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Invariance conditions in non-linear PWM controlled systems

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A geometric approach is used to demonstrate that Pulse-Width-Modulation (PWM) controlled systems exhibit perturbation invariance properties identical to those exhibited by dynamical systems undergoing sliding motions on discontinuity surfaces. At the core of the result is the existence of an ideal equivalence among sliding regimes of Variable Structure Feedback (VSF) control and Pulse-Width-Modulated (PWM) control responses.

1. Introduction

In this article the perturbation invariance properties of sliding regimes associated with systems regulated by Variable Structure Feedback (VSF) control (Drazenovic 1969, Utkin 1981), is shown to be sheared by the responses of systems with PWM control loops (Skoog and Blankenship 1970, Tsypkin 1984).

An ideal equivalence among VSF and PWM is derived under the assumption of high frequency control switchings. PWM controlled responses are shown to locally sustain sliding motions on an integral manifold associated with a suitably defined ideal average system. As an ideal feedback law, the equivalent control, associated with the corresponding ideal sliding motion, coincides with the prescribed duty ratio. Conversely, a given discontinuity surface locally qualifies as an integral manifold of a PWM controlled system provided a local sliding motion exists on it, with an associated equivalent control coincident with the prescribed duty ratio.

The above equivalence is exploitable in PWM design problems by replacing the synthesis of duty ratios (as feedback laws) by simpler switching laws leading to the equivalent sliding mode behaviour on the suitable integral manifold (Sira-Ramirez 1987). The above equivalence immediately reveals the perturbation invariance properties of average PWM controlled trajectories, and the invariance properties of such motions with respect to input or integral manifold coordinates transformations. These properties are inherited from the corresponding ones of the equivalent sliding mode existing on the integral manifold of the average PWM controlled system.

Section 2 analyses, in a unified geometric fashion, non-linear systems controlled by means of VSF laws and PWM feedback loops. An ideal equivalence is obtained among PWM control strategies and VSF control options. The invariance conditions of PWM controlled systems are derived, in § 3, as an immediate consequence of the properties of the equivalent sliding motions occurring on the integral manifold of the average PWM controlled system.

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2. A geometric approach to discontinuous control

2.1. *A geometric approach to VSF control*

Consider the following non-linear system:

$$\frac{dx}{dt} = f(x) + g(x)u \tag{2.1}$$

where $x \in X$, an open set of $\mathbb{R}^n$; the scalar control function $u : \mathbb{R}^n \to \mathbb{R}$ is a (possibly discontinuous) feedback control function; while f and g are smooth, local, vector fields defined on X. Let s denote a smooth real-valued function of x defined by $s : X \to \mathbb{R}$. The level set $S = \{x \in \mathbb{R}^n : s(x) = 0\}$ defines an $(n-1)$-dimensional, locally regular manifold of constant rank, i.e. locally integrable (Boothby 1975), addressed as the sliding manifold or discontinuity surface. The gradient of $s(x)$, denoted by ds, is hence assumed to be non-zero in X except, possibly, on a set of measure zero. Here S is oriented in such a way that ds points from the region where $s(x) < 0$ towards that where $s(x) > 0$.

All results in this article are of a local nature, restricted to an open neighbourhood X of $\mathbb{R}^n$ which has a non-empty intersection with S.

Without loss of generality, a VSF control law is obtained by letting the control function u take one of two possible values in $U := \{0, 1\}$, according to the sign of $s(x)$, as defined by

$$u = \begin{cases} 1, & \text{for } s(x) > 0 \\ 0, & \text{for } s(x) < 0 \end{cases} \tag{2.2}$$

Let $L_h s$ denote the directional derivative of the scalar function s with respect to the vector field h. Here $L_h s$ is also denoted as a differential 1-form, $\langle ds, h \rangle$, acting linearly on $T_x X$ and taking values in $\mathbb{R}$ (see Arnold 1984, p. 174). By Ker ds is meant $\{h : \langle ds, h \rangle = 0, x \in S\}$. In local coordinates

$$\langle ds, h \rangle = \left(\frac{\partial s}{\partial x} \right)^{\mathrm{T}} h$$

Ker ds is also known as the tangent distribution to S, frequently denoted by Δ_S, associating with each point x of S an $(n-1)$-dimensional subspace of $T_x X$. The assumptions about S imply that Δ_S is a constant-dimensional and involutive distribution (Boothby 1975).

