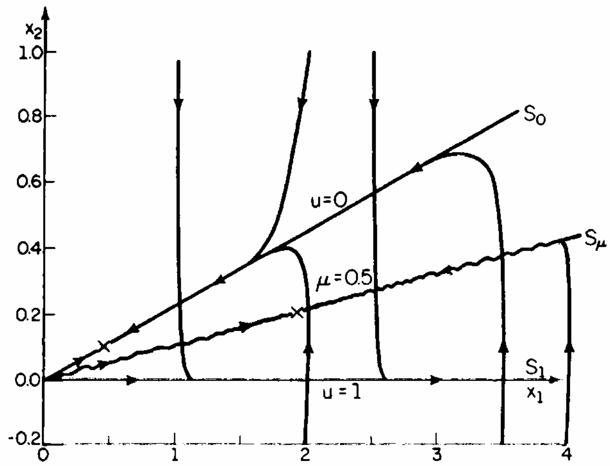

Figure 7. Simulated PWM-controlled response and slow manifolds.

In terms of the surface coordinate value s, the switching surface is defined from, say, (3.12) as

$$S_\mu = \left\{ x \in R^2 : s = x_2 - \frac{(1-\mu)w_0}{w_1 + \lambda_2(\mu)} x_1 - \frac{b}{(1-\mu)w_0} \frac{\lambda_2(\mu)}{\lambda_2(\mu) + w_1} = 0 \right\} \tag{3.18}$$

The VSC law is of the form

$$u = \begin{cases} u^+ & \text{for } s > 0 \\ u^- & \text{for } s < 0 \end{cases} \tag{3.19}$$

with u^+, u^- controls (switch positions) yet to be specified taking values in the discrete set $\{0, 1\}$. Using the definition of s in (3.18), the distribution Ker ds is given by

$$\text{Ker } ds = \text{span} \left\{ \frac{(1-\mu)w_0}{w_1 + \lambda_2(\mu)} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right\} \tag{3.20}$$

The invariance condition $f + u_{EQ}(x)g \in \text{Ker } ds$ on $s = 0$ translates, upon use of the corresponding vector field expressions above, into

$$-\frac{w_0}{w_1 + \lambda_2(\mu)} \left[w_1 x_1 + \frac{\lambda_2(\mu) b w_1}{(1-\mu)^2 w_0^2} + b \right] (\mu - u_{EQ}) = 0 \tag{3.21}$$

that is:

$$u_{EQ} = \mu \tag{3.22}$$

which means that, along the valid portion of S_μ, the average behaviour of the PWM controlled system and the ideal sliding dynamics are totally equivalent. As expected, the necessary and sufficient conditions for the existence of a sliding mode on S_μ, obtained from Theorem 2 (or, equivalently, from condition (2.5)) lead to the determination of the region of existence, totally coincident with that found for the local integral manifold of the average PWM controlled system (3.2). The variable structure control law guaranteeing the existence of a sliding motion on S_μ is obtained from condition (2.5) as

$$\frac{w_0 w_1}{w_1 + \lambda_2(\mu)}\left[x_1 + \frac{\lambda_2(\mu)b}{(1-\mu)^2 w_0^2} + \frac{b}{w_1}\right]\{\mu - u^+\} < 0 \tag{3.23}$$

for $s > 0$ and

$$\frac{w_0 w_1}{w_1 + \lambda_2(\mu)}\left[x_1 + \frac{\lambda_2(\mu)b}{(1-\mu)^2 w_0^2} + \frac{b}{w_1}\right]\{\mu - u^-\} > 0 \tag{3.24}$$

for $s < 0$.

It then follows that the variable structure control law guaranteeing the existence of a sliding motion on S_μ is given by

$$u = \begin{cases} u^+ = +1 & \text{for } s > 0 \\ u^- = 0 & \text{for } s < 0 \end{cases} \tag{3.25}$$

whenever

$$\left[x_1 + \frac{\lambda_2(\mu)b}{(1-\mu)^2 w_0^2} + \frac{b}{w_1}\right] > 0 \tag{3.26}$$

which along $s = 0$ is equivalent to

$$x_2 > \frac{\lambda_2(\mu)b}{(1-\mu)w_0 w_1} \tag{3.27}$$

i.e. a sliding motion exists on S_μ to the right of the point P_1 confirming our previous result for the PWM controlled case. In spite of the local character of the sliding mode, the switching logic (3.25) guarantees global reachability of the portion of S_μ where such sliding motion exist, even from the region to the left of the point P_1. As demonstrated generally, to the left of the point P_1 on S_μ, a sliding mode does not exist with the switching logic (3.25). A sliding mode still can be created on S_μ, in this region, with a reversed switching logic to that of (3.25). It is clear that on an open neighbourhood of the point P_1 a sliding mode does not exist in general.

Notice that any of the average PWM controlled trajectories, totally contained within the existence and transversality region determined by (3.14)–(3.16) and leading to S_μ after a fast transient, also qualifies as a one-dimensional integral manifold in R^2 parametrized by the initial conditions. These manifolds globally sustain a sliding motion leading to stability on the slow portion contained in S_μ. However, the analytic expression for these class of manifolds makes them extremely impractical as sliding surfaces due to electronic hardware implementation of transcendental functions of logarithmic nature. The proposed approach, using the slow manifold characterized by a linear variety, exploits simpler expressions for the sliding surface S_μ and involves only measurement of output voltage and input current while being realizable with operational amplifiers and resistors.

4. Conclusions and suggestions for further research

By establishing an ideal equivalence among VSC and PWM control schemes, a design procedure has been found which proposes the creation of a sliding regime on the integral manifold of the average PWM controlled system. The advantages of a VSC option over a PWM control alternative lie in the hardware simplicity and closedloop robustness. If, on the contrary, a PWM control scheme is still preferred, the proposed design procedure allows for the computation of the necessary duty ratio as a truly feeedback control law. The duty ratio is obtained as the equivalent control associated with the designed sliding manifold. Using either scheme, the ideal average behaviour of the controlled plant is guaranteed to be identical. Necessary and sufficient conditions for the existence of a local integral manifold for the average PWM controlled system are represented by the existence of a local sliding mode on such an integral manifold. Alternatively, the existence of a sliding mode on a certain manifold guarantees that such a surface also qualifies as an integral manifold for an idealized average PWM controlled response. In both cases the equivalence of the duty ratio to the equivalent control was established. This result unifies both approaches and renders an equivalence which can be used to cast the design issues, associated with PWM control techniques, into a geometric framework with the advantages of a more intuitive and systematic methodology.

In realistic applications, high-frequency switchings or high-frequency duty cycles are necessary to approximate the ideal behaviour exploiting the equivalence among both approaches.

The method was used to obtain a VSC law leading to a sliding mode on an affine variety containing the slow manifold of the boost converter. Such a possibility was derived from desirable time-scale separation properties among the LC input filter natural frequency and the RC output circuit time constant. Being only a local integral manifold, the affine variety does not globally satisfy the sliding mode existence conditions. However, its simplicity makes it attractive for hardware implementation since the switch position is determined only from sign information (i.e. on bit data) about an affine functional of the converter's state.

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