INT. J. CONTROL, 1989, VOL. 50, NO. 4, 1487–1506

Sliding regimes in general non-linear systems: a relative degree approach

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In this paper we treat the problem of inducing local sliding regimes on non-linear smooth surfaces defined in the state space of general non-linear controlled systems. A suitable notion of relative degree is found to be of crucial importance in establishing the most salient properties of non-linear systems undergoing sliding motion. The relevance of sliding modes in several control problems as complementary 'outer loop' feedback is examined.

1. Introduction

The notion of the *structure at infinity* of dynamical controlled systems plays a fundamental role in the understanding of non-linear controlled dynamics. Thus far, this concept has allowed the evolution into a non-linear setting of many basic and long standing automatic control problems. Among these problems we find local stabilization, feedback linearization, disturbance decoupling, interaction decoupling, systems invertibility and non-linear adaptive control (see Byrnes and Isidori 1984, Isidori 1985 a and the excellent introductory material in Isidori 1987, which is closely followed in this work).

In this paper, we examine the relevance of the notion of the structure at infinity (or *relative degree*) and of its associated state coordinates transformation into *normal form* coordinates, in general single-input single-output non-linear variable-structure controlled systems operating in *sliding mode* (Utkin 1978).

It is found that a sliding regime exists locally on the zero level set of the output function, if and only if the non-linear system has relative degree equal to 1 (i.e. it exhibits the simplest structure at infinity). The corresponding $(n-1)$-dimensional *zero dynamics* precisely portray the qualitative features of the *ideal sliding dynamics* (Utkin 1978) in local surface coordinates. The problem of inducing sliding regimes on systems with relative degree higher than one is also examined. The implications of the relative degree concept in sliding mode disturbance decoupling, variable-structure control of feedback linearizable systems and model-matching problems, via sliding modes, are also analysed.

In § 2 we present background material on a generalization of the relative degree concept, normal forms and zero dynamics. New results on sliding motion for general non-linear systems are also presented in that section. Section 3 presents applications of sliding modes in control areas such as disturbance decoupling, feedback linearization and model matching. Section 4 contains the conclusions and suggestions for further work in this area.

Received 12 July 1988. Revised 27 October 1988.

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0020-7179/89 $3.00

2. Background and main results

2.1. *Relative degree, normal forms and zero dynamics*

Consider the non-linear smooth system of the form:

$$\left.\begin{aligned} dx/dt &= X(x, u) \\ y &= h(x) \end{aligned}\right\} \tag{2.1}$$

locally defined for all $x \in \mathbf{O}$, an open set in R^n, u: $\mathbf{O} \to R$, is a (possibly discontinuous) scalar feedback input function, while, for each fixed smooth feedback control $u(x)$, X represents a locally smooth controlled vector field defined on $\mathbf{O}$. The output function h: $\mathbf{O} \to R$ is a locally smooth scalar function of the state. We often refer to (2.1) as the pair (X, h).

The level set $h^{-1}(0) = \{x \in \mathbf{O} : h(x) = 0\}$, locally defines a smooth $(n-1)$-dimensional locally *regular manifold* of constant rank (i.e. an *integrable manifold*. (See Boothby (1975); addressed as the *sliding manifold*.) The gradient of $h(x)$, denoted by dh, is assumed locally to be non-zero on $h^{-1}(0)$ except possibly on a set of measure zero. $h^{-1}(0)$ is oriented in such a way that dh points locally from the region where $h(x) < 0$ towards that where $h(x) > 0$.

We shall refer to a property as *local around* x^0, whenever it is valid on an open vicinity N of a given point $x^0 \in \mathbf{O}$, with $\mathbf{O} \supset N$. If the point is located on $h^{-1}(0)$ we say the property is valid *locally on* $h^{-1}(0)$ if it is valid on an open set M of the submanifold $h^{-1}(0)$ (i.e. on an open subset of $h^{-1}(0) \cap N$).

The *Lie derivative* of a scalar function $\phi(x)$ with respect to a smooth vector field X locally defined on $\mathbf{O}$, is denoted by $L_X\phi$. One recursively defines, for any positive integer k: $L_X^k\phi(x) = L_X[L_X^{k-1}\phi(x)]$.

Definition 2.1

The pair (X, h) has locally around x^0 a *zero at infinity of multiplicity* r if

$$\left.\begin{aligned} &L_{\partial X/\partial u}L_X^k h(x) = 0 \quad \text{for all } x \text{ in } N, \text{ and all } k < r-1 \\ &L_{\partial X/\partial u}L_X^{r-1} h(x^0) \neq 0 \end{aligned}\right\} \tag{2.2}$$

The integer r is also called the local *relative degree* of (X, h) at x^0.

Example

Consider the non-linear system (2.1) with $X(x, u) = f(x) + g(x)u$. Then $\partial X/\partial u = g(x)$. Suppose the system has local relative degree r at x^0. Using Definition 2.1 we compute, for any x in N: $L_{\partial X/\partial u}h(x) = L_g h = 0$; $L_{\partial X/\partial u}L_X h(x) = L_g[L_{f+gu}h(x)] = L_g L_f h(x) + uL_g^2 h(x) = L_g L_f h(x) = 0$; $L_{\partial X/\partial u}L_X^2 h(x) = L_g L_{f+gu}[L_f h(x) + uL_g h(x)] = L_g[L_f^2 h(x) + uL_g L_f h(x) + uL_f L_g h(x) + u^2 L_g^2 h(x)] = L_g L_f^2 h(x) = 0$ .... Generally, one obtains, $L_{\partial X/\partial u}L_X^k h(x) = L_g L_f^k h(x) = 0$, for all $k < r-1$ and all x in N. Finally, $L_{\partial X/\partial u}L_X^{r-1}h(x^0) = L_g L_f^{r-1}h(x^0) \neq 0$. Definition 2.1 thus generalizes the usual definition of local relative degree (see Byrnes and Isidori 1984).

Remark

The relative degree of (2.1) is interpreted as the minimum number of times one has to differentiate y, with respect to time, in order to have the derivative of y depend explicitly on u. Notice that if $y^{(k)}$ $(0 \leqslant k < r)$ is independent of u, then $\partial y^{(k)}/\partial u = 0$. Since $y^{(k)} = L_X^k h(x) = L_X[L_X^{k-1} h(x)] = \{\partial[L_X^{k-1} h(x)]/\partial x\} X(x, u) = \{\partial y^{(k-1)}/\partial x\} X(x, u)$ and since $y^{(k-1)}$ is assumed to be independent of u, it then follows that, $\partial[L_X^k h(x)]/\partial u = \partial(\{\partial[L_X^{k-1} h(x)]/\partial x\} X(x, u))/\partial u = \{\partial[L_X^{k-1} h(x)]/\partial x\} \partial X/\partial u = L_{\partial X/\partial u}[L_X^{k-1} h(x)] = 0$. If $y^{(r)}$ is the first time-derivative that explicitly depends on u then, in general, $\partial y^{(r)}/\partial u = L_{\partial X/\partial u}[L_X^{r-1} h(x)] \neq 0$ (i.e. not identically zero). This completely justifies our definition of relative degree.

Proposition 2.2

Let (2.1) have local relative degree r on x^0. Set $\phi_i(x) = L_X^{i-1} h(x)$ for $i = 1, 2, \ldots, r$, while the functions $\phi_{r+j}(x), j = 1, 2, \ldots, n - r$, are chosen to be *functionally independent* of the first r functions, with the only additional requirement that, locally around x^0, $L_{\partial X/\partial u}\phi_{r+j} = 0$ for all js. Define new z coordinates as $z = \Phi(x)$ with $\Phi(x) = \mathrm{col}\,[\phi_1(x), \ldots, \phi_n(x)]$ being a local diffeomorphism on N. The system (2.1) is locally expresssed, around $z^0 = \Phi(x^0)$ as

$$\left.\begin{aligned} dz_i/dt &= z_{i+1}, \quad i = 1, 2, \ldots, r-1 \\ dz_r/dt &= [L_X^r h](\Phi^{-1}(z), u) \\ dz_{r+j}/dt &= q_j(z), \quad j = 1, 2, \ldots, n-r \\ y &= z_1 \end{aligned}\right\} \tag{2.3}$$

Proof

Obvious from the choice of coordinates (see also Isidori (1987)). □

System (2.3) is said to be in local *normal form* coordinates. A block diagram depicting the structure of the system (2.3) is shown in Fig. 1.