Definition 1

A sliding regime is said to exist locally on S, if and only if, as a result of the control policy (2.2), the state trajectories of (2.1) satisfy (Utkin 1981, Sira-Ramirez 1988)

$$\lim_{s \to +0} L_{f+g} s = \lim_{s \to +0} \langle ds, f + g \rangle < 0 \tag{2.3}$$

$$\lim_{s \to -0} L_f s = \lim_{s \to -0} \langle ds, f \rangle > 0 \tag{2.4}$$

Lemma 1

If a sliding regime locally exists on S, then, necessarily, the transversality condition

$$L_g s = \langle ds, g \rangle = \left(\frac{\partial s}{\partial x}\right)^{\mathrm{T}} g(x) < 0$$

is locally satisfied on the manifold S.

Proof

The proof is evident upon subtraction, on S, of (2.4) from (2.3). □

Definition 2

Let $\langle ds, g \rangle$ and $\langle ds, f \rangle$ be non-identically zero on X. We say that S is a local integral manifold for (2.1), with $u(x)$ a given smooth control function, if S is locally integrable and for every initial state specified on S the state trajectories of (2.1) locally remain on S.

It follows that if S is a local integral manifold for (2.1) with $u = u(x)$ then along the trajectories of the controlled system

$$s = 0, \quad \langle ds, f + gu(x) \rangle = 0 \tag{2.5}$$

or equivalently,

$$s = 0, \quad f + gu(x) \in \operatorname{Ker} ds := \Delta_S$$

On an integral manifold S the smooth controlled vector field $f + gu(x)$ is tangent to the manifold S.

Theorem 1 (Utkin 1981, Sira-Ramirez 1988)

A necessary and sufficient condition for the local existence of a sliding mode on S is that there exists, locally on S, a smooth control function $u_{EQ}(x)$, which turns S into a local integral manifold for (2.1), such that

$$0 < u_{EQ}(x) < 1 \tag{2.6}$$

The above theorem actually provides a definition of the ideal (average) sliding motion on the manifold S, known as the ideal sliding dynamics. The smooth control function $u_{EQ}(x)$ is called the equivalent control and according to its definition and (2.5) it satisfies $\langle ds, f + gu_{EQ}(x) \rangle = 0$, i.e.

$$u_{EQ}(x) = -\frac{\langle ds, f \rangle}{\langle ds, g \rangle}, \quad x \in S \tag{2.7}$$

The transversality condition of Lemma 1 is therefore justified on the grounds of existence of the equivalent control. Thus, existence of the equivalent control is also a necessary condition for the existence of a sliding regime (Sira-Ramirez 1988). Notice that if $\langle ds, g \rangle = 0$ on an open set of X, then a sliding motion may still exist on a proper submanifold of S, provided $\langle ds, f \rangle = 0$ locally in X. Such sliding motions are termed singular (Utkin 1981) and will not be considered here.

From (2.7) it follows upon formal substitution in (2.1) that the motions starting on

S, due to the equivalent control—ideal sliding motions—are governed by

$$s = 0, \quad \frac{dx}{dt} = f + g u_{EQ}(x) = f - g \frac{\langle ds, f \rangle}{\langle ds, g \rangle} \tag{2.8}$$

This procedure constitutes the Method of the Equivalent Control (Utkin 1981).

2.2. *A geometric approach to PWM control*

In a PWM control option for system (2.1), the scalar control u, taking values in $U = \{0, 1\}$, is switched *once* within a duty cycle of fixed small duration Δ. The instants of time at which the switchings occur are determined by the sample value of the state vector at the beginning of each duty cycle. The fraction of the duty cycle on which the control holds a fixed value, say 1, is known as the duty ratio and it is denoted by $D(x(t))$. The duty ratio is usually specified as a smooth function of the state vector x. The duty ratio evidently satisfies $0 < D(x) < 1$.

On a typical duty cycle interval, the control input u is defined as (Fig. 1)

$$u = \begin{cases} 1, & \text{for } t \leqslant \tau < t + D(x(t))\Delta \\ 0, & \text{for } t + D(x(t))\Delta \leqslant \tau \leqslant t + \Delta \end{cases} \tag{2.9}$$

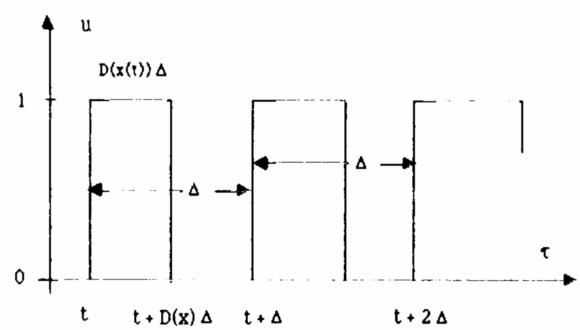

Figure 1. Typical duty cycles in PWM control.

