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Exact linearization in switched-mode DC-to-DC power converters

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Three popular DC-to-DC switched mode power converters—Buck, Boost, and Buck-Boost—are shown to be in the same orbit of structural equivalence as a second-order controllable circuit in Brunovsky canonical form. The equivalence is achievable under non-linear feedback and explicit local diffeomorphic state co-ordinate transformations. The implications of local exact linearizability of this class of systems in the design of Variable Structure Feedback Control laws or Pulse-Width-Modulation strategies is thoroughly discussed.

1. Introduction

DC-to-DC switched mode power converters constitute simple yet efficient means of DC power regulation by discontinuous feedback control action. Rather than attempting a complete survey of such a vast field, the reader is referred to the multi-volume series by Middlebrook and Cuk (1981), the books by Severns and Bloom (1985), and by Czaki *et al.* (1983), where detailed background material can readily be found.

The fundamental property of DC-to-DC power converters is their capability for feedback regulation of the output voltage via abrupt topological changes commanded by suitable switching arrangements utilizing diodes and transistors. Traditionally, the feedback control of these circuits was designed by means of Pulse-Width-Modulation (PWM) techniques and only more recently the theory of Variable Structure Systems (VSS) and their associated sliding regimes (Utkin 1978) has been proposed for the control of these systems (see Venkataramanan *et al.* 1985, Bilalovic *et al.* 1983, Sanders *et al.* 1986, and Sira-Ramirez 1987, 1988). In the work by Sira-Ramirez (1988) a general ideal equivalence has been found among Variable Structure Feedback Control (VSFC) laws leading to sliding regimes and PWM control strategies.

Motivated by the exact linearization result given by Sanders *et al.* (1986), in connection with the Buck-Boost converter, we study the equivalence of the Buck, Boost, and Buck-Boost converter circuits under non-linear feedback and local diffeomorphic state coordinate transformation. These circuits are shown to be in the same structural orbit of a second-order controllable linear system in Brunovsky's canonical form. The local linearizability property and the basic limitations of the linearization approach for feedback design of switched converters complete the work started by Sanders *et al.* (1986) and establishes a definitive relationship among the three circuits.

The implications of the exact linearization of the converter circuits is analysed in the context of discontinuous feedback control design and some of the limitations of the proposed scheme are discussed in detail.

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Section 2 is devoted to a brief presentation of the state space models of the three converters. Section 3 presents general considerations about feedback linearization closely following the work by Hunt *et al.* (1983) and Marino and Spong (1986), and then presents the application of the results to the exact linearization of the three converters. Section 4 is devoted to a careful consideration of the possibilities of using the exact linearization results in the design of feedback control strategies for the regulation of the power converters in the context of VSFC and PWM.

2. State space models of DC-to-DC switch mode power converters

2.1. *Buck converter*

Consider the Buck converter, shown in Fig. 1, with state variables defined as $x_1 = I\sqrt{L}$, $x_2 = V\sqrt{C}$, and parameters $w_0 = 1/\sqrt{LC}$, $w_1 = 1/RC$ and $b = E\sqrt{L}$. The *linear* state space model of the controlled circuit is then

$$\left.\begin{aligned} \frac{dx_1}{dt} &= -w_0 x_2 + ub \\ \frac{dx_2}{dt} &= w_0 x_1 - w_1 x_2 \end{aligned}\right\} \tag{2.1}$$

where u is the control variable representing the switch position and taking values in the discrete set $U = \{0, 1\}$.

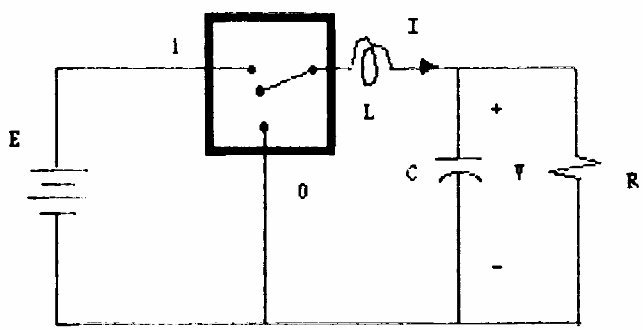

Figure 1. Buck converter circuit.

It has been shown by Sira-Ramirez (1988) that the average response of a switched controlled network to a PWM control strategy with constant duty ratio $0 < \mu < 1$ (or equivalently constant equivalent control μ) can be obtained from the system equations by formally replacing the control variable u by the constant value μ. In particular, the equilibrium point corresponding to such a control strategy is given by

$$x_{2,ss} = \mu b w_0^{-1}, \quad x_{1,ss} = \mu b w_1 w_0^{-2} \tag{2.2}$$

The total steady-state stored energy corresponding to a constant duty ratio is simply

$$0{\cdot}5\{x_{1,ss}^2 + x_{2,ss}^2\} = 0{\cdot}5\left(\frac{\mu b}{w_0}\right)^2\left[1 + \left(\frac{w_1}{w_0}\right)^2\right]$$

Eliminating the parameter μ from (2.2) one demonstrates that the locus of equilibrium points of the system is contained in the straight line

$$x_{2,ss} = w_0 w_1^{-1} x_{1,ss} \tag{2.3}$$

The steady state DC gain is defined as $V_{ss}/E = x_{2,ss}/\sqrt{C}E = \mu$, denoting the 'step down' character of the Buck converter.

