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Pseudolinearization in DC-to-DC power supplies

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A method is proposed for designing regulating feedback control laws in switched-mode DC-to-DC power supplies such as the boost converter. A physically meaningful local state coordinate transformation is proposed on the average pulse-width-modulated controlled converter which results in a perturbed linear tangent model independent of the operating point. Such a model exhibits a linear structure in Brunovsky's canonical form. Conceptually, the feedback regulation scheme is thus considerably simplified.

1. Introduction

Pulse-width-modulation (PWM) control techniques or, alternatively, variable structure feedback (VSF) strategies constitute popular means of feedback regulation for switched controlled bilinear networks (see Venkataramanan *et al.* 1985, Sira-Ramirez 1987 a,b, Sanders *et al.* 1986, Sira-Ramirez and Ilic-Spong 1988, Severns and Bloom 1985, Middlebrook and Cuk 1981).

In Sira-Ramirez (1987 b), a general relationship was established among average PWM controlled responses and ideal VSF controlled trajectories. This equivalence asserts that integral manifolds of average PWM controlled networks qualify as sliding surfaces on which the corresponding equivalent control (Utkin 1978) coincides with the duty ratio associated with the PWM control scheme. In general, average PWM models are obtained from a non-linear system model simply by substituting the switch position function by the duty ratio. Similarly, the ideal sliding dynamics are obtained by replacing the same control function by the equivalent control. The underlying assumption made in obtaining both average models is the high (infinite) frequency switching assumption.

In this article, the feedback control of bilinear switched networks such as the Boost DC-to-DC power converter is examined from the perspective of pseudolinearization (Reboulet and Champetier, 1984). Pseudolinearization of the non-linear discontinuous model is not possible, since the switch control function takes values on a discrete set and its values cannot be perturbed. For this reason, the average PWM controlled network is used and its tangential behaviour examined on a controllable submanifold of stable operating points. A state coordinate transformation, expressible in terms of average stored energy and average consumed power, is found which turns the average perturbed model into a Brunovsky canonical form, independent of the operating point. The specification of a stabilizing feedback loop, whose control action regulates the duty ratio around its nominal operating value, is based on the linearized tangent model. A similar, but conceptually different, approach was taken by Sanders *et al.* (1986) for the variable structure control of the boost converter using feedback linearization (see Hunt *et al.* 1983).

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Section 2 presents the pseudolinearization of the boost converter. Section 3 develops a feedback control method based on the linearized perturbed model. Section 4 contains the conclusions of the article.

2. Method of pseudolinearization

The following presentation closely follows that of Reboulet and Champetier (1984).

Consider the bilinear single-input system

$$\dot{x} = Ax + vBx \tag{2.1}$$

with $x \in R^n$, Ax and Bx are smooth local vector fields defined on an open neighbourhood X of R^n and v is a, possibly discontinuous, scalar control function $v: R^n \to R$. The set of stable operating points is expressed as

$$C_{x,v} := \{(x_0, v_0) \in R^{n+1} \text{ s.t. } [A + v_0 B]x_0 = 0\} \tag{2.2}$$

with local projection on X given by

$$C_x = \{x_0 \in R^n \text{ s.t. } \exists\, v_0 \text{ s.t. } [A + v_0 B]x_0 = 0\} \tag{2.3}$$

In the neighbourhood of an operating point (x_0, v_0), the dynamic behaviour of the system is approximately linear and described by

$$\delta\dot{x} = [A + v_0 B]\,\delta x + [Bx_0]\,\delta v \tag{2.4}$$

where $\delta x \in R^n$ and $\delta v \in R$. One assumes that for any $(x_0, v_0) \in C_{x,v}$, the pair $[A + v_0 B, Bx_0]$ is controllable.

It was shown by Reboullet and Champetier (1984) that, for a given v, the set of operating points $\mathscr{C}_{x,v}$, in the neighbourhood of which the linear tangent model of system (2.1) is controllable, constitutes a one-dimensional submanifold of R^{n+1} and it is open in $C_{x,v}$. $\mathscr{C}_x$ denotes the projection of $\mathscr{C}_{x,v}$ on the x-state space. Evidently, $\mathscr{C}_x$ is contained in C_x. Notice that, necessarily, at every point $(x_0, v_0) \in \mathscr{C}_{x,v}$, the vector $Bx_0 \neq 0$.