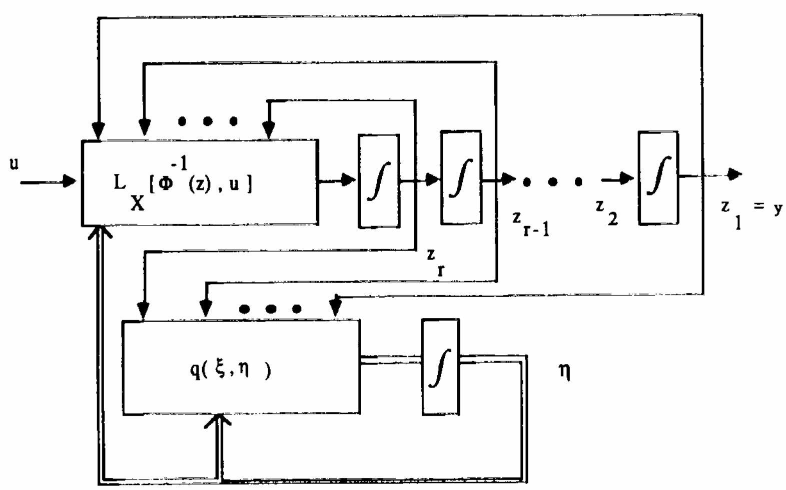

Figure 1. Block diagram of a non-linear system in normal form.

Remark

It is easy to see, from the definition of the normal coordinates, that if initial conditions of (2.1) are set on M (i.e. on $z_1 = 0$), then the components $z_1, \ldots, z_r$ of the normal coordinate vector z are all zero. Hence, any point x^0 on M is expressed, in normal coordinates, as $(0, \eta)$, where $\eta = \text{col}(z_{r+1}, \ldots, z_n)$. If, furthermore, a feedback control $u = \alpha(z)$ is used such that $[L_X^r h](\Phi^{-1}(z), u(z)) = 0$ locally around z^0, the evolution of the controlled system locally remains on $h^{-1}(0)$. (Note that from the definition of local relative degree and the implicit function theorem (Isidori 1985 a), such an $\alpha(z)$ locally exists around z^0 and it is uniquely defined.)

Definition 2.3

The description, in normal coordinates, of (2.1) with initial conditions prescribed on an open set M of $h^{-1}(0)$ and feedback control input $\alpha(z)$ such that $dz_r/dt = [L_X^r h](\Phi^{-1}(z), \alpha(z)) = 0$, locally on M,

$$d\eta/dt = q(0, \eta) = q_0(\eta) \tag{2.4}$$

is addressed as the *zero dynamics.*

The qualitative behaviour of the zero dynamics is entirely governed by $q_0(\eta)$. The system is said to be *minimum phase* if the dimension of the *stable manifold* (Guckenheimer and Holmes 1983), around an equilibrium point in $h^{-1}(0)$, is $n-r$. The system is *globally minimum phase* if it is minimum phase and (2.4) is globally asymptotically stable (Byrnes and Isidori 1984). From now on, we assume that $y = h(x)$ has been chosen to render the system globally minimum phase, i.e. the internal behaviour of the system, while being forced to exhibit locally zero output value, is asymptotically stable to an equilibrium point located on the manifold $h^{-1}(0)$.

Lemma 2.4

Let $u = u(x, v)$ be a *regular* feedback law (i.e. $\partial u/\partial v \neq 0$). The relative degree of a system is a feedback invariant under regular feedback.

Proof

For a system with regular feedback, $u = u(x, v)$, the time-derivatives of y are locally independent of u if and only if they are locally independent of v. □

Remark

Lemma 2.4 implies, in particular, that for a non-linear system, with local relative degree r around x^0, given by $dx/dt = X(x, u)$; $y = h(x)$, and feedback control law, $u = \alpha(x, v)$, which in closed-loop form is written as $dx/dt = X(x, \alpha(x, v)) = X^{\alpha}(x, v)$, the equality: $L_{X^\alpha}^k h = L_X^k h$, holds valid for $k = 0, 1, 2, \ldots, r-1$ and all x in a neighbourhood of x^0.

2.2. Generalities about local sliding regimes

A local variable-structure feedback control law for (X, h) is obtained by letting the control function u take one of two possible feedback function values in the set $U = \{u^+(x), u^-(x)\}$, with $u^+(x) > u^-(x)$ locally defined on a neighbourhood N of x^0, according to the sign of the scalar output function $h(x)$, i.e.

$$u = \begin{cases} u^+(x) & \text{for } h(x) > 0 \\ u^-(x) & \text{for } h(x) < 0 \end{cases} \tag{2.5}$$

The feedback structures $u^+(x)$ and $u^-(x)$ are usually fixed beforehand, but they may also be part of the design problem.

Definition 2.5 (Utkin 1978, Sira-Ramirez 1987, 1988 a)

A *sliding regime* is said to exist locally on an open set M of $h^{-1}(0)$, if, as a result of the control policy (2.5), the state trajectories of (2.1) satisfy

$$\left.\begin{aligned} \lim_{h\to 0^+} L_{X(x,u^+(x))}h &= \lim_{h\to 0^+} \langle dh, X(x, u^+(x))\rangle < 0 \\ \lim_{h\to 0^-} L_{X(x,u^-(x))}h &= \lim_{h\to 0^-} \langle dh, X(x, u^-(x))\rangle > 0 \end{aligned}\right\} \tag{2.6}$$

Theorem 2.6

A sliding regime locally exists on an open set M of $h^{-1}(0)$, if and only if the system (X, h), has local relative degree equal to 1 (i.e. (X, h) has one zero at infinity on a point $x^0 \in M$).

Proof

If $L_{X(x,u)}h$ does not depend locally on u (i.e. $L_{\partial X/\partial u}h(x) = \partial\{L_X h\}/\partial u = 0$ for all x in N) then, changing the control u from $u^+(x)$ to $u^-(x)$, in the vicinity N of x^0, does not have any effect on the local sign of $L_{X(x,u)}h$. Therefore, a sliding regime can not locally exist on M.

To prove sufficiency, suppose $L_{\partial X/\partial u}h(x) = \partial\{L_X h\}/\partial u \neq 0$ locally around a neighbourhood N of x^0. Let $\varepsilon^-(x)$ be a smooth, locally strictly positive function of x. Then, by virtue of the implicit function theorem, the equation $[L_X h](x, u) = \varepsilon^-(x)$ locally has a unique smooth solution $u = u^{-\varepsilon}(x)$ such that $L_{X(x,u^{-\varepsilon}(x))}h(x) = \varepsilon^-(x) > 0$. Similarly, by the same arguments, given a smooth locally strictly negative function $\varepsilon^+(x)$, a smooth control law $u = u^{+\varepsilon}(x)$ locally exists around x^0 such that $L_{X(x,u^{+\varepsilon}(x))}h(x) = \varepsilon^+(x) < 0$. Hence, a sliding regime locally exists on an open set M of $h^{-1}(0)$ for the found variable-structure feedback control law:

$$u = \begin{cases} u^+(x) = u^{-\varepsilon}(x) & \text{for } h(x) < 0 \\ u^-(x) = u^{+\varepsilon}(x) & \text{for } h(x) > 0 \end{cases} \qquad \square$$

Condition $L_{\partial X/\partial u}h \neq 0$, is a generalized local *transversality condition* (Sira-Ramirez 1988 a).

Example

For systems of the form $X(x, u) = f(x) + g(x)u$; $y = h(x)$, the local transversality condition on an open set M in $h^{-1}(0)$ takes the form $L_g h < 0$. To see this, simply subtract the sliding regime conditions (2.6) on any point x of M: $L_f h + u^+ L_g h < 0$ and $L_f h + u^- L_g h > 0$, to obtain $[u^+ - u^-]L_g h < 0$.

