Nonlinear Variable Structure Systems in Sliding Mode: The General Case

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***Abstract*—The problem of inducing local sliding regimes on smooth state-space surfaces of nonlinear single-input single-output controlled systems is addressed in full generality. The notion of relative degree, in its more general form is used in establishing the most salient features of nonlinear controlled systems undergoing sliding motions, including their characteristic disturbance rejection properties.**

I. Introduction

In this note, we examine the properties of general single-input single-output nonlinear variable structure controlled systems operating in sliding mode [1].

It is found that a sliding regime locally exists on the zero level set of the output function, if and only if the nonlinear system has *relative degree* [2], [3] equal to one. The corresponding $n - 1$ dimensional *zero dynamics* precisely coincides with the *ideal sliding dynamics* [1] in local surface coordinates. Using the ideas in [3], the problem of inducing sliding regimes on systems with relative degree higher than one is examined. The disturbance rejection properties of nonlinear systems undergoing sliding motions are also analyzed and a general matching condition is found.

In Section II we present some background material about a generalization of the relative degree concept, normal forms, and zero dynamics. New and general results about sliding motions, in general nonlinear systems, are presented in Section III. Section IV contains the conclusions.

II. Background and Main Results

A. Relative Degree, Normal Forms, and Zero Dynamics

Consider the smooth single-input single-output nonlinear system of the form

$$dx/dt = X(x, u)$$
$$y = h(x) \tag{2.1}$$

locally defined for all $x \in \mathbf{0}$, an open set in R^n. We often refer to (2.1) as the pair (X, h).

The smooth level set $h^{-1}(0) := \{x \in \mathbf{0}: h(x = 0\}$, locally defines the *sliding manifold*. The gradient of $h(x)$, denoted by dh, is locally assumed to be nonzero almost everywhere on $h^{-1}(0)$ and it is oriented in such a way that dh locally points from the region where $h(x) < 0$ towards that where $h(x) > 0$.

The *Lie derivative* [4] of a scalar function $\phi(x)$, with respect to a smooth vector field X, locally defined on $\mathbf{0}$, is denoted by $L_X\phi$. In local coordinates, $L_X\phi$ is expressed as $L_X\phi = (\partial\phi/\partial x)X$. For any positive integer k, one recursively defines $L_X^k\phi(x) = L_X[L_X^{k-1}\phi(x)] = [\partial(L_X^{k-1}\phi(x))/\partial x]X$.

Remark: Lie derivatives constitute an efficient shorthand notation for time derivatives of scalar functions along solutions of differential equations. For instance, with reference to (2.1), $dy/dt = (\partial h/\partial x)dx/dt = (\partial h/\partial x)X =: L_Xh$ (also expressed as $<dh, X>$). $d^2y/dt^2 = [\partial(L_Xh)/\partial x]X = L_X(L_Xh) =: L_X^2h$. For any integer r, $d^ry/dt^r = L_X^rh$. Suppose d^jy/dt^j is locally independent of u, for $j = 0, 1, 2, \cdots, k$, then $\partial[d^ky/dt^k]/\partial u = 0$, i.e., $\partial[L_X^kh]/\partial u = \partial(L_X(L_X^{k-1}h))/\partial u = \partial[(\partial(L_X^{k-1}h)/\partial x)X]/\partial u = [\partial(L_X^{k-1}h)/\partial x](\partial X/\partial u) = L_{\partial X/\partial u}L_X^{k-1}h = 0$.

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Definition 2.1: The pair (X, h) has *relative degree* r at x^o, if d^ky/dt^k does not explicitly depend upon u, locally around x^o, for all $k \le r - 1$. In other words

$$L_{\partial X/\partial u}L_X^kh(x) = 0 \qquad \text{for all } x \text{ in } N, \text{ and all } k < r-1$$
$$L_{\partial X/\partial u}L_X^{r-1}h(x^0) \neq 0 \tag{2.2}$$

with N being an open vicinity of x^o contained in $\mathbf{0}$.