It follows then that, generally

$$x(t+\Delta) = x(t) + \int_t^{t+D(x(t))\Delta} [f(x(\tau)) + g(x(\tau))]\, d\tau + \int_{t+D(x(t))\Delta}^{t+\Delta} f(x(\tau))\, d\tau$$

The ideal average model of the PWM controlled system response is obtained by allowing the duty cycle frequency to tend to infinity with the duty cycle length Δ approaching zero. In the limit, the above relation yields

$$\lim_{\Delta \to 0} \frac{[x(t+\Delta) - x(t)]}{\Delta} = \lim_{\Delta \to 0} \left[\int_t^{t+\Delta} f(x(\tau))\, d\tau + \int_t^{t+D(x(t))\Delta} g(x(\tau))\, d\tau \right] \frac{1}{\Delta}$$

i.e.

$$\frac{dx}{dt} = f(x) + g(x)D(x) \tag{2.10}$$

As the duty cycle frequency tends to infinity, the ideal average behaviour of the PWM controlled system is represented by the smooth response of the system (2.1) to the smooth control function constituted by the duty ratio $D(x)$. The duty ratio $D(x)$ replaces the discrete control function u in (2.1) in the same manner as the equivalent control $u_{EQ}(x)$, of the VSC scheme, replaces u in (2.1) to obtain (2.8).

We refer to (2.10) as the average PWM controlled system.

Lemma 2

Let $\Sigma = \{x \in \mathbb{R}^n : \sigma(x) = 0\}$ be a local integral manifold for the average PWM controlled system (2.10) then

$$0 < D(x) = -\frac{\langle d\sigma, f \rangle}{\langle d\sigma, g \rangle} < 1 \tag{2.11}$$

Proof

The inequalities are obvious from the definition of the duty ratio. The expression for $D(x)$ is obtained from the fact that if Σ is an integral manifold of (2.10), then from Definition 2, $\langle d\sigma, f + gD(x) \rangle = 0$, locally on Σ. From here (2.11) follows immediately. □

Equation (2.11) constitutes a *geometric-based definition* of the duty ratio.

Notice that $\langle d\sigma, g \rangle = 0$, on an open set of X, makes $D(x)$ unbounded unless $\langle d\sigma, f \rangle$ is also zero, in which case Σ is an integral manifold of (2.10) for any conceivable $D(x)$. To avoid this we assume, without loss of generality, that $\langle d\sigma, g \rangle < 0$ locally on Σ. Notice that if $\langle d\sigma, g \rangle > 0$, and Σ is an integral manifold of (2.10), then, redefining Σ as $\{x \in \mathbb{R}^n : \sigma_1(x) = 0\}$ with $\sigma_1(x) = -\sigma(x)$, one obtains $\langle d\sigma_1, g \rangle < 0$ and the assumption would now be valid.

Lemma 3

If Σ is a local integral manifold for (2.10), and $\langle d\sigma, g \rangle < 0$, then, in the region of interest, $D(x)$ is unique.

Proof

Suppose $D_1(x) \neq D(x)$ are duty ratios for which Σ is a local integral manifold of (2.1). It follows from Definition 2 that, locally on S,

$$\langle d\sigma, f + D(x)g \rangle = \langle d\sigma, f + D_1(x)g \rangle = 0$$

From this equality it follows that

$$\langle d\sigma, (D(x) - D_1(x))g \rangle = (D(x) - D_1(x))\langle d\sigma, g \rangle = 0$$

Since by hypothesis $\langle d\sigma, g \rangle < 0$ then, necessarily, $D(x) = D_1(x)$ locally on Σ. This is a contradiction. □

Theorem 2

Suppose the transversality condition $\langle d\sigma, g \rangle < 0$ holds locally true on Σ, then a necessary and sufficient condition for Σ to be a local integral manifold of (2.10) is that locally on Σ

$$\langle d\sigma, f + g \rangle < 0 \quad \text{and} \quad \langle d\sigma, f \rangle > 0 \tag{2.12}$$