For all initial states located on a vicinity M of x^0 in $h^{-1}(0)$, the unique control function, $u^{EQ}(x)$, locally constraining the system trajectories to the zero level set of $h(x)$ in the region of existence of a sliding regime is known as the *equivalent control* (i.e. the equivalent control locally turns the open set M in $h^{-1}(0)$ into an integral manifold for the controlled system trajectories starting on M). The resulting dynamics, ideally constrained to M, is the *ideal sliding dynamics* (Utkin 1978). A coordinate-free description of such dynamics is

$$dx/dt = X(x, u^{EQ}(x)) \tag{2.7}$$

It follows from the definition of local integral manifold that on an open set M of $h^{-1}(0)$ the controlled vector field $X(x, u^{EQ}(x))$ satisfies

$$L_{X(x,u^{EQ}(x))}h(x) = \langle dh, X(x, u^{EQ}(x))\rangle = 0 \tag{2.8}$$

Equation (2.8) actually constitutes a definition of the equivalent control law. To see that such ideal feedback control law is well defined, we must prove that (2.8) locally defines $u^{EQ}(x)$ in a unique way. Moreover, it will be established that the equivalent control thus defined is necessarily locally intermediate among the extreme feedback control laws $u^+(x)$ and $u^-(x)$.

Theorem 2.7

A necessary condition for the local existence of a sliding regime on an open set M of $h^{-1}(0)$ is that there locally exists a unique smooth equivalent control, satisfying

$$u^-(x) < u^{EQ}(x) < u^+(x) \tag{2.9}$$

for the given smooth feedback functions $u^-(x)$ and $u^+(x)$.

Proof

First we prove uniqueness. If a sliding motion locally exists on M, then (X, h) has local relative degree 1, i.e. $L_{\partial X/\partial u}h(x) = \partial L_X(h)/\partial u \neq 0$ for all x in a vicinity N of x^0. By the implicit function theorem the equation $[L_X h](x, u) = 0$ has, locally, a unique solution $u^{EQ}(x)$ in M for which (2.8) holds true. In other words, if a sliding regime exists, the equivalent control locally exists and is uniquely defined.

Suppose now that a sliding regime locally exists on M for the switching feedback control law (2.5). Then, locally on M, the following three relations hold valid:

$$L_{X(x,u^+(x))}h = \langle dh, X(x, u^+(x))\rangle < 0 \tag{2.10}$$

$$L_{X(x,u^{EQ}(x))}h = \langle dh, X(x, u^{EQ}(x))\rangle = 0 \tag{2.11}$$

$$L_{X(x,u^-(x))}h = \langle dh, X(x, u^-(x))\rangle > 0 \tag{2.12}$$

Subtracting (2.11) from (2.10) and (2.12) from (2.11) one obtains

$$\left.\begin{aligned} \langle dh, X(x, u^+(x)) - X(x, u^{EQ}(x))\rangle < 0 \\ \langle dh, X(x, u^{EQ}(x)) - X(x, u^-(x))\rangle < 0 \end{aligned}\right\} \tag{2.13}$$

From the mean value theorem (Boothby 1975), there exists smooth functions $u_0^+(x)$

and $u_0^-(x)$, such that locally on M

$$\left.\begin{aligned}\langle dh, X(x, u^+(x)) - X(x, u^{\mathrm{EQ}}(x))\rangle &= [u^+(x) - u^{\mathrm{EQ}}(x)] \\ &\quad \times \langle dh, \partial X(x, u_0^+(x))/\partial u\rangle < 0 \\ \langle dh, X(x, u^{\mathrm{EQ}}(x)) - X(x, u^-(x))\rangle &= [u^{\mathrm{EQ}}(x) - u^-(x)] \\ &\quad \times \langle dh, \partial X(x, u_0^-(x))/\partial u\rangle < 0\end{aligned}\right\} \quad (2.14)$$

where $u_0^+(x)$ and $u_0^-(x)$, respectively, satisfy $u^{\mathrm{EQ}}(x) < u_0^+(x) < u^+(x)$ and $u^-(x) < u_0^-(x) < u^{\mathrm{EQ}}(x)$, i.e. locally on M, $u^-(x) < u^{\mathrm{EQ}}(x) < u^+(x)$. □

Example

For the linear control case, if a local sliding motion exists on an open set M of $h^{-1}(0)$ then necessarily $L_g h < 0$. Since $L_{X(x,u^{\mathrm{EQ}})} h = L_f h + u^{\mathrm{EQ}} L_g h = 0$ then $u_{\mathrm{EQ}} = -L_f h / L_g h$, i.e. the equivalent control locally exists and is uniquely defined. Also, from the existence conditions (2.5), we locally have on M, $L_f h + u^+ L_g h < 0$ and $L_f h + u^- L_g h > 0$. This implies that there exist smooth functions $a(x) > 0$ and $b(x) > 0$, such that $a(x)[L_f h + u^+ L_g h] + b(x)[L_f h + u^- L_g h] = [a(x) + b(x)] L_f h + [a(x)u^+(x) + b(x)u^-(x)]g(x) = 0$, i.e. $L_f h + L_g h[a(x)u^+(x) + b(x)u^-(x)]/[a(x) + b(x)] = 0$, i.e. $u^-(x) < [a(x)u^+(x) + b(x)u^-(x)]/[a(x) + b(x)] = u^{\mathrm{EQ}}(x) < u^+(x)$ locally on M.

Remark

It should be stressed that the local existence of an equivalent control is not sufficient to guarantee the local existence of a sliding motion on $h^{-1}(0)$. Additional conditions must be satisfied by such an equivalent control in order to be able to conclude local existence of a sliding regime. The next theorem establishes the nature of such conditions.

Definition 2.8

Let (X, h) have local relative degree 1 on $x^0 \in h^{-1}(0)$. The system (X, h) is said to exhibit a local *control foliation property* about the manifold $h^{-1}(0)$, if and only if given any smooth feedback functions $u_1(x) < u_2(x) < u_3(x)$, defined on M, $[L_X h](x, u_1) > [L_X h](x, u_2) > [L_X h](x, u_3)$, locally on M. (See also Sira-Ramirez 1988 b where the crucial importance of this assumption in the sliding regimes of singularly perturbed systems is established.)

Theorem 2.9

Control law (2.5) locally induces a sliding regime, for system (X, h), on an open set M of $h^{-1}(0)$, if and only if (X, h) exhibits a control foliation property about $h^{-1}(0)$ and there exists a feedback control, $u^{\mathrm{EQ}}(x)$, locally satisfying (2.8) and (2.9) on M.

Proof

Suppose that, thanks to a control action of the form (2.5), the system locally exhibits a sliding regime on an open set M of $h^{-1}(0)$. Then according to Theorems 2.7 and 2.8 there necessarily exists a unique smooth $u^{\mathrm{EQ}}(x)$ satisfying $u^-(x) < u^{\mathrm{EQ}}(x) <$

$u^+(x)$ locally on M. Since a sliding motion exists on M, it follows that $[L_X h](x, u^-(x)) > 0$, $[L_X h](x, u^{EQ}(x)) = 0$ and $[L_X h](x, u^+(x)) < 0$, locally on M, i.e. $[L_X h](x, u^-(x)) > [L_X h](x, u^{EQ}(x)) > [L_X h](x, u^+(x))$ on M. Hence, (X, h) exhibits a control foliation property.