2.2. *Boost converter*

Consider now the Boost converter, shown in Fig. 2, where the state variables are again defined as $x_1 = I\sqrt{L}$, $x_2 = V\sqrt{C}$ and the parameters $w_0 = 1/\sqrt{LC}$, $w_1 = 1/RC$ and $b = E/\sqrt{L}$. With u representing the switch position, taking values in $U = \{0, 1\}$, we obtain the following bilinear state-space model for the converter

$$\left.\begin{aligned} \frac{dx_1}{dt} &= -w_0 x_2 + u w_0 x_2 + b \\ \frac{dx_2}{dt} &= w_0 x_1 - w_1 x_2 - u w_0 x_1 \end{aligned}\right\} \tag{2.4}$$

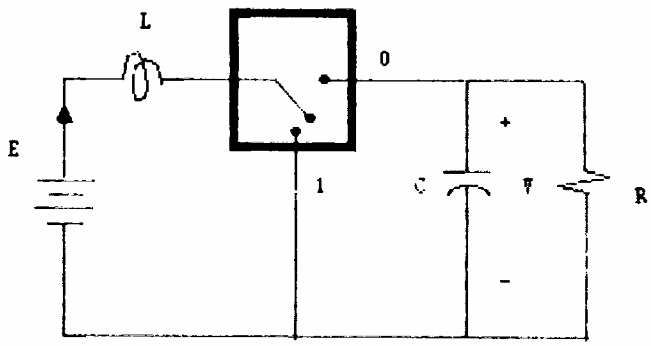

Figure 2. Boost converter circuit.

As before, the equilibrium point obtained from the systems response to a constant duty ratio PWM feedback control (or constant equivalent control) of value $0 < \mu < 1$ is given by

$$x_{1,ss} = w_1 b[(1-\mu)w_0]^{-2}, \quad x_{2,ss} = b[(1-\mu)w_0]^{-1} \tag{2.5}$$

The total stored energy under such steady state conditions is given by

$$0{\cdot}5\{x_{1,ss}^2 + x_{2,ss}^2\} = 0{\cdot}5\left[\frac{b}{(1-\mu)w_0}\right]^2\left[1 + \left(\frac{w_1}{(1-\mu)w_0}\right)^2\right]$$

Elimination of the parameter μ in (2.5) leads to the locus of equilibrium points of the controlled system, contained in this case by the parabola

$$x_{1,ss} = (w_1 b^{-1}) x_{2,ss}^2 \tag{2.6}$$

The steady state DC gain is $V_{ss}/E = 1/(1-\mu)$ indicating the 'step-up' character of the Boost converter. In practice, infinite DC gains are precluded by the non-ideal nature of sources, coils, and condensers.

2.3. *Buck-Boost converter*

Using the same definitions for the state variables and parameters of the previous cases, the dynamical system representing the Buck-Boost converter, shown in Fig. 3, is given by the following bilinear system of differential equations:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= w_0 x_2 + u(-w_0 x_2 + b) \\ \frac{dx_2}{dt} &= -w_0 x_1 - w_1 x_2 + u w_0 x_1 \end{aligned}\right\} \tag{2.7}$$

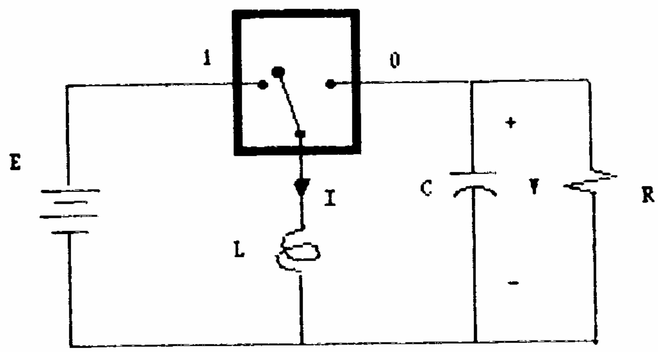

Figure 3. Buck-Boost converter circuit.

The equilibrium point of the constant duty ratio controlled response is given by

$$x_{1,ss} = w_1 \mu b[(1 = \mu) w_0]^{-2}, \quad x_{2,ss} = -\mu b[(1-\mu) w_0]^{-1} \tag{2.8}$$

with $0 < \mu < 1$ being the duty ratio.