Lemma 2.1 (Reboullet and Champetier 1984)

Let there exist a local C^1-diffeomorphism $\phi = (\phi_1, \ldots, \phi_n)^{\mathrm{T}}$ from R^n into R^n such that $\mathscr{C}_{x,v}$ is given by $\phi_2(x) = \ldots = \phi_n(x) = 0$, then for any 1-form α given along $\mathscr{C}_{x,v}$ there locally exists a function T of class C^1 such that

$$\alpha = dT|_{\mathscr{C}_{x,v}} \tag{2.5}$$

i.e. the 1-form α is locally integrable along $\mathscr{C}_{x,v}$.

Theorem 2.2 (Reboullet and Champetier 1984)

Let the non-linear system (2.1) have a controllable tangent model (2.4) along $C_{x,v}$ and let $\mathscr{C}_{x,v}$ be given as in Lemma 2.1 for some C^1-diffeomorphism ϕ, then there exist mappings $z = T(x)$, $v = T_{n+1}(x, v) = \alpha(x) + \beta(x)v$, with $\beta(x) \neq 0$ in $\mathscr{C}_x$, such that the tangent model of the transformed system in the z-state space

$$\left.\begin{aligned} \delta\dot{z}_i &= \delta z_{i+1}, \quad i = 1, 2, \ldots, n-1 \\ \delta\dot{z}_n &= \delta v \end{aligned}\right\} \tag{2.6}$$

is independent of the operating point.

To obtain the tangent model (2.6) the following equations must be locally satisfied on $C_{x,v}$.

$$\alpha_1[A + v_0 B]^{i-1} B x_0 = 0, \quad i = 1, 2, \ldots, n-1 \tag{2.7}$$

$$\alpha_i = \alpha_1[A + v_0 B]^{i-1}, \quad i = 2, 3, \ldots, n \tag{2.8}$$

$$\alpha_{n+1} = \alpha_n[A + v_0 B, B x_0] \tag{2.9}$$

The problem then becomes one of finding 1-forms $\alpha_1, \ldots, \alpha_n$ (and α_{n+1}) satisfying equations (2.7)–(2.9) at any point of C_x (and of $C_{x,v}$) such that there exist mappings $T_i(x)$ ($i = 1, 2, \ldots, n$) and $T_{n+1}(x, v)$ for which

$$\left.\begin{aligned} \alpha_i &= dT_i|_{C_x}, \quad i = 1, 2, \ldots, n \\ \alpha_{n+1} &= dT_{n+1}|_{C_{x,v}} \end{aligned}\right\} \tag{2.10}$$

The procedure for finding the linearizing transformations consists of three basic steps:

(i) choose any arbitrary covector α_1 along the direction determined by (2.7);

(ii) compute $\alpha_2, \ldots, \alpha_n$ from (2.8) and (2.9);

(iii) integrate α_i ($i = 1, 2, \ldots, n$) along C_x and α_{n+1} along $C_{x,v}$ to obtain the linearizing mappings $T_1, \ldots, T_{n+1}$.

3. Pseudolinearization of the boost converter model

In this section the problem is posed of obtaining a linear tangent model of the average PWM controlled DC-to-DC switchmode power supply such as the boost converter. This network is represented in state space by a bilinear dynamical model (Wood 1974, Sira-Ramirez 1987 a). A linearized model, in Brunovsky canonical form, is obtained for the boost converter that is independent of the operating point. The set of equilibrium points $C_{x,v}$, C_x and the sets $\mathscr{C}_{x,v}$ and $\mathscr{C}_x$ are clearly identified and the linearizing transformation is readily obtained following the procedure indicated in § 2. These linearizing transformations turn out to have an interesting physical meaning in terms of the total average stored energy and the total average consumed power.

Consider the boost converter model shown in Fig. 1

$$\left.\begin{aligned} \dot{y}_1 &= b - \omega_0 y_2 + u\omega_0 y_2 \\ \dot{y}_2 &= \omega_0 y_1 - \omega_1 y_2 - u\omega_0 y_1 \end{aligned}\right\} \tag{3.1}$$

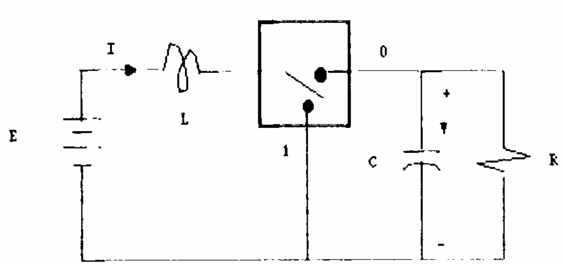

Figure 1. Boost converter.

where $y_1 = \sqrt{L}i$, $y_2 = \sqrt{C}V$, $b = E/\sqrt{L}$, $\omega_0 = 1/\sqrt{LC}$, $\omega_1 = 1/RC$ and u denotes the switch position function, acting as a control input, and taking values in the discrete set $U = \{0, 1\}$. The vector fields f and g are given by $f = (b - w_0 y_2)\partial/\partial y_1 + (w_0 y_1 - w_1 y_2)\partial/\partial y_2$, $g = w_0 y_2 \partial/\partial y_1 - w_0 y_1 \partial/\partial y_2$ where $\partial/\partial y_1$ and $\partial/\partial y_2$ are the unit direction vectors in the tangent space of R^2.