If on the other hand the system (X, h) exhibits a control foliation proprty and $u^{EQ}(x)$ exists such that locally on an open set M in $h^{-1}(0)$, $[L_X h](x, u^{EQ}(x)) = 0$ and $u^+(x) > u^{EQ}(x) > u^-(x)$. Then, it follows that $[L_X h](x, u^-(x)) > [L_X h](x, u^{EQ}(x)) = 0$ $[L_X h](x, u^-(x))$. Hence, necessarily, $[L_X h](x, u^+(x)) < 0$ and $[L_X h](x, u^-(x)) > 0$ holds true on M. It follows that there exists an open neighbourhood N on $x^0 \in M$, with non-empty intersection with $h^{-1}(0)$, where conditions (2.6) are satisfied. Thus, a sliding regime locally exists on $h^{-1}(0)$. □

Example

Notice that for $X(x, u) = f(x) + ug(x)$, and if the transversality condition $L_g h < 0$ holds locally true on an open set M of $h^{-1}(0)$, then the control foliation property is automatically satisfied. Since $L_X h = L_f h + uL_g h$, then for $u^+ > u^{EQ} > u^-$, $L_f h + u^+ L_g h < L_f h + u^{EQ} L_g h = 0 < L_f h + u^+ L_g h$. It follows that, for the class of affine systems, the condition $u^- < u^{EQ} < u^-$ is both a necessary and sufficient condition for the existence of a sliding regime (Sira-Ramirez 1988 a, 1989).

Example

Consider the non-linear system

$$dx_1/dt = \cos(ux_2) - x_1^2 =: X_1(x, u)$$

$$dx_2/dt = \sin(ux_1) =: X_2(x, u)$$

$$y = x_2 =: h(x)$$

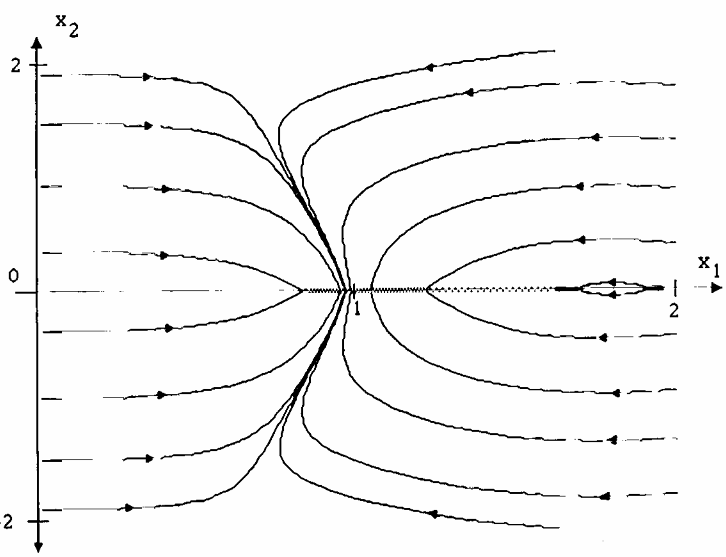

Figure 2. Controlled trajectories with local sliding motion on $x_2 = 0$.

In this case, the vector field $\partial X/\partial u = -x_2 \sin(ux_2)\partial/\partial x_1 + x_1 \cos(ux_1)\partial/\partial x_2$. Since $L_{\partial X/\partial u} h = x_1 \cos(ux_1)$, the local relative degree of the system, with respect to the output $y = x_2$, is equal to 1 everywhere except on the line $x_1 = 0$. Thus, a local sliding motion exists on the manifold $x_2 = 0$ by use of an appropriate variable-structure control law. Indeed, from $L_X h = \sin(ux_1)$, it is seen that the feedback control law $u = -x_1 \operatorname{sign} x_2$ locally creates a sliding regime on $x_2 = 0$, $0 < x_1 < \sqrt{\pi}$. The ideal sliding dynamics is obtained for the control law satisfying $L_X h = \sin(u^{EQ}(x)x_1) = 0$ on $x_2 = 0$, i.e. $u^{EQ}(x) = 0$. It follows that $dx_1/dt = 1 - x_1^2$, locally describes the ideal sliding motion. Figure 2 depicts the controlled phase trajectories.

2.3. *Sliding regimes in variable-structure systems with relative degree higher than one*

If for the proposed output function $y = h(x)$ the system locally exhibits relative degree r higher than 1 on x^0, then a sliding regime does not locally exist on $h^{-1}(0)$. However, using the ideas of Isidori (1987) for *local feedback stabilization*, then an alternative means to create a local sliding motion which eventually reaches $h^{-1}(0)$ is obtained. The idea is to use the auxiliary output function

$$w = k(x) = L_X^{r-1} h(x) + c_{r-2} L_X^{r-2} h(x) + \dots + c_1 L_X h(x) + c_0 h(x) \tag{2.15}$$

or in normal form coordinates

$$w = z_r + c_{r-2} z_{r-1} + \dots + c_1 z_2 + c_0 z_1 \tag{2.16}$$

Evidently, $L_{\partial X/\partial u} k(x^0) = L_{\partial X/\partial u} L_X^{r-1} h(x^0) \neq 0$, i.e. the system (X, k) has local relative degree 1, and a sliding motion can now be locally created on an open set of $k^{-1}(0)$. Then, ideally, $w = 0$ and $z_r = -c_{r-2} z_{r-1} - \dots - c_1 z_2 - c_0 z_1$. The ideal sliding system, associated with the new sliding surface $k^{-1}(0)$, is expressed as

$$\left.\begin{aligned} &dz_i/dt = z_{i+1}, \quad i = 1, 2, \dots, r-2 \\ &dz_{r-1}/dt = z_r = -c_{r-2} z_{r-1} - \dots - c_1 z_2 - c_0 z_1 \\ &dz_{r+j}/dt = q(z_1, z_2, \dots, z_{r-1}, -(c_{r-2} z_{r-1} + \dots \\ &\qquad + c_1 z_2 + c_0 z_1), z_{r+1}, \dots, z_n) \\ &\quad j = 1, \dots, n-r \\ &\quad y = z_1 \\ &\quad w = 0 \end{aligned}\right\} \tag{2.17}$$

It is easy to see that by suitable choice of the parameters $c_0, \dots, c_{r-2}$, an asymptotically stable motion can be obtained for the first $r - 1$ coordinates, z_1 to z_{r-1} (and hence, for z_r too). Thus, while a sliding motion locally takes place on $k^{-1}(0)$, the original output y and its first $r - 1$ derivatives asymptotically approach zero (i.e. the state vector of the original system approaches the manifold $h^{-1}(0)$, as originally desired).

The corresponding equivalent control is now locally given, in original coordinates, as the unique solution of

$$[L_X k](x, u^{EQ}(x)) = [L_X^r h + c_{r-2} L_X^{r-1} h + \dots + c_1 L_X^2 h + c_0 L_X h](x, u^{EQ}(x)) = 0 \tag{2.18}$$

Notice that when $h^{-1}(0)$ is reached by the sliding controlled trajectory, the equivalent control locally becomes the unique solution of

$$[L_X k](x, u^{EQ}(x)) = [L_X^r h](x, u^{EQ}(x)) = 0 \quad (2.19)$$

Example

For systems with affine vector fields, $X = f + ug$ and output $y = h(x)$, which exhibit relative degree r on x^0, the function $w = k(x)$ is given by $k(x) = L_f^{r-1} h(x) + c_{r-2} L_f^{r-2} h(x) + \dots + c_1 L_f h(x) + c_0 h(x)$ and the equivalent control (2.18) is computed as

$$u^{EQ}(x) = -L_f k / L_g k = [L_f^r h + c_{r-2} L_f^{r-1} h + \dots + c_1 L_f^2 h + c_0 L_f h] / L_g L_f^{r-1} h$$

Locally, on an open set M of $h^{-1}(0)$, this expression takes the form $u^{EQ}(x) = L_f^r h / L_g L_f^{r-1} h$.

The use of the auxiliary output $w = k(x)$ implies the possibility of either being able to completely measure the original state variables, and proceed to use (2.15), or else being able to generate $r-1$ derivatives of the original output function y. The last possibility is usually accomplished by means of a high-gain phase-lead 'post-processor' (this idea is taken from Isidori (1987), Chap. 2, § 5) fed by the output signal $y(t)$. The transfer function of such a post-processor is given by

$$\frac{-Kn(s)}{(1+Ts)^r} \quad (2.20)$$

with T being a sufficiently small positive constant, K a sufficiently large gain with, locally, the same sign as $L_{\partial X/\partial u} L_X^{r-1} h(x)$. $n(s)$ is a stable polynomial built as $n(s) = s^{r-1} + c_{r-2} s^{r-2} + \dots + c_1 s + c_0$ (see Fig. 3).