Proof

Let Σ be a local integral manifold for (2.10), then using the hypothesis that $\langle d\sigma, g\rangle < 0$, it follows from the right-hand side of (2.11) that

$$-\langle d\sigma, f\rangle > \langle d\sigma, g\rangle$$

and therefore $\langle d\sigma, f+g\rangle < 0$. On the other hand, using the first inequality of (2.11), it follows that

$$-\langle d\sigma, f\rangle < 0, \quad \text{i.e. } \langle d\sigma, f\rangle > 0$$

To prove sufficiency, suppose (2.12) holds true locally on Σ. Then, there exists strictly positive smooth functions $a(x)$ and $b(x)$ such that on the region of interest

$$a(x)\langle d\sigma, f+g\rangle + b(x)\langle d\sigma, f\rangle = 0$$

Rearranging the above expression

$$\left\langle d\sigma, f + \left[\frac{a(x)}{(a(x)+b(x))}\right] g\right\rangle = 0$$

i.e. there exists a smooth control function

$$0 < D(x) = \frac{a(x)}{[a(x)+b(x)]} < 1$$

such that, locally on Σ, $\langle d\sigma, f + D(x)g\rangle = 0$. In other words, in X, Σ is a local integral manifold of (2.10). □

Theorem 3

A sliding regime of (2.1) locally exists on an integrable manifold Σ if and only if Σ is a locally integral manifold of an average PWM controlled system whose duty ratio coincides with the equivalent control.

Proof

Suppose Σ is an integral manifold for the average PWM controlled system (2.10), then Theorem 2 applies and (2.12) holds true. It follows that locally on Σ

$$\langle d\sigma, f+g\rangle = \lim_{\sigma\to -0} \langle d\sigma, f+g\rangle < 0$$

and

$$\langle d\sigma, f\rangle = \lim_{\sigma\to -0} \langle d\sigma, f\rangle > 0$$

i.e. the variable structure control law: $u = 1$ for $\sigma(x) > 0$ and $u = 0$ for $\sigma(x) < 0$ applied on system (2.1) creates a sliding mode locally on Σ. Then, necessarily the transversality condition $\langle d\sigma, g\rangle < 0$ holds, according to Lemma 1. The corresponding equivalent control $u_{EQ}(x)$ satisfies $\langle d\sigma, f + gu_{EQ}(x)\rangle = 0$ and because, by hypothesis, Σ is an integral manifold of (2.10), $\langle ds, f + gD(x)\rangle = 0$ also holds locally. It follows that

$$(u_{EQ}(x) - D(x))\langle d\sigma, g\rangle = 0, \quad \text{i.e. } u_{EQ}(x) = D(x)$$

Suppose now that a sliding motion exists locally on Σ, then (2.3) and (2.4) hold true locally on Σ. Therefore, the hypothesis of Theorem 2 are also valid. Hence Σ

qualifies as a local integral manifold of the average PWM controlled system (2.10) for some $D(x)$. Notice that from Theorem 1

$$0 < u_{EQ}(x) < 1$$

is satisfied in the region of interest. By definition, the equivalent control, $u_{EQ}(x)$ also turns Σ into a local integral manifold in the region of existence of a sliding regime. By virtue of the uniqueness of the duty ratio of Lemma 3, the duty ratio $D(x)$ coincides with the equivalent control $u_{EQ}(x)$ as a smooth feedback function of the state vector. □

3. Invariance properties of PWM controlled systems

In this section we centre attention on the properties of the average PWM controlled systems. Using the results of the previous section it will be shown that the average PWM dynamics exhibits perturbation invariance properties which are obtained from the geometric-based definition of the duty ratio. It is shown, in particular, that perturbations affecting the controlled system do not influence the average dynamics provided a *matching condition* is satisfied. If the duty ratio does not saturate to one of its extreme values on an open set of the state space of the system, such an invariance property is always exhibited. Similarly for VSF control, average responses of PWM controlled systems are invariant with respect to coordinate transformations performed on the input space coordinates or on the integral manifold defining coordinate function.