An idealized PWM control strategy, is defined as one in which the switchings to the $u = 1$ position are assumed to occur at the beginning of each period known as the duty cycle and turning to the $u = 0$ position once within the duty cycle according to a switching policy determined by a smooth feedback function of the average state vector x, known as the duty ratio, and denoted by $v(x)$. The duty cycle is assumed to be periodical with infinitesimally small period (i.e. infinitely large frequency) and the duty ratio is the fraction of the duty cycle on which the switch position is at $u = 1$. Hence $0 < v(x) < 1$ (see Fig. 2). Under the high frequency assumption, it was rigorously demonstrated by Sira-Ramirez (1987 b) that the average model is given by the non-linear model

$$\left.\begin{aligned} \dot{x}_1 &= b - [1 - v(x)]\omega_0 x_2 \\ \dot{x}_2 &= [1 - v(x)]\omega_0 x_1 - \omega_1 x_2 \end{aligned}\right\} \tag{3.2}$$

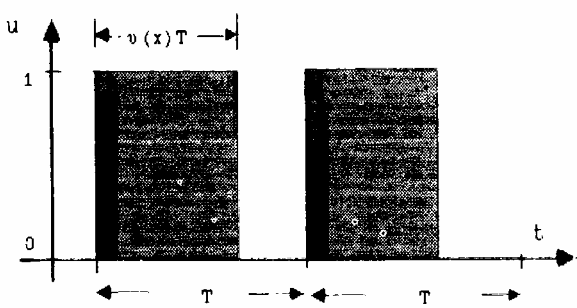

Figure 2. Typical duty cycle and duty ratio.

If an equilibrium point x_0 exists then $v_0 = v(x_0)$ is a constant scalar satisfying $0 < v_0 < 1$. The equilibrium point of (3.2), if it exists for the given $v(x)$, is given by

$$x_{10} = \omega_1 b[\omega_0(1 - v_0)]^{-2}, \quad x_{20} = b[\omega_0(1 - v_0)]^{-1} \tag{3.3}$$

Then, eliminating v_0 in (2.13), it follows from the definitions of § 2 that the sets $C_{x,v}$ and C_x are given by

$$C_{x,v} = \{(x_0, v_0) \in R^3 : x_{10} = \omega_1 b^{-1} x_{20}^2, 0 < v_0 < 1\} \tag{3.4}$$

$$C_x = \{x_0 \in R^2 : \exists\, v_0 \in (0, 1) \text{ in } R \text{ s.t. } x_{10} = \omega_1 b^{-1} x_{20}^2\} \tag{3.5}$$

while the sets

$$\begin{aligned} \mathscr{C}_{x,v} = \{(x_0, v_0) \in R^3 : x_{10} = \omega_1 b^{-1} x_{20}^2, \omega_1 b\omega_0^{-2} < x_{10} < \infty \\ b\omega_0^{-1} < x_{20} < \infty; 0 < v_0 < 1\} \end{aligned} \tag{3.6}$$

$$\begin{aligned} \mathscr{C}_x = \{x_0 \in R^2 : \exists\, v_0 \in (0, 1) \text{ in } R \text{ s.t. } x_{10} = \omega_1 b^{-1} x_{20}^2 \\ \omega_1 b\omega_0^{-2} < x_{10} < \infty, b\omega_0^{-1} < x_{20} < \infty\} \end{aligned} \tag{3.7}$$

These sets are all shown in Fig. 3.

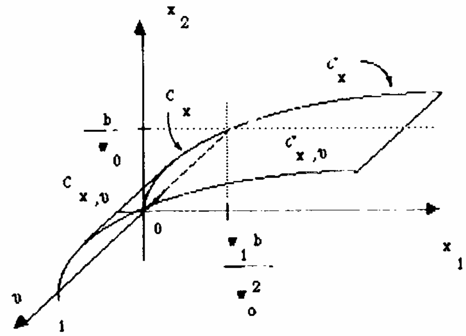

Figure 3. Submanifold of controllable stable operating points in R^3 and its projections in R^2 for the boost converter.

