Systems & Control Letters 13 (1989) 177–181
North-Holland

Distributed sliding mode control in systems described by quasilinear partial differential equations *

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Received 1 March 1989
Revised 8 May 1989

Abstract: The theory of variable structure systems and their associated Sliding Regimes is extended to controlled dynamical systems described by first order quasilinear partial differential equations.

Keywords: Variable structure systems; sliding regimes; distributed dynamical systems.

1. Introduction

The theory of variable structure systems (VSS) and their associated sliding regimes constitutes an interesting field of the control systems discipline with a vast number of applications. A detailed account of the basic elements of the theory, as applied to dynamical systems described by ordinary differential equations (ODE), is contained in the work of Utkin [10–12].

There are only few instances where the sliding mode control method has been applied to systems described by partial differential equations (PDE). In Orlov and Utkin [5], sliding modes were proposed for the regulation of a distributed thermal process described by a second-order PDE of the parabolic type. The control scheme resorted to a finite dimensional approximation of the distributed process. The sliding mode creation problem was defined on the associated *finite* dimensional controlled system approximation, characterized by a set of ODE's. In Orlov and Utkin [6], the theory of sliding mode control was extended to infinite dimensional systems described by differential equations defined in Banach spaces. Applications were given, in that article, for a multi-dimensional heat process. In Breger et al. [2] discontinuous control is proposed for the heat equation using averaging theory.

In this article the theory of VSS, and their associated sliding regimes, is extended to dynamical systems described by first order quasilinear PDE's (FOQPDE). The key idea is to exploit the geometric features of the flows associated to the *characteristic direction field* of a controlled FOQPDE. The sliding mode conditions are thus characterized in terms of a finite dimensional sliding mode existence problem defined on the *controlled characteristic equation*. The results may then be easily particularized for the case of controlled systems described by first order linear PDE's, of the homogeneous and nonhomogeneous type.

Section 2 presents the definitions and main results. Section 3 is devoted to a simple illustrative example. Background material and terminology on the geometric aspects of QPDE's are directly taken from Chapter 2 of Arnold's book [1]. For deeper background on the subject of PDE's, the reader is referred to the extensive treatise written by Courant and Hilbert [4].

2. Main results

2.1. *Sliding regimes in systems described by controlled FOQPDE's*

Consider a dynamical system described by a feedback-controlled FOQPDE:

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} X_i(v, x, t, u) = b(v, x, t, u), \tag{2.1a}$$

$$y = h(v, x, t) \tag{2.1b}$$

* This work was supported by the Consejo de Desarrollo Cientifico, Humanistico y Tecnologico of the Universidad de Los Andes, Mérida, Venezuela, under Research Grant I-280-87.

where y is the scalar-valued output function, x represents the vector of local spatial coordinate functions x_i defining points on an open set in R^n, t denotes time, while $u = u(v, x, t)$ is a distributed feedback control law taking values in R. The function v is the unknown scalar function, regarded as the distributed 'state' of the controlled system. For each smooth solution v of (2.1), the X_i's are the smooth components of a time-varying control-parametrized vector field X, which is assumed to be locally nonzero and defined on an open set of R^n. The function $b: R^{n+3} \to R$ and the function $h: R^{n+3} \to R$ are locally smooth functions of their arguments. Condition $y = 0$ is assumed to locally define an isolated smooth manifold solution $v = \phi(x, t)$, i.e.,

$$h(\phi(x, t), x, t) \equiv 0.$$

The graph of v is assumed to be a smooth time-varying surface with locally nonzero gradient except possibly on a set of measure zero. This surface is addressed as the *sliding manifold*, or the *sliding surface*, and is locally defined as

$$S = \{(v, x, t) \in R^{n+2}: v = \phi(x, t)\}.$$

All our considerations and results are of a *local* character on a given open set N of R^{n+2} described by the local coordinate functions (v, x, t). The projection of such an open set N onto R^{n+1} is labeled as M and such a set is equipped with local coordinates (x, t). For a given smooth feedback function $u = u(v, x, t)$, and a corresponding solution v of (2.1), the vector field

$$\mathrm{col}[X(v, x, t, u), 1]$$

is a smooth vector field locally defined on M.

Definition 1 [1]. Given an n-dimensional surface γ in M and a (not necessarily smooth) function $\psi: \gamma \to R$, the *Cauchy data*, or the *initial condition*, of the FOQPDE (2.1) is constituted by the pair (ψ, γ). The n-dimensional submanifold Γ in N, represented by the graph of ψ on γ, is called the *initial submanifold*. Given a smooth feedback function u, an initial submanifold Γ is *noncharacteristic* at the point (x_0, t_0) in ψ, if the vector

$$\mathrm{col}[X(v_0, x_0, t_0, u(v_0, x_0, t_0)), 1]$$

in R^{n+1} is not tangent to γ at the point (x_0, t_0), with $v_0 = \phi(x_0, t_0)$.