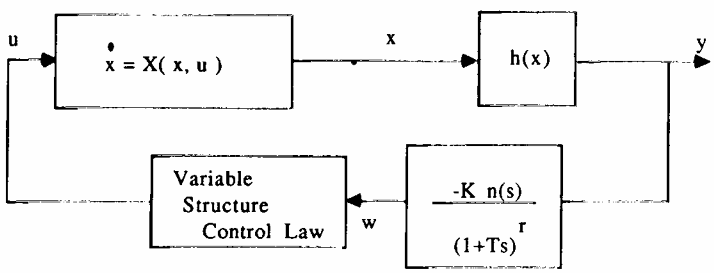

Figure 3. Sliding regime creation in systems of relative degree > 1.

Example

A simplified model of a spacecraft attempting a soft lunar landing is given by (Cantoni and Finzi 1980)

$$dx_1/dt = x_2; \quad dx_2/dt = g - (\sigma/x_3)u; \quad dx_3/dt = -u$$

where x_1 is the position coordinate, oriented downwards with origin on the ground, x_2 is the downward velocity and x_3 represents the combined mass of the spacecraft and the residual fuel. σ is a constant of the relative ejection velocity. The control parameter u represents the rate of ejection per unit time and it is assumed to take values on the discrete set $\{0, \alpha\}$ with α a given constant such that $\sigma\alpha$ is the maximum thrust of the braking engine. Consider the output to be $y = h(x) = x_1$.

In this example, $X = x_2 \partial/\partial x_1 + [g - (\sigma/x_3)u]\partial/\partial x_2 - u\partial/\partial x_3$ and $L_X h = x_2$ while $\partial X/\partial u = -(\sigma/x_3)\partial/\partial x_2 - \partial/\partial x_3$. Hence, $L_{\partial X/\partial u} h = 0$. $L_X^2 h = g - (\sigma/x_3)u$ and $L_{\partial X/\partial u} L_X h = -(\sigma/x_3) \neq 0$, i.e. the system has relative degree 2. Consider then the auxiliary output $w = k(x) = x_2 + cx_1$, with $c > 0$. Then $L_X k = cx_2 + g - (\sigma/x_3)u$, and $L_{\partial X/\partial u} k = -(\sigma/x_3)$. A local sliding regime can now be created on $k^{-1}(0) := \{x \in R^3 : x_2 = -cx_1\}$. The equivalent control is found to be $u^{EQ} = (x_3/\sigma)[cx_2 + g]$. A local sliding motion exists on $k^{-1}(0)$ provided $0 < u^{EQ} = (x_3/\sigma)[cx_2 + g] = (x_3/\sigma)[-c^2 x_1 + g] < \alpha$. The first inequality is obviously satisfied since $x_1 < 0$ before landing, and the second inequality states that the net average descending force $x_3[g - c^2 x_1]$ along the sliding line is to be bounded by the maximum thrust $\sigma\alpha$. Notice that since $x_2 = 0$ when $x_1 = 0$ on the sliding surface, the position coordinate of the ideal sliding dynamics, regulated by the asymptotically stable dynamics, $dx_1/dt = -cx_1$, guarantees a soft lunar landing. On the sliding line the mass evolution is governed by $dx_3/dt = -(x_3/\sigma)[g - c^2 x_1]$.

3. An 'outer loop' sliding mode control approach to some non-linear control problems

In this section the relevance of sliding regimes is examined in several non-linear control problems. The basic ideas are taken from Isidori (1987). The substantial part of his development is closely followed, if only in the context of the more general case represented by (2.1).

3.1. *Robust stabilization in feedback linearizable systems*

Suppose the local relative degree of the system (X, h) is n at x^0, with n being the dimension of the system state. It follows that, in normal form coordinates, the system may be locally expressed around z^0 as

$$\left.\begin{aligned} dz_i/dt &= z_{i+1}; \quad i = 1, 2, \ldots, n-1 \\ dz_n/dt &= [L_X^n h](\Phi^{-1}(z), u) \\ y &= z_1 \end{aligned}\right\} \tag{3.1}$$

where by definition of relative degree $\partial\{[L_X^n h](\Phi^{-1}(z^0), u)\}/\partial u = L_{\partial X/\partial u} L_X^{n-1} h \neq 0$. It follows, from the implicit function theorem, that given any external independent scalar input function v, the equation $[L_X^n h](\Phi^{-1}(z), u) = v$ locally has a unique smooth solution $u = \alpha(z, v)$ around z^0. Hence, using such a control law on (3.1), one obtains the linear controllable system:

$$\left.\begin{aligned} dz_i &= z_{i+1}; \quad i = 1, 2, \ldots, n-1 \\ dz_n/dt &= v \\ y &= z_1 \end{aligned}\right\} \tag{3.2}$$

If one defines a new auxiliary linear output in new coordinates as

$$w = k(z) = z_n + c_{n-2} z_{n-1} + \ldots + c_1 z_2 + c_0 z_1 \tag{3.3}$$

or in original coordinates as

$$w = k(x) = L_X^{n-1}h(x) + c_{n-2}L_X^{n-2}h(x) + \dots + c_1 L_X h(x) + c_0 h(x) \tag{3.4}$$

then, the relative degree of (3.1) with respect to w is 1, as it is easily checked. Hence, a sliding motion can be created on an open set of $k^{-1}(0)$ by a suitable choice of smooth (possibly linear) 'outer loop' feedback structures $v^+(z)$ and $v^-(z)$ such that on an open set of $k^{-1}(0)$, $v^-(z) < v^{EQ}(z) < v^+(z)$, with

$$\begin{aligned} v^{EQ}(z) &= (c_{n-2}^2 - c_{n-3})z_{n-1} + (c_{n-2}c_{n-3} - c_{n-4})z_{n-2} \\ &\quad + \dots + (c_{n-2}c_1 - c_0)z_2 + c_{n-2}c_0 z_1 \end{aligned} \tag{3.5}$$

If such a sliding regime is created on $k^{-1}(0)$, w is ideally set to zero and hence $z_n = -c_{n-2}z_{n-1} - \dots - c_1 z_2 - c_0 z_1$. The ideal sliding dynamics is governed by

$$\left.\begin{aligned} dz_i/dt &= z_{i+1}; \quad i = 1, 2, \dots, n-2 \\ dz_{n-1}/dt &= -c_{n-2}z_{n-1} - \dots - c_1 z_2 - c_0 z_1 \\ y &= z_1 \\ w &= 0 \end{aligned}\right\} \tag{3.6}$$

Evidently, system (3.6) can be made locally asymptotically stable by suitable choice of the design parameters c_i. The result is the possibility of locally reaching the original surface $h^{-1}(0)$ while a sliding motion locally takes place on an open set of $k^{-1}(0)$. The variable-structure control policy provides some degree of robustness to the exactly feedback-linearized dynamics (see Spong and Sira-Ramirez 1986 and Sira-Ramirez 1988 a).

3.2. *Disturbance rejection properties of systems undergoing sliding regimes*

Consider a smooth non-linear perturbed system of the form

$$\left.\begin{aligned} dx/dt &= X(x, u, w) \\ y &= h(x) \end{aligned}\right\} \tag{3.7}$$

where w is a scalar perturbation signal affecting the system behaviour. Let us assume that the system locally has relative degree 1 on x^0, and that $\partial(L_X h)/\partial w = L_{\partial X/\partial w}h(x) = 0$ (i.e. the input w is assumed to have local relative degree higher than 1 with respect to the output function h). In normal form coordinates, the perturbed system is written as

$$\left.\begin{aligned} dz_1/dt &= [L_X h](\Phi^{-1}(z), u, w) = [L_X h](\Phi^{-1}(z), u) \\ d\eta/dt &= q(z_1, \eta, w) \\ y &= z_1 \end{aligned}\right\} \tag{3.8}$$

If a sliding motion can be locally created on $z_1 = 0$, the equivalent control $u^{EQ}(z)$ obtained by zeroing the first equation of (3.8) is clearly unaffected by the perturbation signal w. Only the ideal sliding dynamics is influenced by the perturbation input w. The next lemma follows immediately.