The total stored energy in the steady state is given by

$$0{\cdot}5\{x_{1,ss}^2 + x_{2,ss}^2\} = 0{\cdot}5\left[\frac{\mu b}{(1-\mu)w_0}\right]^2 \left\{1 + \left[\frac{w_1}{(1-\mu)w_0}\right]^2\right\}$$

As before, the locus of equilibrium points is contained in an arc of a parabola, given in this case by

$$x_{1,ss} = (w_1 b^{-1})(x_{2,ss} - b w_0^{-1}) x_{2,ss} \tag{2.9}$$

Finally, the DC gain is given by $-\mu/(1-\mu)$ which shows that, modulo a polarity reversal of the input source voltage, the Buck-Boost converter can 'step down' for $\mu \in (0, 0{\cdot}5)$ and 'steps up' for $\mu \in (0{\cdot}5, 1)$.

3. Exact feedback linearization of DC-to-DC converters

3.1. *General results about exact linearization*

Consider the non-linear smooth system defined over an open set X in $\mathbb{R}^n$ containing the origin of local coordinates

$$\frac{dx}{dt} = f(x) + u g(x) \tag{3.1}$$

with f and g smooth (C^∞) vector fields locally defined in X with $f(0) = 0$. The following remarkable result is given by Hunt *et al.* (1983). In the following developments we closely follow the work by Marino and Spong (1986).

Definition 1

System (3.1) is locally exactly linearizable, by non-linear feedback, in a neighbourhood of the origin if there exists a diffeomorphism $T: X \to \mathbb{R}^{n+1}$ such that the change of coordinates $y_i = T_i(x)$, $v = T_{n+1}(x, u)$ results in a linear controllable system in Brunovsky canonical form

$$\frac{dy_i}{dt} = y_{i+1}, \quad i = 1, 2, \ldots, n-1, \quad \frac{dy_n}{dt} = v \tag{3.2}$$

Theorem 1 (Hunt *et al.* 1983, Marino and Spong 1986)

The necessary and sufficient conditions for (3.1) to be locally feedback linearizable are as follows:

(*a*) sp $\{g, \mathrm{ad}\,fg, \ldots, \mathrm{ad}\,f^{n-1}g\} = \mathbb{R}^n$ in X;

(*b*) sp $\{g, \mathrm{ad}\,fg, \ldots, \mathrm{ad}\,f^{n-2}g\}$ is an involutive set (Boothby 1975) of constant rank in X;

where sp is the span of the involved vector fields over the field of smooth functions defined in X

$$\mathrm{ad}\,fg := \left[\frac{\partial g}{\partial x}\right] f - \left[\frac{\partial f}{\partial x}\right] g \quad \text{and} \quad \mathrm{ad}\,f^i g := \mathrm{ad}\,f(\mathrm{ad}\,f^{i-1}g)$$

with ad $f^0 g := g$.

The required change of coordinates is determined from the (non-unique) solution of the set of linear partial differential equations

$$L_g T_1 = 0, \quad L_{\mathrm{ad}fg} T_1 = 0, \quad \ldots, \quad L_{\mathrm{ad}f^{n-2}g} T_1 = 0 \tag{3.3}$$

which is guaranteed to exist in X by virtue of the involutivity condition (*b*) given above (Boothby 1975). In general, $L_h s$ designates the 'Lie derivative' or directional derivative of the smooth scalar function s in the direction of the vector field h. Once T_i is determined, the rest of the components of the transformation, $T_2, \ldots, T_n, T_{n+1}$ are obtained from the recursive relation

$$T_{i+1}(x) = L_f T_i(x), \quad i = 1, 2, \ldots, n-1 \quad \text{and} \quad T_{n+1}(x, u) = L_{f+ug} T_n \tag{3.4}$$

The T_i, $i = 1, 2, \ldots, n$, are independent functions of x satisfying $T_i(0) = 0$ while $T_{n+1}(x, u)$ is of the form $\alpha(x) + u\beta(x)$ with $\beta(x)$ non-singular on X and $\alpha(0) = 0$.

The class of systems (3.1) which are transformable to the same linear system in Brunovsky's canonical form is said to be in the same *structural orbit* or they are in the same *feedback equivalence class.*

3.2. *Exact linearization in DC-to-DC switch-mode power converters*

In this section we shall apply the results of feedback linearization to the converter circuits of § 2. The linearizing transformations will be represented by local diffeomorphisms in which the new coordinates have the meaning of total stored energy and rate of change of stored energy, respectively.