Proposition 2.2 [3]: Let (2.1) have local relative degree r on x^o. Set $\phi_i(x) = L_X^{i-1}h(x)$ for $i = 1, 2, \cdots, r$, while the functions $\phi_{r+j}(x)$, $j = 1, 2, \cdots, n - r$, are chosen to be *functionally independent* of the first r functions, with the only additional requirement that, locally around x^o, $L_{\partial X/\partial u}\phi_{r+j} = 0$ for all j's. Define new z coordinates as $z = \Phi(x)$ with $\Phi(x) = \text{col}\,[\phi_1(x), \cdots, \phi_n(x)]$ being a *local diffeomorphism* [4] on N. The transformed system, said to be in *normal coordinates*, is locally expressed, around $z^o = \Phi(x^o)$ as

$$dz_i/dt = z_{i+1} \qquad i = 1, 2, \cdots, r-1$$
$$dz_r/dt = [L_X^rh](\Phi^{-1}(z), u)$$
$$dz_{r+j}/dt = q_j(z) \qquad j = 1, 2, \cdots, n-r; \quad y = z_1. \tag{2.3}$$

Proof: Obvious from the choice of coordinates. □

Remark: If initial conditions of (2.1) are set on M (i.e., on $z_1 = 0$), then the components $z_1, \cdots, z_r$ of the normal coordinate vector z are all zero. Hence, any point x^o on M is expressed, in normal coordinates, as: $(0, \eta)$ where $\eta = \text{col}\,(z_{r+1}, \cdots, z_n)$. If, furthermore, a feedback control $v = \alpha(z)$ is used such that $[L_X^rh]\,(\Phi^{-1}(z), u(z)) = 0$ locally around z^o, the evolution of the controlled system locally remains on $h^{-1}(0)$. (By virtue of the definition of local relative degree, and the Implicit Function theorem [4], such an $\alpha(z)$ is locally guaranteed to exist around z^o and, moreover, it is uniquely defined.)

Definition 2.3 [2]: The dynamic behavior of the system (2.1), with initial conditions set on $h^{-1}(0)$, and feedback control input $\alpha(z)$ such that the quantity $dz_r/dt = [L_X^rh]\,(\Psi^{-1}(z), \Phi(z)) = 0$, described in normal coordinates by

$$d\eta/dt = q(0, \eta) = q_0(\eta) \tag{2.4}$$

is addressed as the *zero dynamics*. □

Assumption: It is assumed throughout that $y = h(x)$ has been chosen to make the system (2.4) asymptotically stable (i.e., it is *globally minimum phase* [2]).

Lemma 2.4: The relative degree of a system is a feedback invariant.

Proof: For a system with feedback $u = u(x, v)$, the time derivatives of y are locally independent of u if and only if they are locally independent of v. □

III. Sliding Regimes in General Nonlinear Systems

A. Generalities about Local Sliding Regimes

A local variable structure feedback control law for (X, h) is obtained by letting

$$u = \begin{cases} u^+(x) & \text{for } h(x) > 0 \\ u^-(x) & \text{for } h(x) < 0 \end{cases} \tag{3.1}$$

with $u^+(x) > u^-(x)$, locally on a neighborhood N of x^o.

Definition 3.2 [1], [5]: A *sliding regime* is said to locally exist on an open set M of $h^{-1}(0)$, if, as a result of the control policy (3.1), the state trajectories of (2.1) satisfy

$$\lim_{h\to 0^+} dy/dt = \lim_{h\to 0^+} L_{X(x,u^+(x))}h$$
$$= \lim_{h\to 0^+} \langle dh, X(x, u^+(x))\rangle < 0;$$

$$\lim_{h\to 0^-} dy/dt = \lim_{h\to 0^-} L_{X(x,u^-(x))}h$$
$$= \lim_{h\to 0^-} \langle dh, X(x, u^-(x))\rangle > 0. \tag{3.2}$$

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Theorem 3.3: A sliding regime locally exists on an open set M of $h^{-1}(0)$ for some discontinuous control law of the form (3.1), if and only if the system (X, h) has local relative degree equal to 1.