Lemma 3.1

Let (X, h) have local relative degree 1. The existence of a local sliding motion on $h^{-1}(0)$ is independent of the perturbation signal w if and only if w has local relative degree at least equal to 2 with respect to the output function $h(x)$.

However, notice that w, in general, does affect the evolution of the ideal sliding dynamics (zero dynamics), unless the normal form coordinates $z_2 = \phi_2(x), \ldots, z_n = \phi_n(x)$, are chosen in such a way that $L_{\partial X/\partial w}\phi_i = 0$ for $i = 2, \ldots, n$. But, owing to the fact that the normal form description demanded that the ϕ_is were chosen to also satisfy $L_{\partial X/\partial u}\phi_i = 0$, for $i = 2, \ldots, n$, it follows that $\partial X/\partial w$ is at least locally exactly in the range of $\partial X/\partial u$. On the other hand, $\partial X/\partial w$ is locally in the range of $\partial X/\partial u$ if and only if there exists a smooth function $b(x)$ such that locally on M, $\partial X/\partial w = b(x)[\partial X/\partial u]$. Hence, for $i = 2, 3, \ldots, n$ we have $L_{\partial X/\partial w}\phi_i = L_{[b(x)\partial X/\partial u]}\phi_i = b(x)L_{\partial X/\partial u}\phi_i = 0$. It follows that w does not affect the zero dynamics. We have thus proved the following general theorem.

Theorem 3.2

System (3.7) is totally unaffected by perturbation signals w, of any kind, if and only if the *matching condition*

$$\partial X/\partial w \in \text{range}\ \{\partial X/\partial u\} \tag{3.9}$$

is satisfied.

Remark

For the case of affine vector fields of the form $X(x, u, w) = f(x) + g(x)u + p(x)w$. Condition (3.9) is equivalent to $p(x) \in \text{range}\ g(x)$ which is a well known 'invariance condition' (see Drazenovic 1969 for the linear time-invariant case and also Sira-Ramirez 1988 c for the non-linear case).

Notice however that even in the case of a perturbation signal with relative degree equal to 1, it is still possible to create a local sliding motion on the zero level set of the output function. For this, bounds are to be known for the perturbation signal. In general, the extreme values of the variable-structure control law u^+, u^- will depend on the bounds of the perturbation signal. Usually, however, the feedback functions $u^+(x)$, $u^-(x)$ are fixed at the outset. In this case the following theorem applies.

Theorem 3.3

Let the system $dx/dt = X(x, u, w)$ and $y = h(x)$ have local relative degree 1 both in u and w, and let the system exhibit a local control foliation property on $h^{-1}(0)$. Suppose the scalar perturbation w is known to be restricted to the bounded interval $W = [w_{\min}, w_{\max}]$ of the real line. A sliding regime locally exists on $h^{-1}(0)$ if and only if for all $w \in W$

$$u^-(x) < u^{EQ}(x, w) < u^+(x)$$

Example

Consider the dynamic model of an ideal separately excited direct-current motor

(Rugh (1981), pp. 98–99):

$$dx_1/dt = -(R_a/L_a)x_1 - (K/L_a)x_2u + (V_a/L_a) = X_1(x, u)$$
$$dx_2/dt = -(B/J)x_2 + (K/J)x_1u + (1/J)T^L = X_2(x, u)$$
$$y = x_2 = h(x)$$

with x_1 being the armature current, x_2 the angular velocity of the motor shaft moving against a viscous torque characterized by a damping coefficient B, and J is the moment of inertia of the mechanical load. The control u is the controlled current in the field circuit. V_a is the constant armature voltage. R_a and L_a represent armature circuit resistance and inductance while K is the torque constant of the motor. T^L is a load perturbation torque.

Here, $\partial X/\partial u = -(K/L_a)x_2\partial/\partial x_1 + (K/J)x_1\partial/\partial x_2$ and $L_{\partial X/\partial u}h = (K/J)x_1$ and the system has local relative degree equal to 1 everywhere except on $x_1 = 0$. However, the perturbation torque, which also acts as an input, exhibits relative degree also equal to 1, since $\partial X/\partial T^L = (1/J)\partial/\partial x_2$ and $L_{\partial X/\partial T^L}h = 1/J \neq 0$. Moreover, since $(1/J)\partial/\partial x_2 \notin$ range $\partial X/\partial u$, the load perturbation torque T^L cannot be decoupled from the angular velocity output. A sliding regime does exist on $x_2 = 0$, but its creation has to take into account the magnitude bounds of the perturbation torque. The control foliation property is trivially satisfied in this example and thus a sliding regime can be created whenever a variable-structure feedback field current law with extreme values $u^+(x)$, $u^-(x)$ can be prescribed such that

$$u^+(x) < \min_{w \in W} (Bx_2 - w)/Kx_1 < \max_{w \in W} (Bx_2 - w)/Kx_1 < u^+(x)$$

In spite of this possibility, the ideal sliding dynamics cannot be made totally independent of the perturbation load torque.

3.3. *Robust disturbance decoupling in the absence of the matching condition*

Suppose that the input w in (3.7) has local relative degree larger than r with respect to the output function $h(x)$. Let the vector ξ denote the first r normal coordinates

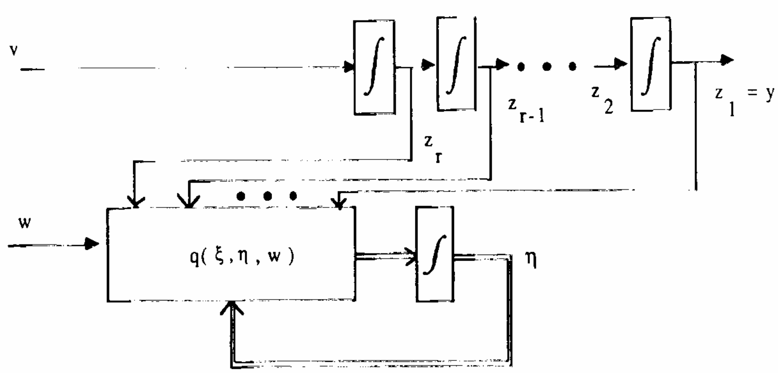

Figure 4. Confinement of perturbations to zero dynamics block.

$z_1, z_2, \ldots, z_r$. In such coordinates, the system (3.7) is expressed locally around z^0 as

$$\left.\begin{aligned} dz_i/dt &= z_{i+1}, \quad i = 1, 2, \ldots, r-1 \\ dz_r/dt &= [L_X^r h](\Phi^{-1}(z), u) \\ d\eta/dt &= q(\xi, \eta, w); \quad y = z_1 \end{aligned}\right\} \tag{3.10}$$

Given a smooth feedback control law $u = \alpha(z, v)$ such that, locally on M, $[L_X^r h](\Phi^{-1}(z), \alpha(z, v)) = v$, with v being an external independent scalar input, then the output y is totally decoupled from the perturbation input w (see Fig. 4). Notice, once more, that such a scalar control law exists by virtue of the implicit function theorem and the general definition of relative degree. This proves the 'if' part of the following theorem, which generalizes the linear version in Isidori (1987, Chap. 2, § 6).