3.2.1. *Transformation of the Buck converter model to canonical form.* The Buck converter differential equations are already linear and controllable, hence they are trivially transformable to Brunovsky's canonical form. For this, simply consider $f = \mathrm{col}\,(-w_0 x_1, w_0 x_1 - w_1 x_2)$, $g = \mathrm{col}\,(b, 0)$. Over the field of smooth functions, the set $\{g, [f, g]\}$—controllability matrix columns—constitutes a linearly independent set of constant rank 2. Solving (3.3)

$$L_g T_1 = \left(\frac{\partial T_1}{\partial x}\right) g = 0$$

leads to $T_1 = T_1(x_2)$. A possible solution is then given by

$$T_1 = x_2, \quad T_2 = w_0 x_1 - w_1 x_2$$
$$T_3 = v = (w_1^2 - w_0^2) x_2 - w_0 w_1 x_1 + u b w_0 = -w_1 T_2 - w_0^2 T_1 + u b w_0 \tag{3.5}$$

hence

$$\frac{dT_1}{dt} = T_2, \quad \frac{dT_2}{dt} = T_3 = v \tag{3.6}$$

$T_i(0) = 0$, $i = 1, 2$, while $T_3(x, u)$ is of the form $\alpha(x) + \beta(x)u$ with $\beta(x) = bw_0$, non-singular, and $\alpha(0) = 0$. In x_1, x_2 coordinates, the set of points in the plane for which $T_2 = 0$ coincides with the locus of equilibrium points (2.3) obtained with constant duty ratio μ in a PWM control scheme.

The above transformation is global but, in general, it is not unique. Indeed taking, for instance, $T_1 = 0{\cdot}5x_1^2$, the stored energy in the capacitor T_2 results in the input power minus the output power (or the rate of change of capacitor stored energy)

$$T_2 = 0{\cdot}5 w_0 x_1 x_2 - 0{\cdot}5 w_1 x_2^2$$

and

$$T_3 = [(2w_1^2 - w_0^2)x_2^2 - 3w_0 w_1 x_1 x_2] + (b w_0 x_2)u$$
$$= 2(3w_1^2 - w_0^2)T_1 - 3T_2 w_1 + b w_0 \sqrt{2T_1}\, u$$

Here T_3 is of the form $\alpha(x) + \beta(x)u$ with $\beta(x)$ singular along $x_2 = 0$. Notice that T, defined this way, is not a global diffeomorphism since it is not globally one-to-one; T_1 and T_2 fail to be functionally independent along $x_2 = 0$ and the transformed system becomes uncontrollable on this line. The use of this transformation makes the system only locally equivalent to a Brunovsky canonical form on, say, the open set $x_2 > 0$.

3.2.2. *Exact linearization of the Boost converter model.* The vector field $f(x)$ in (2.4) does not satisfy the hypothesis of Theorem 1 by which $f(0)$ is to be zero. Hence, a previous linear change of coordinates is to be performed on the system equations

$$z_1 = x_1 - w_1 b w_0^{-2}, \quad z_2 = x_2 - b w_0^{-1} \tag{3.7}$$

This transformation defines input current z_1 and output voltage z_2 measured relative to the steady-state values of the input current and output voltage, respectively, resulting from a fixed position of the switch at $u = 0$. The vector fields are now given by

$$f = \text{col}\,[-w_0 z_2, w_0 z_1 - w_1 z_2]$$
$$g = \text{col}\,[w_n(z_2 + bw_0^{-1}), -w_0(z_1 + w_1 bw_0^{-2})] \tag{3.8}$$

with $f(0) = 0$ as demanded by the theorem.

The set $\{g, [f, g]\}$ is in this case

$$\{g, [f, g]\} = \begin{bmatrix} w_0(z_2 + bw_0^{-1}) & -w_1 w_0(z_2 + bw_0^{-1}) \\ -w_0(z_1 + w_1 bw_0^{-2}) & -w_1 w_0(z_1 + w_1 bw_0^{-2}) - bw_0 \end{bmatrix}$$

which, over the field of smooth functions, is everywhere a linearly independent set of rank 2, except on the straight lines $z_2 = -bw_0^{-1}$ and $z_1 = -w_1 bw_0^{-2} b/(2w_1)$. Since $\{g\}$ is trivially involutive, the system is exactly linearizable by means of *local* diffeomorphic state coordinates transformation and non-linear feedback. The analysis of the necessary and sufficient conditions for exact linearization reveals that the required change of coordinates is not a global diffeomorphism but only a local one, hence linearization is restricted to the open set where the conditions of Theorem 1 are satisfied and therefore the lines specified above are to be excluded from our considerations.