Proof: Necessity is obvious from (3.2). To prove sufficiency, suppose $L_{\partial X/\partial u}h(x) = \partial(L_X h)/\partial u \neq 0$ locally around x^o. Let $\epsilon^-(x)$ be a smooth, locally strictly positive function of x. Then, by virtue of the Implicit Function theorem, the equation $[L_X h](x, u) = \epsilon^-(x)$ locally has a unique smooth solution $u = u^{-\prime}(x)$ such that $L_{X(x,u^{-\prime}(x))}h(x) = \epsilon^-(x) > 0$. By the same arguments, given a smooth locally strictly negative function $\epsilon^+(x)$, a smooth control law $u = u_o^{+\prime}(x)$ locally exists around x^o such that $L_{X(x,u^{+\prime}(x))}h(x) = \epsilon^+(x) < 0$. Hence, a sliding regime locally exists on an open set H of $h^{-1}(0)$ since (3.2) is satisfied by the variable structure feedback control law $u^+(x) = u^{-\prime}(x)$ for $h(x) < 0$ and $u^-(x) = u^{+\prime}(x)$ for $h(x) > 0$. □

Condition $L_{\partial X/\partial u}h \neq 0$, is a general local *transversality condition* [5].

Example: For systems of the form $X(x, u) = f(x) + ug(x)$, $y = h(x)$, the local transversality condition takes the form $L_g h < 0$.

For all initial states located on a vicinity M of x^o in $h^{-1}(0)$, the unique control function, $u^{EQ}(x)$, locally constraining the system trajectories to the manifold $y = h(x) = 0$, in the region of existence of a sliding regime, is known as the *equivalent control*. The resulting dynamics, ideally constrained to M, is the *ideal sliding dynamics* [1], and it evidently coincides with the zero dynamics of (X, h). A coordinate free description of such dynamics is

$$dx/dt = X(x, u^{EQ}(x)). \tag{3.3}$$

Since M becomes, locally an *integral manifold* [4] for (3.3). It follows that the gradient of h is locally orthogonal to the controlled vector field $X(x, u^{EQ}(x))$, i.e.,

$$L_{X(x,u^{EQ}(x))}h(x) = \langle dh, X(x, u^{EQ}(x))\rangle = 0. \tag{3.4}$$

Theorem 3.4: A necessary condition for the local existence of a sliding regime on an open set M of $h^{-1}(0)$ is that there locally exists a unique smooth equivalent control, $u^{EQ}(x)$, satisfying

$$u^-(x) < u^{EQ}(x) < u^+(x) \tag{3.5}$$

for the given smooth feedback functions $u^-(x)$ and $u^+(x)$.

Proof: To prove uniqueness, notice that if a sliding motion locally exists on M, then (X, h) has local relative degree 1, i.e., $L_{\partial X/\partial u}h(x) \neq 0$ for all x in a vicinity N of x^o. By the Implicit Function theorem the equation $[L_X h](x, u) = 0$ has, locally, a unique solution $u^{EQ}(x)$ in M for which (3.4) holds valid. Hence, if a sliding regime exists, the equivalent control locally exists and is uniquely defined.

Suppose now that a sliding regime locally exists on M for the switching feedback control law (3.1). Then, locally on M, the following three relations hold valid:

$$L_{X(x,u^+(x))}h = \langle dh, X(x, u^+(x))\rangle < 0 \tag{3.6}$$

$$L_{X(x,u^{EQ}(x))}h = \langle dh, X(x, u^{EQ}(x))\rangle = 0 \tag{3.7}$$

$$L_{X(x,u^-(x))}h = \langle dh, X(x, u^-(x))\rangle > 0. \tag{3.8}$$

Subtracting (3.7) from (3.6) and (3.8) from (3.7) one obtains

$$\langle dh, X(x, u^+(x)) - X(x, u^{EQ}(x))\rangle < 0$$

$$\langle dh, X(x, u^{EQ}(x)) - X(x, u^-(x))\rangle < 0. \tag{3.9}$$

From the Mean Value theorem, there exists smooth functions $u_o^+(x)$ and $u_o^-(x)$, such that locally on M

$$\langle dh, X(x, u^+(x)) - X(x, u^{EQ}(x))\rangle = [u^+(x) - u^{EQ}(x)]\langle dh, \partial X(x, u_0^+(x))/\partial u\rangle < 0$$