It will be assumed throughout that for a given smooth distributed feedback control $u(v, x, t)$ and a given Cauchy data (represented by the *noncharacteristic* initial condition submanifold Γ in R^{n+2}), the graph of the solution v of (2.1) is locally smooth, with nonzero gradient everywhere on the open set N where we carry our considerations, except, possibly, on a set of measure zero. This assumption is satisfied in several classical physical examples. (See, for instance, Arnold [1], p. 62.)

Available to the controller is a *distributed variable structure feedback switching law*:

$$u = \begin{cases} u^+(v, x, t) & \text{for } y > 0, \\ u^-(v, x, t) & \text{for } y < 0, \end{cases} \tag{2.2}$$

with $u^+(v, x, t) > u^-(v, x, t)$, locally.

Definition 2. A distributed sliding regime is said to locally exist on an open set $\mathcal{N}$ of the manifold S if and only if the *total derivative* of the output function of the controlled system (2.1)–(2.2) satisfies (see [10]):

$$\lim_{y \to +0} \frac{\mathrm{d}y}{\mathrm{d}t} < 0 \quad \text{and} \quad \lim_{y \to -0} \frac{\mathrm{d}y}{\mathrm{d}t} > 0. \tag{2.3}$$

To simplify notation we introduce the vector $z = \mathrm{col}(v, x, t)$ of local coordinate functions and the control-parametrized vector field

$$\xi = \mathrm{col}[b(z, u), X(z, u), 1]$$

referred to as the *characteristic direction field* of (2.1). The *Lie derivative* of a scalar function $h(z)$ with respect to the vector field ξ, for a given feedback control input $u = u(z)$, is denoted by $L_{\xi(z,u(z))}h$ or simply by $L_\xi h$. In local coordinates:

$$L_\xi h = (\partial h/\partial v) b(z, u) + (\partial h/\partial x) X(z, u) + \partial h/\partial t.$$

Theorem 1. *For a given Cauchy data* (ψ, γ) *defining an initial submanifold* Γ *with nonempty intersection with* N*, a distributed sliding regime locally exists for system* (2.1)–(2.2) *on an open set* $\mathcal{N}$ $(:= N \cap S)$ *of* S*, if and only if the* phase flows *corresponding to the controlled characteristic direction field of* (2.1)*, which arise from the initial submanifold* Φ*, exhibit such a local sliding regime on* $\mathcal{N}$ *under the influence of the switching law* (2.2).

Proof. Suppose a distributed sliding mode locally exists for (2.1)–(2.2) on an open set $\mathcal{N}$ of S. Then, the total time derivatives of y, at any point z in N, belonging to the graph of the solution of the controlled equation, can be computed in terms of the directional derivatives along the controlled characteristic direction field ξ. These derivatives are given by:

for $y>0$:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= [\partial h/\partial v]\mathrm{d}v/\mathrm{d}t + [\partial h/\partial x]\mathrm{d}x/\mathrm{d}t + [\partial h/\partial t]\\ &= [\partial h/\partial v]b(v, x, t, u^{+})\\ &\quad + [\partial h/\partial x]X(v, x, t, u^{+}) + [\partial h/\partial t]\\ &= L_{\xi(z,u^{+}(z))}h < 0;\end{aligned}$$

for $y<0$:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= [\partial h/\partial v]\mathrm{d}v/\mathrm{d}t + [\partial h/\partial x]\mathrm{d}x/\mathrm{d}t + [\partial h/\partial t]\\ &= [\partial h/\partial v]b(v, x, t, u^{-})\\ &\quad + [\partial h/\partial x]X(v, x, t, u^{-}) + [\partial h/\partial t]\\ &= L_{\xi(z,u^{-}(z))}h > 0.\end{aligned}$$

In other words, the controlled dynamical system described by the following set of ordinary differential equations:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \xi(z, u), \tag{2.4a}$$

$$y = h(z), \tag{2.4b}$$

(also known as the *controlled characteristic equation* (2.1)), with initial conditions taking values in Γ, exhibits a local sliding regime on the open set $\mathcal{N}$ of the sliding manifold S, determined by $y=0$, when u is governed by the switching law (2.2). Sufficiency follows easily by assuming that a sliding mode exists for the controlled characteristic system and hypothesizing, at the same time, that a distributed sliding mode *does not* exist. By reversing the arguments presented above, a contradiction is easily established. □

Local sliding regimes, on subsets of S, of the distributed controlled system (2.1), (2.2) are, hence, completely characterized in terms of the local sliding motions – on the same manifold S – of the finite dimensional time-varying system (2.4) controlled by a switching law of the form (2.2).