3.1. *Perturbation invariance properties of PWM controlled responses*

From the geometric-based expression for the duty ratio, found in (2.11), it follows that the motions on Σ, due to the duty ratio $D(x)$, are governed by

$$\frac{dx}{dt} = f + gD(x) = \left\{ I - g\left[\left(\frac{\partial s}{\partial x}\right)^{\mathrm{T}} g\right]^{-1}\left(\frac{\partial s}{\partial x}\right)^{\mathrm{T}}\right\} f \tag{3.1}$$

Equation (3.1) represents an idealized version of the motions occurring about the integral manifold Σ and they constitute an 'average' description for the behaviour of the controlled trajectories of (2.11) on Σ. We denote by Δ_Σ the tangent distribution associated with Σ. Notice that for any vector field in span $\{g(x)\}$—i.e. any vector of the form $g(x)u$ with $u = u(x)$ a smooth control function—it follows that

$$\left\{ I - g\left[\left(\frac{\partial \sigma}{\partial x}\right)^{\mathrm{T}} g\right]^{-1}\left(\frac{\partial \sigma}{\partial x}\right)^{\mathrm{T}}\right\} g(x)u = 0 \tag{3.2}$$

Proposition 1

The matrix

$$F := \left[I - g\left[\left(\frac{\partial \sigma}{\partial x}\right)^{\mathrm{T}} g\right]^{-1}\left(\frac{\partial \sigma}{\partial x}\right)^{\mathrm{T}}\right]$$

is a projection operator (i.e. it satisfies $F^2 = F$) taking any vector in the tangent space T_xX onto the tangent distribution Δ_Σ along span $\{g(x)\}$.

Proof

It is easy to see that $F^2 = F$. For this, let

$$b(x) = \langle d\sigma, g \rangle = \left(\frac{\partial \sigma}{\partial x}\right)^{\mathrm{T}} g$$

and compute F^2 as

$$\begin{aligned} F^2 &= F - g(x)b^{-1}(x)\left[\left(\frac{\partial \sigma}{\partial x}\right)^{\mathrm{T}} F\right] = F - \left\{g(x)b^{-1}(x)\left[\left(\frac{\partial \sigma}{\partial x}\right)^{\mathrm{T}}\right]\right\} \\ &\quad + \left\{g(x)b^{-1}(x)\left[\left(\frac{\partial \sigma}{\partial x}\right)^{\mathrm{T}}\right]\right\}\left\{g(x)b^{-1}(x)\left[\left(\frac{\partial \sigma}{\partial x}\right)^{\mathrm{T}}\right]\right\} \\ &= F - g(x)b^{-1}(x)\left[\left(\frac{\partial \sigma}{\partial x}\right)^{\mathrm{T}}\right] + g(x)b^{-1}(x)\left[\left(\frac{\partial \sigma}{\partial x}\right)^{\mathrm{T}}\right] = F \end{aligned}$$

From the integral manifold condition (2.5) $Ff \in \mathrm{Ker}\, d\sigma = \Delta_\Sigma$, F takes the vector field f onto the sliding distribution Δ_Σ. Any vector field h in span $\{g(x)\}$ is of the form $h = gu$ for some u. It follows from the definition of F that $Fh = Fgu = 0 \cdot u = 0$. □

Any component of the vector field f in the span of the input vector field g does not have any influence on the average PWM dynamics. Only the components of f along the distribution Δ_Σ will define the nature of the average PWM dynamics. Figure 2 depicts the geometry of the perturbation invariance property.

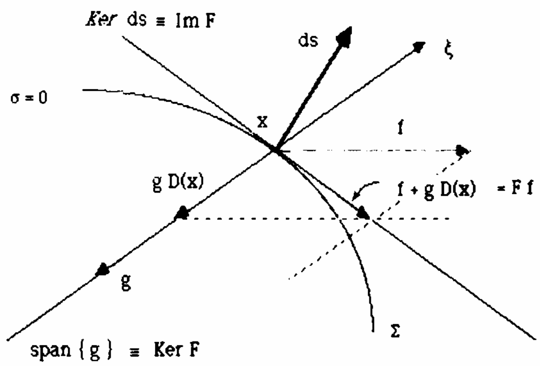

Figure 2. Geometry of the perturbation invariance property.

Consider the following perturbed dynamical system:

$$\frac{dx}{dt} = f(x) + g(x)u + \xi \tag{3.3}$$

where if we let $\xi \equiv \delta f(x)$ then ξ represents 'parametric perturbations' of the normal drift vector field f. If we let $\xi \equiv h(t)$, then ξ will represent a state independent, unstructured, external perturbation.