$$\langle dh, X(x, u^{EQ}(x)) - X(x, u^-(x))\rangle = [u^{EQ}(x) - u^-(x)]\langle dh, \partial X(x, u_0^-(x))/\partial u\rangle < 0 \tag{3.10}$$

where $u_o^+(x)$ and $u_o^-(x)$, respectively, satisfy $u^{EQ}(x) < u_o^+(x) < u_o^+(x)$ and $u^-(x) < u_o^-(x) < u^{EQ}(x)$, i.e., locally on M, $u^-(x) < u^{EQ}(x) < u^+(x)$. □

Example: Let $X(x, u) = f(x) + ug(x)$ be such that, locally on an open set M of $h^{-1}(0)$, $L_g h < 0$. $L_{X(x,u^{EQ})}h = L_f h + u^{EQ} L_g h = 0$ implies $u_{EQ} = -L_f h/L_g h$, i.e., the equivalent control locally exists and is uniquely defined. Furthermore, if a local sliding motion exists on an open set M of $h^{-1}(0)$, we locally have on M: $L_f h + u^- L_g h < 0$ and $L_f h + u^+ L_g h > 0$. This implies that there exists smooth functions $a(x) > 0$ and $b(x) > 0$, such that $a(x)[L_f h + u^+ L_g h] + b(x)[L_f h + u^- L_g h] = [a(x) + b(x)] L_f h + [a(x)u^+(x) + b(x)u^-(x)]g(x) = 0$, i.e., $L_f h + L_g h\,[a(x)u^+(x) + b(x)u^-(x)]/[a(x) + b(x)] = 0$, i.e., $u^-(x) < [a(x)u^+(x) + b(x)u^-(x)]/[a(x) + b(x)] = u^{EQ}(x) < u^+(x)$ locally on M. □

Definition 3.5: Let (X, h) have local relative degree 1 on $x^o \in h^{-1}(0)$. The system (X, h) is said to exhibit a local *control foliation property* about the manifold $h^{-1}(0)$, if and only if given smooth feedback functions $u_1(x) < u_2(x) < u_3(x)$, defined on M, then, $[L_X h](x, u_1) > [L_X h](x, u_2) > [L_X h](x, u_3)$, locally on M. (See [6] for the singularly perturbed case.) □

Theorem 3.6: Control law (3.1) locally induces a sliding regime, for system (X, h), on an open set M of $h^{-1}(0)$, if and only if (X, h) exhibits a control foliation property about $h^{-1}(0)$, and there exists a feedback control $u^{EQ}(x)$ locally satisfying both (3.4) and (3.5) on M.

Proof: Suppose that, thanks to a control action of the form (3.1), the system locally exhibits a sliding regime on an open set M of $h^{-1}(0)$. Then, according to Theorem 3.4, there necessarily exists a unique smooth $u^{EQ}(x)$ satisfying $u^-(x) < u^{EQ}(x) < u^+(x)$ locally on M. Since a sliding motion exists on M, it follows that $[L_X h](x, u^-(x)) > 0$, $[L_X h](x, u^{EQ}(x)) = 0$, and $[L_X h](x, u^+(x)) < 0$, locally on M, i.e., $[L_X h](x, u^-(x)) > [L_X h](x, u^{EQ}(x)) > [L_X h](x, u^+(x))$ on M. Hence, (X, h) exhibits a control foliation property.

If, on the other hand, the system (X, h) exhibits a control foliation property and $u^{EQ}(x)$ exists such that locally on an open set M in $h^{-1}(0)$, $[L_X h](x, u^{EQ}(x)) = 0$ and $u^+(x) > u^{EQ}(x) > u^-(x)$. Then, it follows that $[L_X h](x, u^-(x)) > [L_X h](x, u^{EQ}(x)) = 0 > [L_X h](x, u^-(x))$. Hence, necessarily, $[L_X h](x, u^+(x)) < 0$ and $[L_X h](x, u^-(x)) > 0$ holds true on M. It follows that there exists an open neighborhood M of $x^o \in 0$, with nonempty intersection with $h^{-1}(0)$, where conditions (3.2) are satisfied. Thus, a sliding regime locally exists on $h^{-1}(0)$.