The state coordinate transformation T_1 is obtained from (3.1) as a solution of the partial differential equation

$$\left[\frac{\partial T_1}{\partial z_1}\right] w_0(z_2 + bw_0^{-1}) - \left[\frac{\partial T_1}{\partial z_2}\right] w_0(z_1 + w_1 bw_0^{-2}) = 0 \tag{3.9}$$

Imposing the restriction that $T_1(0) = 0$ we obtain a solution of (3.9) as

$$T_1 = 0{\cdot}5\{[(z_1 + w_1 bw_0^{-2})^2 + (z_2 + bw_0^{-1})^2] - b^2 w_0^{-2}(1 + w_1^2 w_0^{-2})\} \tag{3.10}$$

and using (3.4), T_2 is obtained as

$$T_2 = L_f T_1 = b(z_1 + w_1 bw_0^{-2}) - w_1(z_2 + bw_0^{-1})^2 \tag{3.11}$$

Finally

$$T_3 = v = \left[\frac{\partial T_2}{\partial z}\right](f + ug) = b^2 - bw_0(z_2 + bw_0^{-1}) - 2w_1 w_0(z_1 + w_1 bw_0^{-2})(z_2 + bw_0^{-1})$$
$$+ 2w_1^2(z_2 + bw_0^{-1})^2 + u[b + 2w_1(z_1 + w_1 bw_0^{-2})]w_0(z_2 + bw_0^{-1}) \tag{3.12}$$

Here T_1 represents the total stored energy in the circuit, measured relative to the steady-state value of the total stored energy when $u = 0$; T_2 represents the input power minus the output power, or rate of change of the total stored energy. Furthermore, in x_1, x_2 coordinates, the set of points in the plane for which $T_2 = 0$ coincides with the locus of equilibrium points (2.6) obtained with constant duty ratio μ in a PWM control scheme. This is intuitively clear since in equilibrium, the average total stored energy is to remain constant. Notice that this fact is independent of the value of the duty ratio μ. Notice also that T_3 is of the form $\alpha(z) + \beta(z)u$ with $\beta(z)$ singular over the lines

$$z_1 = -w_1 bw_0^{-2} - b(2w_1)^{-1} \quad \text{and} \quad z_2 = -bw_0^{-1}$$

These lines have already been excluded from our considerations since on them the conditions of Theorem 1 are violated. On these lines the transformed system also becomes uncontrollable and exact linearization cannot be achieved.

Figure 4 shows the level curves of $T_1 = 0$ and $T_2 = 0$ and the region of exact linearization. It is clear that T is not a global diffeomorphism since it is not one-to-one. Due to the local character of the transformation we restrict our attention to the open set

$$z_1 > -w_1 b w_0^{-2} - b(2w_1)^{-1}, \quad z_2 > -b w_0^{-1}$$

On this set the transformation (3.10) and (3.11) is locally invertible, i.e. it is a local diffeomorphism.

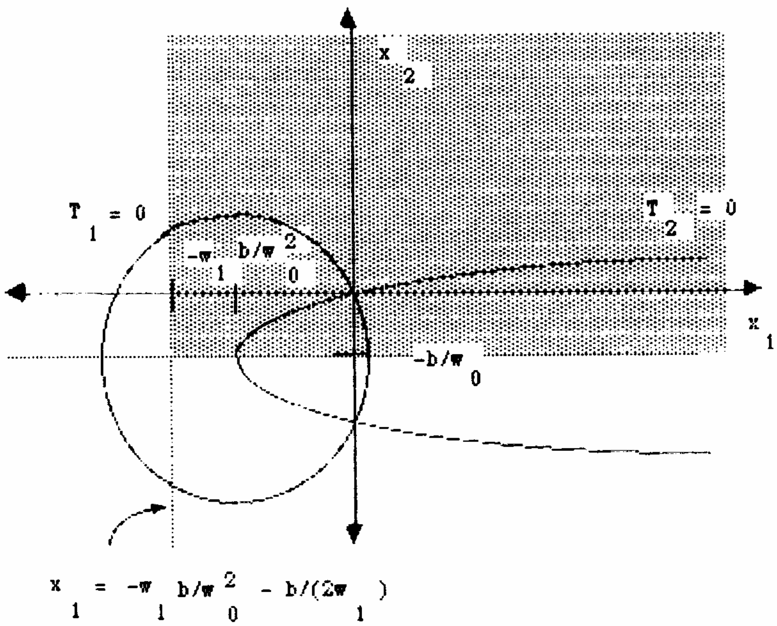

Figure 4. Boost converter curvilinear linearizing coordinates and region of existence of local diffeomorphism.