Theorem 3.4

Let (X, h) have relative degree r on x^0. There exists a smooth feedback control law of the form $u = \alpha(x, v)$, which locally decouples the output $y = h(x)$ from the disturbance input w if and only if the input w has relative degree strictly greater than r on x^0, i.e. for all x in a neighbourhood N of x^0:

$$L_{\partial X/\partial w} L_X^{i-1} h(x) = 0; \quad i = 1, 2, \ldots, r \tag{3.11}$$

Proof

To prove necessity, suppose $u = \alpha(x, v)$ is any feedback control law locally decoupling the output h from the perturbation input w. The closed-loop system is expressed as

$$\left.\begin{aligned} dx/dt &= X(x, \alpha(x, v), w) =: X^\alpha(x, v, w) \\ y &= h(x) \end{aligned}\right\} \tag{3.12}$$

If in the system (3.12) w is locally decoupled from the output then necessarily the normal form coordinates, $z_i = L_{X^\alpha}^{i-1} h$, $i = 1, \ldots, r$, are all locally independent of w. From (3.12) it follows that the quantities $L_X^{i-1} h$, $i = 1, \ldots, r$, are also independent of w. Hence, locally around x^0, $\partial(L_X^{i-1} h)\partial w = L_{\partial X/\partial w} L^{i-2} h = 0$ for $i = 1, 2, \ldots, r$. Since $dz_r/dt = [L_{X^\alpha}^r h](x, v, w)$ must also be locally independent of w, then one has $\partial[L_{X^\alpha}^r h](x, v, w)/\partial w = 0$. Hence $\partial L_X^r h/\partial w = 0$ and therefore $L_{\partial X/\partial w} L_X^{r-1} h = 0$ locally around x^0. □

Since $L_{\partial X/\partial w} L_X^{i-1} h = [\partial(L_X^{i-1} h)/\partial x]\partial X/\partial u = 0$, $i = 1, 2, \ldots, r$, condition (3.11) is equivalent to the condition of having $\partial X/\partial u$ locally contained in the null space of the matrix $\Omega(x)$ given by

$$\Omega(x) = \begin{bmatrix} \partial h/\partial x \\ \partial(L_X h)/\partial x \\ \vdots \\ \partial(L_X^{r-1} h)/\partial x \end{bmatrix} \tag{3.13}$$

which constitutes a generalization of the condition found in Isidori (1987) for systems linear in the control.

The sliding mode disturbance decoupling problem can be formulated as follows. Consider the perturbed system (3.7) with local relative degree r. It is desired to find a variable-structure control law, inducing a local sliding regime on an open set of the zero level set $k^{-1}(0)$ of an auxiliary output function $\psi = k(x)$, such that the original output y is locally stabilized to zero while being unaffected by the perturbation signal w.

The variable-structure control law will constitute an 'outer loop' feedback control, inducing locally desirable robustness properties into an 'inner loop' feedback control law. Such a control law of the form $u = \alpha(x, v)$ is assumed to be devised for 'exact' disturbance decoupling. The sliding mode approach is especially useful in obtaining a robust design with respect to small modelling errors and other external perturbation signals. This discontinuous control scheme can be accomplished by proposing an auxiliary output function of the form

$$\psi = k(x) = L_{X'}^{r-1} h(x) + c_{n-2} L_{X'}^{r-2} h(x) + \dots + c_1 L_{X'} h(x) + c_0 h(x) \tag{3.14}$$

which in normal form coordinates is a linear output function given by

$$\psi = k(z) = z_r + c_{r-2} z_{r-1} + \dots + c_1 z_2 + c_0 z_1 \tag{3.15}$$

Devising a variable-structure control law that locally creates a sliding regime on an open set of $k^{-1}(0)$, the 'outer loop' closed-loop system in normal form coordinates would be expressed locally around z^0 as

$$\left.\begin{aligned} dz_i/dt &= z_{i+1}; \quad i = 1, 2, \dots, r-2 \\ dz_{r-1}/dt &= z_r = w - c_{r-2} z_{r-1} - \dots - c_1 z_2 - c_0 z_1 \\ dz_r/dt &= v = 0{\cdot}5[1 + \operatorname{sign} k(z)] v^+(z) + 0.5[1 - \operatorname{sign} k(z)] v^-(z) \\ d\eta/dt &= q(\xi, \eta, w) \\ y &= z_1; \quad \psi = k(z) \end{aligned}\right\} \tag{3.16}$$

The corresponding ideal sliding dynamics is thus

$$\left.\begin{aligned} dz_i/dt &= z_{i+1}, \quad i = 1, 2, \dots, r-2 \\ dz_{r-1}/dt &= -c_{r-2} z_{r-1} - \dots - c_1 z_2 - c_0 z_1 \\ d\eta/dt &= q(\xi, \eta, w); \quad y = z_1; \quad \psi = 0 \end{aligned}\right\} \tag{3.17}$$

If the coefficients c_i in (3.15) are appropriately chosen an asymptotically stable motion is obtained towards $z_1 = 0$ which is totally independent of the perturbation input w.

The following trivial lemma will be useful.

Lemma 3.5

Let $m = m(x, u, w)$ with $u = \alpha(x, v, w)$ such that locally $\partial\alpha/\partial w \neq 0$. If m is locally independent of w, then m is also locally independent of u.

If one is allowed to conduct measurements on the disturbance signal w (this will be the case in the model-matching problem), one can relax somewhat the hypothesis imposed on the formulation of the disturbance-decoupling problem.

Indeed, suppose that both the perturbation input w and the control input u have local relative degree r (notice that a smaller local relative degree of w renders the problem unsolvable) and consider the control law $u = \alpha(x, v, w)$. The closed-loop system becomes

$$\left.\begin{aligned} dx/dt &= X(x, u, w) = X(x, \alpha(x, v, w), w) = X^\alpha(x, v, w) \\ y &= h(x) \end{aligned}\right\} \tag{3.18}$$

Let $\Omega^\alpha(x)$ denote the matrix in (3.13) with X substituted by X^α. Using the result of Theorem 3.4, a feedback control law exists that locally solves the disturbance-decoupling problem, if and only if

$$\partial X^\alpha/\partial w \in \text{Null space of } \Omega^\alpha(x) \tag{3.19}$$

It follows, from the definition of local relative degree, that the first $r-1$ entries of the vector $\Omega^\alpha(x)\partial X^\alpha/\partial w$ are all identically zero. The last entry, which must also be zero, is given by

$$[\partial(L_{X^\alpha}^{r-1} h)/\partial x]\partial X^\alpha/\partial w = L_{\partial X^\alpha/\partial w} L_{X^\alpha}^{r-1}\, h = 0 \tag{3.20}$$

Notice that since the quantities $[\partial(L_{X^\alpha}^{i-1}h)/\partial x]\partial X^\alpha/\partial w = L_{\partial X^\alpha/\partial w} L_{X^\alpha}^{i-1} h = \partial\{L_{X^\alpha}^{i} h\}/\partial w = 0$, $i = 0, 1, 2, \ldots, r-1$ are independent of w it follows according to Lemma 3.5 that they are also independent of α. Hence $L_{X^\alpha}^i h = L_X^i h$ for $i = 0, 1, 2, \ldots, r-1$. Condition (3.20) is then rewritten as

$$\begin{aligned} L_{\partial X^\alpha/\partial w} L_X^{r-1}\, h &= L_{[\partial X/\partial w + (\partial X/\partial u)\partial\alpha/\partial w]} L_X^{r-1}\, h \\ &= L_{\partial X/\partial w} L_X^{r-1}\, h + (\partial\alpha/\partial w) L_{(\partial X/\partial u)} L_X^{r-1}\, h = 0 \end{aligned} \tag{3.21}$$

If a control law $\alpha(x, v, w)$ exists satisfying (3.21), then one may essentially eliminate all possible influence of w on the rth differential equation of the normal form model. The solution of (3.21) with respect to α is explicitly found only in special cases, as the next example shows.

Example

If $X(x, u) = f(x) + g(x)u + p(x)w$, condition (3.21) translates to $L_p L_f^{r-1} h + (\partial\alpha/\partial w) L_g L_f^{r-1} h = 0$. In this case $L_g L_f^{r-1} h$ and $L_p L_f^{r-1} h$ are independent of w and hence $\partial\alpha/\partial w = -(L_p L_f^{r-1} h)/L_g L_f^{r-1} h$. Integrating with respect to w one finds that $\alpha(x, v, w) = -(L_p L_f^{r-1} h/L_g L_f^{r-1} h)w + \gamma(x, v)$. Choosing $\gamma(x, v) = -(L_f^r h/L_g L_f^{r-1} h) + (1/L_g L_f^{r-1} h)v$ the controlled system, expressed in normal form coordinates, is reduced to

$$\left.\begin{aligned} dz_i/dt &= z_{i+1}; \quad i = 1, 2, \ldots, r-1 \\ dz_r/dt &= v \\ d\eta/dt &= q(\xi, \eta, w); \quad y = z_1 \end{aligned}\right\} \tag{3.22}$$

Once again, v can be designed as a variable-structure feedback control law switching on the basis of the sign of an auxiliary output function of the form $h(z) = z_r + c_{r-2} z^{r-1} + \ldots + c_0 z_1$ with appropriately chosen coefficients.