In the neighbourhood of an operating point (x_0, v_0) the dynamic behaviour of the average system (3.2) may be considered as linear in the form (2.4) with

$$A + v_0 B = \begin{vmatrix} 0 & -(1-v_0)\omega_0 \\ (1-v_0)\omega_0 & -\omega_1 \end{vmatrix}, \quad Bx_0 = \begin{vmatrix} \omega_0 & x_{20} \\ -\omega_0 & x_{10} \end{vmatrix} \tag{3.8}$$

The pair $[A + v_0 B, Bx_0]$ is controllable everywhere in R^2 except on $x_2 = 0$ and $x_1 = -b(2\omega_1) - 1$. None of these uncontrollability sets intersects $\mathscr{C}_x$. Notice that, from (3.5), C_x and $\mathscr{C}_x$ are contained by the level set $\phi_2(x) = 0$ with ϕ_2 being a local C^1-diffeomorphism given by

$$\phi_2(x) = bx_1 - \omega_1 x_2^2 = 0 \tag{3.9}$$

Hence any 1-form α given along $\mathscr{C}_x$ is integrable.

Let α_i be 1-forms over $\mathscr{C}_x$, $\alpha_i = dT_i|_{\mathscr{C}_x}$, $i = 1, 2$. To obtain a linear tangent model, independent of the operating point the α_i must satisfy, on $\mathscr{C}_{x,v}$, equations (2.7)–(2.9).

According to (2.7), $\alpha_1 g(x) = 0$ on $\mathscr{C}_x$. This results in $\alpha_1 = x_{10}\, dx_1 + x_{20}\, dx_2$ from where

$$z_1 = T_1(x) = 0{\cdot}5(x_1^2 + x_2^2) \tag{3.10}$$

Using (2.8) and the expression (3.8) we obtain after some manipulations involving the expressions for the equilibrium points (3.3) that the 1-form $\alpha_2 = \alpha_1[A + v_0 B] = b\, dx_1 - 2\omega_1 x_{20}\, dx_2$. Integrating this expression along C_x we obtain T_2 as

$$z_2 = T_2(x) = bx_1 - \omega_1 x_2^2 \tag{3.11}$$

Finally, using (2.9) $\alpha_3 = \alpha_2[A + v_0 B, Bx_0] = -2(1 - v_0)\omega_0\omega_1 x_{20}\, dx_1 + [-b(1 - v_0)\omega_0 + 2\omega_1^2 x_{20}]\, dx_2 + [b\omega_0 x_{20} + 2\omega_1\omega_0 x_{10} x_{20}]\, dv$, one obtains after integration and further straightforward manipulations

$$\begin{aligned} v = z_3 = T_3(z, v) &= (b^2 - b\omega_0 x_2 - 2\omega_1\omega_0 x_1 x_2 + 2\omega_1^2 x_2^2) \\ &+ (b\omega_0 x_2 + 2\omega_1\omega_0 x_1 x_2)v =: \alpha(x) + \beta(x)v \end{aligned} \tag{3.12}$$

The transformation $T_1(x)$ represents the total average stored energy by the circuit while its rate of change, $T_2(x)$, is simply the average input power minus the average output power, which we term the 'total average consumed power'. T_3 is then the rate of change of the total average consumed power.

Evidently

$$\dot{T}_1 = T_2, \quad \dot{T}_2 = T_3 := v \quad \text{or} \quad \dot{z}_1 = z_2, \quad \dot{z}_2 = z_3 \equiv v \tag{3.13}$$

i.e. the system is in Brunovsky's controllable canonical form. The tangent model to (3.13) is thus independent of the operating point, and in z-state space such model is expressed as

$$\delta\dot{z}_1 = \delta z_2, \quad \delta\dot{z}_2 = \delta z_3 \equiv \delta v \tag{3.14}$$

The inverse transformation $x = T^{-1}(z)$ does not globally exist in R^3, thus indicating the local character of the diffeomorphism T. However, in the region of interest, $\mathscr{C}_x$, T is indeed a local diffeomorphism with inverse computed as

$$x_1 = b(2w_1)^{-1}[1 + (2w_1 b^{-1})^{-2}(2z_1 + w_1^{-1}z_2)]^{1/2}$$

$$x_2 = \{b^2(2w_1^2)^{-1}\{[1 + (2w_1 b^{-1})^{-2}(2z_1 + w_1^{-1}z_2)]^{1/2} - 1\} - w_1^{-1}z_2\}^{1/2} \tag{3.15}$$