The transformed system is in Brunovsky canonical form

$$\frac{dT_1}{dt} = T_2, \quad \frac{dT_2}{dt} = T_3 = v \tag{3.13}$$

The equilibrium point (2.5) in new coordinates is represented by

$$\left.\begin{aligned} T_{1,ss} &= \frac{\mu(2-\mu)}{(1-\mu)^2}\frac{b^2}{w_0^2}\left[1 + \frac{w_1^2}{w_0^2}\left(1 + \frac{1}{(1-\mu)^2}\right)\right] \\ T_{2,ss} &= 0 \end{aligned}\right\} \tag{3.14}$$

3.2.3. *Exact linearization of the Buck-Boost converter model.* In this case the vector fields are represented by

$$f = \text{col}\,[w_0 x_2, -w_0 x_1 - w_1 x_2], \quad g = \text{col}\left[-w_0\left(x_2 - \frac{b}{w_0}\right), w_0 x_1\right] \tag{3.15}$$

The vector field f satisfies the condition $f(0) = 0$. Hence, a preliminary change of coordinates is unnecessary in this case. As in the previous case, the necessary and sufficient conditions for the existence of a linearizing transformation are not globally satisfied. Indeed, the set $\{g, [f, g]\}$ is given by

$$\{g, [f, g]\} = \begin{vmatrix} -w_0\left(x_2 - \dfrac{b}{w_0}\right) & w_0 w_1 x_2 \\ w_0 x_1 & w_1 w_0\left(x_1 + \dfrac{b}{w_1}\right) \end{vmatrix} \tag{3.16}$$

which is a linearly independent set of rank 2, everywhere except on the curve

$$(x_2 - bw_0^{-1})(x_1 + bw_1^{-1}) + x_1 x_2 = 0 \tag{3.17}$$

The set of points satisfying (3.17) represent a hyperbola with centre at the point $(-b(2w_1)^{-1}, b(2w_0)^{-1})$ and asymptotes given by the straight lines $x_1 = -b(2w_1)^{-1}$, $x_2 = b(2w_0)^{-1}$. Since $\{g\}$ is a trivially involutive set, the system is exactly linearizable by means of *local* diffeomorphic state coordinates transformation and non-linear feedback.

The necessary and sufficient conditions for exact linearization leads to a required change of coordinates which is not a global diffeomorphism but only a local one. Linearization is thus restricted to an open set of the plane where the conditions of Theorem 1 are satisfied. For this reason the curve specified in (3.17) is to be excluded from our considerations.

The first component of the linearizing transformation T_1 is given by a solution of the partial differential equation

$$L_g T_1 = -w_0(x_2 - bw_0^{-1})\frac{\partial T_1}{\partial x_1} + w_0 x_1 \frac{\partial T_1}{\partial x_2} = 0 \tag{3.18}$$

A solution satisfying $T_1(0) = 0$ is given by

$$T_1 = 0{\cdot}5\{x_1^2 + (x_2 - bw_0^{-1})^2 - b^2 w_0^{-2}\} \tag{3.19}$$

From (3.19) and (3.4) one immediately obtains T_2 as

$$T_2 = L_f T_1 = bx_1 - w_1 x_2(x_2 - bw_0^{-1}) \tag{3.20}$$

Finally, T_3 is given by

$$\begin{aligned} T_3 = L_{f+ug} T_2 = &-(1 - w_1^2 w_0^{-2}) b w_0 x_2 - b w_1 x_1 + 2 w_1 w_0 x_1 x_2 + 2 w_1^2 x_2^2 \\ &+ u[(x_2 - bw_0^{-1})(x_1 + bw_1^{-1}) + x_1 x_2] \end{aligned} \tag{3.21}$$

Here T_1 represents the total stored energy in the circuit, with the capacitor voltage measured modulo its steady-state value when a duty ratio $\mu = 0{\cdot}5$ is used in a PWM control scheme; T_2 represents the corresponding rate of change of the total stored energy. As before, the set of points in the plane for which $T_2 = 0$ coincides with the locus of equilibrium points (2.6) obtained with constant duty ratio μ in a PWM control scheme. This is intuitively clear since in equilibrium, the average total stored energy is to remain constant. Notice that this fact is independent of the value of the duty ratio μ. Notice also that T_3 is of the form $\alpha(z) + \beta(z)u$ with $\beta(z)$ singular over the curve (3.17). This curve has already been excluded from our considerations since on it the conditions of Theorem 1 are violated. On this hyperbola the transformed system also becomes uncontrollable and exact linearization cannot be achieved.

Figure 5 shows the region of existence of exactly linearizing transformations and some level curves for T_1 and T_2. Here T is not a global diffeomorphism since it is not one-to-one. Due to the local character of the transformation we restrict our attention to the open set $x_1 > -b(2w_1)^{-1}$, $x_2 < b(2w_0)^{-1}$. On this set, the state coordinate transformation (3.19)–(3.21) is locally invertible, i.e. it is a local diffeomorphism.