Definition 3

The average PWM dynamics is said to exhibit a strong invariance property with respect to the perturbation signal ξ, if it is independent of the perturbation signal ξ.

The following theorem is an immediate consequence of the projective nature of the map F, defined above, governing the average PWM dynamics.

Theorem 4

The average PWM motions of the perturbed system (3.3), satisfy the strong invariance property with respect to ξ, if and only if the disturbance vector ξ satisfies the matching condition

$$\xi \in \text{span}\,\{g(x)\} \tag{3.4}$$

Proof

For the dynamical system (3.3), the ideal average PWM behaviour is governed by $dx/dt = F(f+\xi)$. Sufficiency follows from the fact that if $\xi \in \text{span}\,\{g(x)\}$ then $\xi = g(x)v(x)$ for some smooth scalar function $v(x)$. In this case, the projection operator F annihilates the influence of ξ on the average dynamics. Necessity follows from the fact that if

$$F\xi = \left[I - g\left[\left(\frac{\partial\sigma}{\partial x}\right)^{\mathrm{T}} g\right]^{-1}\left(\frac{\partial\sigma}{dx}\right)^{\mathrm{T}}\right]\xi = 0$$

then, necessarily,

$$\xi = g\left[\left(\frac{\partial\sigma}{\partial x}\right)^{\mathrm{T}} g\right]^{-1}\left[\left(\frac{\partial\sigma}{dx}\right)^{\mathrm{T}}\xi\right] =: \mu(x)g$$

which implies that $\xi \in \text{span}\,\{g(x)\}$. □

Definition 4

The average PWM dynamics is said to exhibit a weak invariance property with respect to the perturbation signal ξ, whenever ξ does not satisfy the matching condition (i.e. whenever it does not satisfy the strong invariance property).

The justification for this definition stems from the fact that if $\xi \notin \text{span}\,\{g(x)\}$ then ξ can be uniquely decomposed as the sum of two vectors; one along the span of $g(x)$ and the other along the distribution, ker $d\sigma$, i.e.

$$\xi = g(x)v + \eta(X)$$

for some v and $\eta(x) \in \text{Ker}\, d\sigma$. Notice that the average PWM motion is unaffected by $g(x)v$ while $\eta(x)$ is tangential to the integral manifold Σ, i.e. $\eta(x)$ does not affect the local existence of an average motion on the integral manifold—i.e. $\eta(x)$ does not 'pull' the state trajectories out of Σ—while it indeed affects the average dynamics within its constrained motions on Σ.

Remark 1

Notice that the matching condition (3.4) does not affect the average PWM trajectories on Σ provided Σ does not include an open set where the duty ratio $D(x)$ saturates and the switch position function u adopts the extreme values of either 1 or 0. Indeed, suppose $u = 0$ on such an open set of Σ, then the unperturbed system trajectories would locally satisfy $\langle d\sigma, f\rangle = 0$. However, a perturbation ξ in (3.3) would produce $\langle d\sigma, f+\xi\rangle = \langle d\sigma, \xi\rangle$ which is, in general, non-zero, specially, if the matching condition is satisfied since then, for some v, $\langle d\sigma, \xi\rangle = v\langle d\sigma, g\rangle$ which according to the transversality condition is non-zero. If, on the other hand, the switch position u is set to $u = 1$ on an open set of Σ, the unperturbed system would satisfy $\langle d\sigma, f+g\rangle = 0$. Hence, the response of (3.3) would now produce

$$\langle d\sigma, f+g+\xi\rangle = \langle ds, \xi\rangle \neq 0$$

The perturbed state trajectories would leave Σ on the open sets where the duty ratio saturates to either 0 or 1.

The situation depicted in Remark 1 exactly corresponds to the fact that in sliding mode control the perturbation invariance properties of the sliding regime are only exhibited precisely on the sliding manifold. Before the sliding regime is achieved on the discontinuity surface, and the control function adopts one of its extreme values, the trajectories have been shown not to be immune to external or parametric perturbations (Drazenovic 1969).

The design of an appropriate PWM strategy to conform to a desired vector field, independent of ξ, can only be done when the matching condition (3.4) is satisfied. This entitles knowledge of the structural properties of the disturbance channels.