Theorem 2. *A distributed sliding regime exists on an open set* $\mathcal{N}$ *of* S *for system* (2.1), (2.2) *if and only if there is an open neighborhood* N *of* S *in* R^{n+2} *where*

$$\frac{\partial}{\partial u}L_{\xi}h \neq 0. \tag{2.5}$$

Proof. If $L_{\xi}h$ does not depend locally on u then, changing the control u from $u^{+}(z)$ to $u^{-}(z)$ at points z of $\mathcal{N}$ does not have any effect on the sign of $L_{\xi}h$. Therefore, there exists an open set N in R^{n+2}, containing $\mathcal{N}$, where the existence conditions (2.3) are violated and a sliding regime can not locally exist on $\mathcal{N}$.

To proof sufficiency, suppose $L_{\xi}h(z)$ explicitly depends on u, locally around $\mathcal{N}$ in N. Let $\varepsilon^{-}(z)$ be a smooth, locally strictly positive function of z. Then, by virtue of the *implicit function theorem*, the equation

$$L_{\xi}h(z, u) = \varepsilon^{-}(z)$$

locally has a unique smooth solution $u = u^{-\varepsilon}(z)$ such that

$$L_{\xi(z,u^{-\varepsilon}(z))}h(z) = \varepsilon^{-}(z) > 0.$$

Similarly, by the same arguments, given a smooth locally strictly negative function $\varepsilon^{+}(z)$, a smooth control law $u = u_0^{+\varepsilon}(z)$ locally exists such that

$$L_{\xi(z,u^{+\varepsilon}(z))}h(z) = \varepsilon^{+}(z) < 0.$$

Hence, conditions (2.3) are locally valid around N and a sliding regime exists on the open set $\mathcal{N}$ of S for the found distributed variable structure feedback control law:

$$u = \begin{cases} u^{+}(z) = u^{+\varepsilon}(z) & \text{for } h(z) > 0,\\ u^{-}(z) = u^{-\varepsilon}(z) & \text{for } h(z) < 0. \end{cases} \quad \square$$

Definition 3. For all initial states z located on the open set $\mathcal{N}$ of S, the unique distributed control function, $u^{\mathrm{EQ}}(z)$, locally constraining the distributed trajectories to the sliding manifold S, in the region of existence $\mathcal{N}$ of the sliding motion, is known as the *distributed equivalent control*. (i.e., the equivalent control turns the open set $\mathcal{N}$ of S into a *ocal integral manifold* of the characteristic controlled direction field defined on $\mathcal{N}$ for some given initial Cauchy data defined on $\mathcal{N}$). The resulting characteristic dynamics, ideally con-

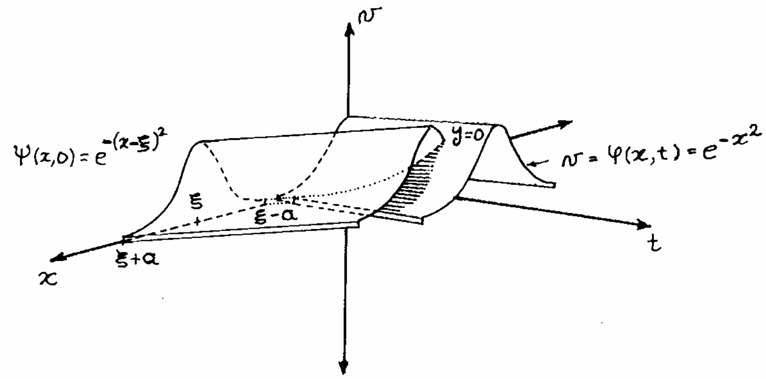

Fig. 1. Partial distributed sliding mode of $v_t + uv_x = 0$, on $v = \exp(-x^2)$.

strained to S, will be addressed as the *characteristic ideal sliding dynamics*. (See the original concept in Utkin [10] for ODE's.) A coordinate-free description of such dynamics in S is:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \xi(z, u^{\mathrm{EQ}}(z)), \quad h(z) = 0. \tag{2.6}$$

The direction field $\xi(z, u^{\mathrm{EQ}}(z))$ will be referred to as the *equivalent direction field*.