Example: Notice that for $X(x, u) = f(x) + ug(x)$, and if the transversality condition $L_g h < 0$ holds locally true, the control foliation property is trivially satisfied. It follows that, for affine systems, the condition $u^- < u^{EQ} < u^-$ is both a necessary and sufficient condition for the existence of a sliding regime (see [5]).

B. Sliding Regimes in Variable Structure Systems with Relative Degree Higher Than One

If, for the proposed output function $y = h(x)$, the system locally exhibits relative degree r, higher than 1, on x^o, then, an alternative to create a local sliding motion, which eventually reaches $h^{-1}(0)$, is to use the auxiliary output function (see [3] for the basic idea in *local feedback stabilization*)

$$w = k(x) = L_X^{r-1}h(x) + c_{r-2}L_X^{r-2}h(x) + \cdots + c_1 L_X h(x) + c_0 h(x) \tag{3.11}$$

or, in normal form coordinates

$$w = z_r + c_{r-2}z_{r-1} + \cdots + c_1 z_2 + c_0 z_1. \tag{3.12}$$

Evidently, $L_{\partial X/\partial u}k(x^o) = L_{\partial X/\partial u}L_X^{r-1}h(x^o) \neq 0$, i.e., the system (X, k) has local relative degree 1, and a sliding motion can now be locally created on an open set of $k^{-1}(0)$. Then, ideally, $w = 0$ and $z_r = -c_{r-2}z_{r-1} - \cdots - c_1 z_2 - c_0 z_1$. The ideal sliding system, associated with the new sliding surface $k^{-1}(0)$, is expressed as

$$dz_i/dt = z_{i+1}; \quad i = 1, 2, \cdots, r-2$$

$$dz_{r-1}/dt = z_r = -c_{r-2}z_{r-1} - \cdots - c_1 z_2 - c_0 z_1$$

$$dz_{r+j}/dt = q(z_1, z_2, \cdots, z_{r-1}, -(c_{r-2}z_{r-1} + \cdots + c_1 z_2 + c_0 z_1), z_{r+1}, \cdots, z_n) \quad j = 1, \cdots, n-r$$

$$y = z_1;\ w = 0 \tag{3.13}$$

It is easy to see that by suitable choice of the parameter $c_0, \cdots, c_{r-2}$, an asymptotically stable motion can be obtained for the first $r - 1$ normal coordinates. Thus, while a sliding motion locally takes place on $k^{-1}(0)$, the original output y and its first $r - 1$ derivatives asymptotically approach zero (i.e., the state vector of the system approaches the manifold $h^{-1}(0)$, as initially desired). The corresponding equivalent control is now locally given, in original coordinates, as the unique solution of $[L_X h](x, u^{EQ}(x)) = 0$.

Example: For systems with affine vector fields $X = f + u g$ and output $y = h(x)$ which exhibit relative degree r on x^o, the function $w = k(x)$ is given by $k(x) = L_f^{r-1} h(x) + c_{r-2} L_f^{r-2} h(x) + \cdots + c_1 L_f h(x) + c_0 h(x)$ and the equivalent control is computed as $u^{EQ}(x) = -L_f k / L_g k = [L_f^r h + c_{r-2} L_f^{r-1} h + \cdots + c_1 L_f^2 h + c_0 L_f h] / L_g L_f^{r-1} h$. Locally, on an open set M of $h^{-1}(0)$, the last expression takes the form $u^{EQ}(x) = L_f^r h / L_g L_f^{r-1} h$. □

The use of the auxiliary output $w = k(x)$ implies the possibility of either being able to completely measure the original state variables and proceed to use (3.11), or else being able to generate $r - 1$ derivatives of the original output function y. The last possibility is usually accomplished by means of a high gain, phase lead, "post-processor," as proposed in [3], fed by the output signal $y(t)$. The transfer function of such a post-processor is given by

$$\frac{-Kn(s)}{(1+Ts)^r} \tag{3.14}$$

with T being a sufficiently small positive constant, K a sufficiently large gain with, locally, the same sign as $L_{\partial X/\partial u} L_X^{r-1} h(x)$. $n(s)$ is a stable polynomial built as $n(s) = s^{r-1} + c_{r-2} s^{r-2} + \cdots + c_1 s + c_0$.