3.4. *Sliding mode stabilization in non-linear model-matching schemes*

In the *model-matching problem* (Isidori 1985 a, b, 1987) one wishes to obtain a feedback control law for the system $dx/dt = X(x, u)$, $y = h(x)$ such that the input–

output behaviour coincides with that of a linear system characterized by $dz/dt = Az + bw$, $y = cz$. In the design of such a feedback control law we are allowed to measure the input w of the reference linear system and its state vector z, i.e. $u = \alpha(x, z, w)$.

The problem can be solved by seeking the required feedback so that the output error signal, $e = h(x) - Cz$, is decoupled from the input w. One can also impose, after a local decoupling law has been found, that the non-linear system output y be robustly stabilized to zero from the input w.

Following Isidori (1987, Chap. 2, § 6), consider the 'extended' system $dx^e/dt = X^e(x^e, u, w)$; $y = h^e(x^e)$, with $x^e = \text{col}\,[z, x]$ given by

$$\frac{d}{dt}\begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} Az \\ X(x, u) \end{bmatrix} + \begin{bmatrix} bw \\ 0 \end{bmatrix}; \quad e = h(x) - cz \tag{3.23}$$

If we let $u = \alpha(x, z, w)$ and denote $X^\alpha(x^e, w) = X(x, \alpha(x, z. w))$, it is easy to see that the matrix $\Omega(x)$ of (3.13), corresponding to the extended system (3.23), and denoted by $\Omega^e(x)$ is given by

$$\Omega^e(x) = \begin{bmatrix} \partial h^e/\partial x^e \\ \partial(L_{X^e} h^e)/\partial x^e \\ \vdots \\ \partial(L_{X^e}^{r-1} h^e)/\partial x^e \end{bmatrix} = \begin{bmatrix} -c & \partial h/\partial x \\ -cA & \partial(L_{X^\alpha} h)/\partial x \\ \vdots & \vdots \\ -cA^{r-1} & \partial(L_{X^\alpha}^{r-1} h)/\partial x \end{bmatrix} \tag{3.24}$$

For the extended system $\partial X^e/\partial w = \text{col}\,[b, \partial X^\alpha/\partial w]$. Using the result of Theorem 3.4, this column vector must belong to the null space of $\Omega^e(x)$. This implies that the following conditions must be satisfied:

$$-cb + L_{\partial X^\alpha/\partial w} h = 0; \quad -cAb + L_{\partial X^\alpha/\partial w} L_{X^\alpha} h = 0; \quad \dots; \quad -cA^{r-1} b + L_{\partial X^\alpha/\partial w} L_{X^\alpha}^{r-1 h} = 0 \tag{3.25}$$

Since the relative degree is invariant under feedback, the terms of the form $L_{\partial X^\alpha/\partial w} L_{X^\alpha}^k h$ $(k = 1, \dots, r-2)$ in (3.25) vanish. This means that necessarily $cb = cAb =$, $\dots$, $cA^{r-2} b = 0$. The linear reference model must exhibit at least the same relative degree as the non-linear system. Since by Lemma 3.5, $L_{X^\alpha}^k h = L_X^k h$, for $k = 0, 1, \dots, r-1$. The last equality in (3.25) implies that the nonlinear feedback control law must satisfy

$$\begin{aligned} -cA^{r-1} b + L_{\partial X^\alpha/\partial w} L_{X^\alpha}^{r-1 h} &= -cA^{r-1} b + L_{(\partial X/\partial u)(\partial \alpha/\partial w)} L_X^{r-1} h \\ &= cA^{r-1} b + (\partial \alpha/\partial w) L_{(\partial X/\partial u)} L_X^{r-1} h = 0 \end{aligned}$$

i.e. $(\partial \alpha/\partial w) = (cA^{r-1} b)/L_{\partial X/\partial u} L_X^{r-1} h$. Since in this case $\partial X/\partial u$ is independent of w (and so is $L_X^{r-1} h$) one can integrate with respect of w to obtain

$$\alpha(x, z, w) = [cA^{r-1} b / L_{\partial X/\partial u} L_X^{r-1} h] w + \gamma(x, z) \tag{3.26}$$

To determine the unspecified part, $\gamma(x, z)$, of the feedback control law (3.26), one imposes equality among the rth derivatives of the output, y, of the non-linear system, as well as the corresponding one of the reference model output (see Isidori (1987)). This is equivalent to setting to zero the rth differential equation of the normal form model of the extended system. This procedure leads to

$$cA^r z + cA^{r-1} bw = [L_{X^r} h](x, \alpha(x, z, w))$$
$$= [L_{X^r} h](x, [cA^{r-1} b / L_{\partial X/\partial u} L_X^{r-1} h]w + \gamma(x, z)) \qquad (3.27)$$

The definition of relative degree and the implicit function theorem guarantee the local existence of a unique solution for $\gamma(x, z)$, i.e. for $\alpha(x, z, w)$.

Example

For controlled vector fields X of the form $f + gu$, (3.27) results in

$$cA^r z + cA^{r-1} bw = L_{f^r} h + ([cA^{r-1} b / L_{\partial X/\partial u} L_X^{r-1} h]w + \gamma(x, z)) L_g L_f^{r-1} h$$

i.e.

$$\alpha(x, z, w) = [cA^{r-1} b / L_{\partial X/\partial u} L_X^{r-1} h]w + \gamma(x, z)$$
$$= [cA^r z + cA^{r-1} bw - L_f^r h] / L_g L_f^{r-1} h$$

which coincides with the result in Isidori (1987).

The closed-loop system makes the non-linear model behave in the same manner as the linear system from the input–output viewpoint, except for a term depending on the initial condition that can also be appropriately set to zero. Indeed, the output of the non-linear controlled system is of the form

$$y(t) = h(x(t)) = e(t) + cz(t) = e(t) + \int_0^t c \exp[A(t-\sigma)]\, bw(\sigma)\, d\sigma \qquad (3.28)$$

Writing the normal form equations for the extended system it is easy to see that if a control law obtained from (3.27) is used the term $e(t)$, above, only depends on the initial conditions of the extended system. Its effect can therefore be cancelled.

The output $y(t)$ generated by the closed-loop extended system can now be stabilized by appropriate choice of the input w. Since the non-linear system behaves in a linear fashion and responds according to (3.28), a variable-structure control law (which properly takes into account the relative degree of the linear system) can now be devised for robust stabilization of the non-linear system with arbitrarily prespecified eigenvalues. The details are left for the reader.

4. Conclusions and suggestions for further research

In this paper the relevance of the relative degree concept has been examined in the analysis and design issues related to the creation of sliding regimes for general non-linear systems. The results indicate that the simplest possible structure at infinity must be exhibited by non-linear systems undergoing sliding motions on the zero level set of the output feedback function. General necessary, as well as necessary and sufficient, conditions for the existence of sliding regimes have been presented. The disturbance rejection properties of the sliding mode control were examined and a generalization of the matching condition was found. The implications of sliding mode control as an 'outer loop' feedback strategy was also examined in a variety of control problems, including : local stabilization of feedback linearizable systems, disturbance decoupling problems—with and without measurement of the disturbance input—and non-linear model-matching. The basic results and original ideas of Isidori (1987) in this area were shown to easily generalize to the non-linear control case.

Several important research areas may be pursued in the future within the context of the paper. For instance, one may wish to extend the general results about sliding motions to the case of non-linear multivariable systems and non-linear discrete-time systems.

Acknowledgment

This work was supported by the Consejo de Desarollo Científico, Humanístico y Tecnológico of the Universidad de Los Andes (Mérida, Venezuela) under Research Contract I-280-87.

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