Given an arbitrary smooth, noncharacteristic initial n-dimensional submanifold Γ of the zero output manifold $v = \phi(x, t)$, every integral manifold of the equivalent direction field, $\xi(z, u^{\mathrm{EQ}}(z))$, is evidently a local solution, specified by $v = \phi(x, t)$, of the PDE representing the distributed ideal sliding dynamics:

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} X_i(v, x, t, u^{\mathrm{EQ}}(v, x, t)) = b(v, x, t, u^{\mathrm{EQ}}(v, x, t)), \tag{2.7a}$$

$$y(v, x, t) = 0. \tag{2.7b}$$

A necessary and sufficient condition for an open set $\mathscr{N}$ of S to qualify as a local $(n+1)$-dimensional integral manifold of the controlled trajectories (2.6) is that the gradient of h be locally pointwise orthogonal to the smooth equivalent direction field $\xi(z, u^{\mathrm{EQ}}(z))$, i.e.,

$$L_{\xi(z, u^{\mathrm{EQ}}(z))} h(z) = 0 \quad \text{for } z \in \mathscr{N}. \tag{2.8}$$

For an exposition of the results available for the assessment of the existence of sliding regimes in systems of the general form (2.4), the reader is refered to Sira-Ramirez [7] and to [8,9] for other classes of systems.

3. Example

Consider the controlled system described by

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} = 0, \tag{3.1a}$$

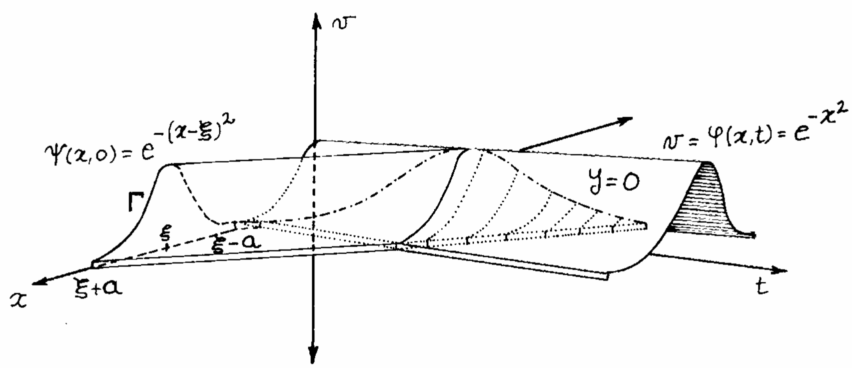

Fig. 2. Total distributed sliding mode of $v_t + uv_x = 0$, on $v = \exp(-x^2)$.

$$y = \begin{cases} -v + \exp(-x^2) & \text{for } -a < x < a, \\ 0 & \text{elsewhere.} \end{cases} \tag{3.1b}$$

Let w be a given positive constant. The distributed variable structure control law

$$u = w \operatorname{sign}\left[-v + \exp(-x^2)\right],$$

excercised along each possible characteristic, creates a distributed sliding regime on the manifold $y = 0$ when the controlled motions start from, say, the initial submanifold defined by:

$$\Gamma = \{(v, x) : v = \phi(x)\}$$

where

$$\phi(x) = \begin{cases} \exp\left[-(x-\xi)^2\right] \\ \quad \text{for } -a + \xi < x < a + \xi, \\ 0 \quad \text{elsewhere} \end{cases}$$

with ξ a given positive constant satisfying $\xi > 2a$.

Figures 1 and 2 depict the nature of the sliding regime creation process on $y = 0$ by means of the distributed controlled motions of (3.1).

4. Conclusions and suggestions for further research

The theory of variable structure systems undergoing sliding motions can be easily extended to controlled systems described by first order quasilinear PDE's. The key property of such class of dynamical systems is the possibility of relating properties of their solution to those of a controlled system described by a set of ordinary differential equations (best known as the characteristic equation). This property was used in this article to establish conditions for the local existence of a distributed sliding regime for a quasilinear dynamical system on a given switching surface. A distributed sliding mode locally exists for the distributed system whenever the corresponding controlled characteristic system exhibits such motion on the sliding surface. The given sliding manifold must also qualify as a local integral manifold of an 'equivalent direction field'. The equivalent direction field is the average controlled direction field prescribed by the equivalent control method on the characteristic system. The case of systems described by implicit nonlinear partial differential equations can also be treated from an entirely geometrical viewpoint using the characteristic surfaces defined by the standard contact structure defined on the manifold of 1-jets of the solution function of the nonlinear controlled PDE (see [1]). Such research direction leads to a complete generalization of the results presented here. The geometric theory of second order PDE's could also be taken as a starting point for the adequate treatment of distributed sliding regimes in controlled systems described by such dynamical models.

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