The transformed system is in Brunovksy canonical form

$$\frac{dT_1}{dt} = T_2, \quad \frac{dT_2}{dt} = T_3 = v \tag{3.22}$$

The equilibrium point (2.8) in new coordinates is represented by

$$\left.\begin{aligned} T_{1,ss} &= -\frac{\mu}{(1-\mu)^2}\frac{b^2}{w_0^2}\left[(2-\mu) + \left(\frac{w_1}{w_0}\right)^2 \frac{\mu}{(1-\mu)^2}\right] \\ T_{2,ss} &= 0 \end{aligned}\right\} \tag{3.23}$$

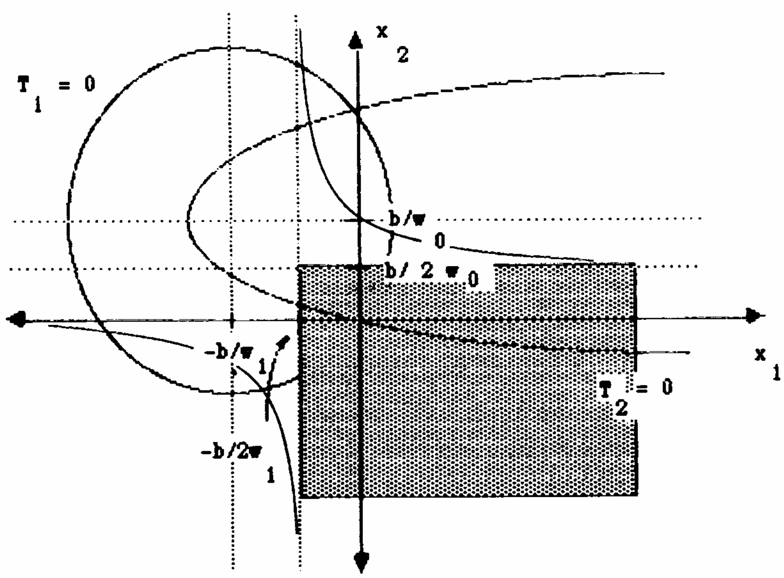

Figure 5. Buck-Boost converter curvilinear linearizing coordinates and region of existence of local diffeomorphism.

4. Variable structure control of linearized converter models

4.1. *Generalities about discontinuous control of linearized converters*

The common model of the three basic converters we have studied is represented by a simple double integrating plant with control input v. In this section we discuss the basic limitations of using the linearized model for the design of feedback stabilizing control strategies for the above class of switch-controlled structurally equivalent systems.

In conventional non-linear controller design through feedback linearization (Meyer and Cicolani 1980) the input v represents an *independent* control input which can be used directly for the specification of a *linear* feedback control policy or as an *outer loop* robust feedback path (Spong and Sira-Ramirez 1986). Once this simple law is synthesized on the basis of the Brunovsky model (by using either pole assignment or optimal control considerations), then the inverse transformation of the control input coordinates generates, in the original control coordinates, a non-linear stabilizing feedback design. However, this is not, in general, the case in switched controlled systems. The crucial observation is that the original non-linear system model assumes that the control input u takes values in the finite set $U = \{0, 1\}$ as it represents the action of an ideal switching element. The control input transformation, specified by

$$T_3 = v = \alpha(x) + u\beta(x)$$

severely restricts the new input v to take values only on a set of two possible non-linear feedback functions: $\alpha(x)$ or $\alpha(x) + \beta(x)$. Hence all feedback control policies corresponding to the simplified design carried on the basis of the models (3.6), (3.13) or (3.22) have to be restricted merely to the specification of a *switching logic* among the two fixed (non-linear) feedback paths represented by $\alpha(x)$ and $\alpha(x) + \beta(x)$. In transformed coordinates T_1, T_2 this equally amounts to two (non-linear) feedback paths of the form $a(T_1, T_2)$ and $a(T_1, T_2) + b(T_1, T_2)$, where a and b are, respectively, the composition of the functions α and β with the corresponding components of the linearizing transformation T. This crucial limitation of the control input v is responsible for the *local* nature of the sliding regime.

The specification of a feedback control law for the Brunovsky model (3.6), (3.13) or (3.22) reduces then to find a switching logic among feedback paths $a(T_1, T_2)$ and $a(T_1, T_2) + b(T_1, T_2)$ such that the controlled trajectories, in transformed coordinates, exhibit a desirable stable motion towards equilibrium. This can be accomplished either by the specification of a *duty ratio*, in a PWM control strategy, or a *sliding surface* with prescribed geometrical properties, within a Variable Structure Control approach. Sira-Ramirez (1988) has shown that both approaches, PWM and VSC, are generally equivalent. Integral manifolds of PWM controlled non-linear systems qualify as sliding manifolds for VSC policies while the equivalent control, associated with any VSC policy, coincides with the corresponding duty ratio which makes of the sliding surface an integral manifold of the PWM controlled system. The equivalence has rigorously been shown to hold true in general under the assumption of high frequency switchings.