Notice that in the z-state space a controller design for stabilization around a desirable transformed equilibrium point (z_1^*, z_2^*) is very simple. First of all, under steady state conditions $z_2^* = 0$. Secondly, for a given set point z_1^*, the required feedback control law for (3.13) is $v(z) = -m_1(z_1 - z_1^*) - m_2 z_2$ with $m_1, m_2 > 0$. The closed-loop system $\dot{z}_1 = z_2$, $\dot{z}_2 = -m_1(z_1 - z_1^*) - m_2 z_2$, is made asymptotically exponentially stable towards $(z_1^*, 0)$ due to freely assignable stable eigenvalues. Once the suitable control law $v(z)$ is designed then the required controller ν is obtained from (3.12) as

$$\nu = \beta^{-1}(x)[v(z) - \alpha(x)]|x = T^{-1}(z) = \beta^{-1}(T^{-1}(z))[v(z) - \alpha(T^{-1}(z))] \tag{3.16}$$

4. Regulation via the average pseudolinearized model

A control scheme, based on the method of dynamic inverses (Meyer and Cicolani 1980), is proposed in this section for the feedback control of DC-to-DC power converters of the boost type. Using the pseudolinearization results of § 3, a simplified regulator is specified which generates a corrective control in a servo-model-following scheme. The comparison among the linearized plant, as seen through the 'trimmap', and the servo-model driven by an external reference input, or setpoint, defines a set of error signals which drive the simplified linear regulator and obtains the required correction term.

The feedback regulation scheme of Fig. 4 synthesizes, after appropriate inversion, a desirable duty ratio geared to achieve desirable steady state stability of the equilibrium point of the average model. The obtained smooth duty ratio is translated then into an appropriate sequence of on-off pulses by means of a PWM. The observed state variables of the plant are processed through a low pass filter to obtain the average (i.e. chattering-free), smooth values of the state functions. The actual states of the regulated plant undergo small, high frequency oscillations around the regulated equilibrium point.

Figure 4 shows the control scheme (see Meyer and Cicolani 1980) for the boost converter. Here x_1^* represents a desirable average value for the steady state input current acting as a set point. Notice that for a given average steady state input current x_1^*, the corresponding capacitor voltage x_2^* is fixed through (3.5) or (3.7). The steady state set point in x-state space must be translated, by means of $T(x)$, to a corresponding one in z-state space where the component z_2^* is invariably zero according to (3.9) and (3.11).

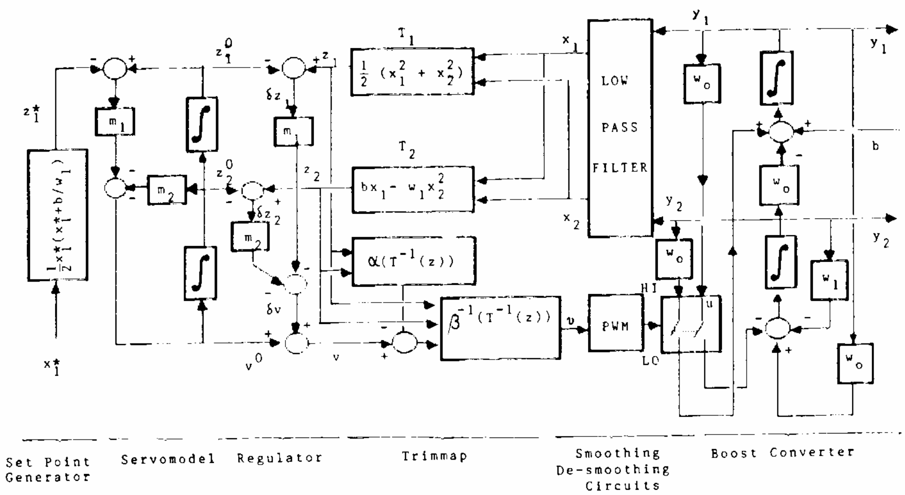

Figure 4. Feedback control scheme for the boost converter.

5. Conclusions and suggestions for further research

A pseudolinearization approach has been applied for the regulator design of DC-to-DC switchmode power supplies of a bilinear nature such as the boost converter. The integration of appropriately chosen 1-forms along smooth one-dimensional controllable steady state submanifolds of the average PWM controlled converter leads to the specification of a local linearizing diffeomorphic state coordinate transformation. This transformation, besides being physically meaningful in terms of total average stored energy and total average consumed power, yields a controllable tangent model in Brunovsky's canonical form. The independency of the tangent model with respect to the average operating point allows the design of a general, smooth feedback control scheme specifying the desirable duty ratio. Such a duty ratio is computed accordingly to an externally specified set point value for the steady state input current (or, equivalently, steady state output voltage). The practical implementation of the ideas reported here are thus far encouraging. They will be fully reported elsewhere.

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