Example 1

In the case of linear time-invariant systems of the form

$$\frac{dx}{dt} = Ax + bu$$

the matching condition (3.4) is simply: $\xi \in \operatorname{im} b$' In the multiple input case

$$\frac{dx}{dt} = Ax + Bu$$

the matching condition translates into $\xi \in \operatorname{im} B$, i.e. rank $B = \operatorname{rank}[B, \xi]$—see the work by Drazenovic (1969) and El-Ghezawi *et al.* (1983) for the counterpart of this result related to sliding mode control.

3.2. *Invariance under input and state coordinates transformations*

Suppose a non-vanishing (i.e. scalar non-singular) transformation is performed on the integral manifold coordinate σ, or on the average PWM system control input space, respectively, by means of

$$s^* = H(x)s, \quad D^* = K(x)D \tag{3.5}$$

Theorem 5

The average PWM dynamics is unaffected by non-vanishing transformations of either integral manifold, or control space, coordinates.

Proof

Suppose a non-singular transformation of the input space coordinate is performed on (2.1) according to (3.5). The equations for the average PWM motion are given by

$$\frac{dx}{dt} = f(x) + g(x)D(x) = f(x) + g(x)K^{-1}(x)D^*$$

From the manifold condition of Lemma 2, and the invertibility of $K(x)$, it follows that the transformed duty ratio is

$$D^*(x) = -[b(x)K^{-1}(x)]^{-1}\left[\left(\frac{\partial\sigma}{\partial x}\right)^{\mathrm{T}} f\right] = -K(x)b^{-1}(x)\left[\left(\frac{\partial\sigma}{\partial x}\right)^{\mathrm{T}} f\right]$$

with $b(x) = (\partial\sigma/\partial x)^{\mathrm{T}} g(x)$. Hence, the average PWM dynamics is governed by

$$\frac{dx}{dt} = f(x) + g(x)D^*(x) = \left\{I - g(x)K^{-1}(x)K(x)b^{-1}(x)\left[\left(\frac{\partial\sigma}{\partial x}\right)^{\mathrm{T}}\right]\right\}f(x)$$

$$= \left\{I - g(x)b^{-1}(x)\left[\left(\frac{\partial\sigma}{\partial x}\right)^{\mathrm{T}}\right]\right\}f(x) = Ff(x)$$

i.e. the average PWM motion equations are invariant with respect to non-vanishing input space coordinates transformations. However, the input-to-integral manifold 'channel', represented by the scalar function $b(x) = (\partial\sigma/\partial x)^{\mathrm{T}} b(x)$, is indeed changed by the transformation H.

Suppose now that a transformation of the form (3.5) is carried out on the surface coordinate defining the integral manifold Σ. Due to the non-singularity of $H(x)$, $\sigma^*(x) = 0$ if and only if $\sigma(x) = 0$. Thus, the integral manifold location is unaffected by the transformation. The invariance condition is now,

$$\langle d\sigma^*, f + gD(x)\rangle = \left[\left(\frac{\partial\sigma^*(x)}{\partial x}\right)^{\mathrm{T}}\right][f(x) + g(x)D(x)]$$

$$= \sigma(x)\left(\frac{\partial H(x)}{\partial x}\right)^{\mathrm{T}}[f(x) + g(x)D(x)]$$

$$+ H(x)\left(\frac{\partial\sigma}{\partial x}\right)^{\mathrm{T}}[f(x) + g(x)D(x)] = 0$$

The first term of the sum being zero on the integral manifold Σ, where $\sigma = 0$. From the remaining expression, and the local non-singularity of $H(x)$, it follows that the duty ratio is still defined as in (2.11). The average PWM dynamics is then invariant with respect to non-vanishing surface coordinate transformations. □

4. Conclusions

The perturbation invariance properties of systems controlled by PWM feedback are explained by demonstrating the existence of an ideal equivalence of the PWM controlled response and that of a sliding mode induced on an integral manifold of the associated average system smoothly controlled by the duty ratio. As in the sliding mode control of systems, a lowering of the sensitivity of the system is achieved when controlled by PWM feedback control. Such an invariance is always present during the period of time when the duty ratio is not saturated to one of its extreme values (zero or one).

The equivalence established in this work constitutes a step towards the systematic treatment of PWM control design via sliding regimes. The advantage of such an equivalence resides in the 'automatic' synthesis of prescribed feedback duty ratios by means of on-the-average equivalent variable structure feedback strategies defined on an appropriate sliding surface. Additional benefits are also drawn from hardware simplicity, and perturbation robustness characteristic of the equivalent sliding mode approach.

All the above results straightforwardly extend to the case of multiple inputs.

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