The necessary and sufficient conditions for the existence of a sliding regime (equivalently for the existence of an integral manifold of the average PWM controlled system) for the above transformed class of switched systems result, in general, in a *local* sliding regime exclusively determined by the specification of the surface (or the duty ratio). The computation of the region of existence is cumbersome, even in the simplest example, and hence the simplifying nature of the exact linearization approach is lost. In the next few paragraphs we consider in detail the sliding mode approach. The corresponding considerations for the PWM control follow along similar lines.

4.2. *VSC control of linearized converters*

Consider the second-order Brunovsky model with non-linear feedback

$$\frac{dT_1}{dt} = T_2, \quad \frac{dT_2}{dt} = v, \quad v = a(T_1, T_2) + ub(T_1, T_2), \quad u \in \{0, 1\} \tag{4.1}$$

Suppose a stable motion is to be induced on the above system such that a prespecified equilibrium point $T_{1,ss}$ is to be reached asymptotically. If we let the sliding surface be of the form

$$S = \{(T_1, T_2) : s = T_2 + m(T_1 - T_{1,ss}) = 0, m > 0\} \tag{4.2}$$

then under ideal sliding conditions ($s = 0, ds/dt = 0$) the governing equation of the *equivalent system* (Utkin 1978) is

$$\frac{dT_1}{dt} = -m(T_1 - T_{1,ss}) \tag{4.3}$$

i.e. an asymptotically exponentially stable motion towards equilibrium is achieved for the total stored energy of the system. The corresponding equivalent control, ideally responsible for this motion, is obtained from the above ideal sliding conditions as

$$v_{EQ} = m^2(T_1 - T_{1,ss}) \tag{4.4}$$

The necessary and sufficient conditions for the existence of a sliding regime on S (Utkin 1978) are simply that on $s = 0$ we have

$$\begin{aligned} \min\{a(T_1, T_2), a(T_1, T_2) + b(T_1, T_2)\} < v_{EQ} &= m^2(T_1 - T_{1,ss}) \\ &= a(T_1, T_2) + u_{EQ} b(T_1, T_2) \\ &< \max\{a(T_1, T_2), a(T_1, T_2) + b(T_1, T_2)\} \end{aligned} \tag{4.5}$$

or equivalently

$$0 < u_{EQ} = [m^2(T_1 - T_{1,ss}) - a(T_1, T_2)]b^{-1}(T_1, T_2)|_{s=0} < 1 \tag{4.6}$$

In original coordinates the regions of existence, along the non-linear sliding manifold

$$S = \{x \in \mathbb{R}^2 : s = T_2(x) + mT_1(x) = 0, m > 0\},$$

are represented by disjoint open sets given by

$$\beta(x_1, x_2) > 0, \quad \alpha(x) < m^2[T_1(x) - T_{1,ss}] < [\alpha(x) + \beta(x)]$$

and

$$\beta(x_1, x_2) < 0, \quad \alpha(x) > m^2[T_1(x) - T_{1,ss}] > [\alpha(x) + \beta(x)] \tag{4.7}$$

The existence of a sliding mode is thus locally restricted to the region of validity of the above inequalities which in turn must be intersected with the region of validity of the linearizing transformations. In the case of the converters, physical reasons related to the location of the point of equilibrium make one select the region $\beta(x_1, x_2) > 0$ for the Buck and Boost cases, while $\beta(x_1, x_2) < 0$ for the Buck-Boost case.

The particular values of $\alpha(x)$, $\beta(x)$ and the corresponding regions of existence of a sliding motion along the stabilizing sliding curve containing the equilibrium point $T_{1,ss}$ can now be obtained from the form of the input coordinate space transformation $T_3(x)$ for each converter circuit, as given by formulas (3.5), (3.12), and (3.21).

5. Conclusions and suggestions for further research

The existence of linearizing locally diffeomorphic transformations of traditional state-space coordinates for three basic DC-to-DC power supplies has been shown to render this class of systems as structurally equivalent to a second-order integrating plant. In all cases the transformed coordinates T_1, T_2 are consistently interpreted as the total stored energy and the input minus the output power or rate of change of the total stored energy. The transformation of the input coordinate space, represented by a topological change of the circuit structure commanded by a controlled switch, leads to a finite set of non-linear control laws. A switching logic for the ideal switch is to be chosen in order to obtain a desirable stable motion of the transformed state-space trajectory. The design problem can therefore be considered in a unified fashion for the three converters by solving a discontinuous non-linear feedback control problem in the linearized coordinates. The synthesis problem is thus restricted to finding the appropriate switching law by means of either Variable Structure Systems theory (and their associated sliding regimes) or Pulse Width Modulations control strategies. The resulting design problem is conceptually simple but computationally cumbersome. Aside from the local nature of the linearizing transformation, which can be fully justified on physical grounds, the existence of a sliding motion is also local. Within this context, we have examined the limitations of the exact feedback linearization approach in the control of switched power supplies. Other bilinear converters, such as the Cuk converter and their useful variations remain to be studied from this viewpoint.

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