C. Disturbance Rejection Properties of Systems Undergoing Sliding Regimes

Consider a smooth nonlinear perturbed system of the form

$$dx/dt = X(x, u, w)$$
$$y = h(x) \tag{3.15}$$

where w is a scalar perturbation signal affecting the system behavior. Let us assume that the system locally has relative degree 1 on x^o, and suppose the input w is assumed to have local relative degree higher than 1 with respect to the output function h. In normal form coordinates, the perturbed system is written as

$$dz_1/dt = [L_X h](\Phi^{-1}(z), u, w) = [L_X h](\Phi^{-1}(z), u)$$
$$d\eta/dt = q(z_1, \eta, w); \qquad y = z_1. \tag{3.16}$$

If a sliding motion can be created on $z_1 = 0$, the equivalent control $u^{EQ}(z)$, is clearly unaffected by the perturbation signal w, although the ideal sliding motion may still be affected. The following lemma follows immediately.

Lemma 3.8: Let (X, h) have local relative degree 1. The existence of a local sliding motion on an open set M of $h^{-1}(0)$ is independent of the perturbation signal w if and only if w has local relative degree, at least, equal to 2, with respect to the output function $h(x)$. □

Theorem 3.9: The ideal sliding dynamics is totally unaffected by perturbation signals w, of any kind, if and only if the *matching condition*

$$\partial X/\partial w \in \text{range } \{\partial X/\partial u\} \tag{3.17}$$

is satisfied.

Proof: If the zero dynamics is unaffected by w, necessarily, $L_{\partial X/\partial w}\phi_i = 0$ for $i = 2, \cdots, n$. Since the ϕ_i's were chosen to also satisfy $L_{\partial X/\partial u}\phi_i = 0$, for $i = 2, \cdots, n$, it follows that $\partial X/\partial w$ is, at least locally, exactly in the range of $\partial X/\partial u$. On the other hand, $\partial X/\partial w$ is locally in the range of $\partial X/\partial u$ if and only if there exists a smooth function $b(x)$ such that, locally on M, $\partial X/\partial w = b(x)\,[\partial X/\partial u]$. Hence, for $i = 2, 3, \cdots, n$ we have $\partial L_X \phi_i/\partial w = L_{\partial X/\partial w}\phi_i = L_{[b(x)\partial X/\partial u]}\phi_i = b(x)\, L_{\partial X/\partial u}\, \phi_i = 0$. Hence, w does not affect the zero dynamics.

IV. Conclusions

In this note, the relevance of the relative degree concept has been examined in the analysis and design issues related to the creation of sliding regimes for general nonliner systems. The results indicate that the simplest possible structure at infinity must be exhibited by nonlinear systems undergoing sliding motions on the zero level set of the output feedback function. General necessary as well as necessary and sufficient conditions for the existence of sliding regimes have been presented. The disturbance rejection properties of sliding mode control were also examined and a generalization of the matching condition was found.

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Error Bounds in the Averaging of Hybrid Systems

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***Abstract*—This note analyzes the errors introduced by the averaging of hybrid systems. These systems involve linear systems which can take a number of different realizations based on the state of an underlying finite state process. The averaging technique (based on a formula from Lie algebras known as the Backer-Campbell-Hausdorff (BCH) formula) provides a single system matrix as an approximation to the hybrid system. The two errors discussed are: 1) the error induced by the truncation of the BCH series expansion; and 2) the error between the actual hybrid system and its average. A simple sufficient stability test is proposed to check the asymptotic behavior of this error. In addition, conditions are derived that allow the use of state feedback to arrive at a time-invariant system matrix instead of averaging.**

I. Introduction and Problem Formulation

Hybrid systems are a special class of piecewise constant time-varying systems. Such models switch at different time instants among a finite set of linear time-invariant realizations. Systems of this type can be used

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