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Global sliding motions on compact manifolds

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Within the context of non-linear variable-structure controlled systems, a general geometric characterization is introduced for the global existence of sliding motions on compact manifolds. As a necessary condition for the existence of global sliding motions, sign conditions are derived for the volume integrals of the divergence of the available feedback controlled vector fields. The ideal sliding dynamics are characterized in terms of a volume preserving flow or, equivalently, by a total zero-divergence smoothly controlled vector field. Several illustrative examples are given throughout the article.

1. Introduction

The flow map associated with a vector field defining a smooth non-linear dynamical system continuously transforms compact regions of the state-space into equally compact regions of a rather complicated nature. However, at each instant of time, the rate of change of volume of such an evolving region is equal to the volume integral over the transformed region of the divergence of the vector field. Such a fundamental result (Arnold 1985, p. 198) constitutes a stronger version of Liouville's theorem for linear systems by which the determinant of the transition matrix equals the integral of the trace of the infinitesimal generator (Arnold 1985, p. 195)—see also Arnold (1978, p. 69, Lemma 1). Using this fundamental result, a general geometric characterization of global sliding regimes is proposed, in this article, for non-linear variable structure systems (VSS), defined in $\mathbb{R}^n$ (Utkin 1978, Itkis 1976), which adopt smooth manifolds bounding compact regions of $\mathbb{R}^n$ as sliding surfaces (henceforth such manifolds are called 'compact manifolds'). For general background on VSS, readers are referred to De Carlo *et al.* (1988), Utkin (1987), or Sira-Ramirez (1988).

The necessary and sufficient conditions for the existence of a sliding regime on a compact manifold are translated into set inclusion conditions on the set-valued flow map generated by each possible structure of the controlled vector field. A simple necessary condition for the global existence of a sliding regime is derived from this alternative, but particular, characterization. The condition involves a difference in sign of the rate of change of the controlled volume for each available feedback structure, i.e. a difference in sign of the volume integral of the divergence of the controlled vector field for each possible feedback structure. Smooth responses obtained from formal application of the equivalent control (Utkin 1978) to the original controlled dynamics results in having the flow map, associated with the ideal sliding dynamics, preserve the volume of the compact region.

Section 2 contains some background definitions and general results about sliding motions on compact manifolds. Illustrative examples are given throughout the section. Section 3 contains the conclusions of the paper.

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2. Basic definitions and main results

2.1. *Background results*

Consider a non-linear dynamical system, defined on $\mathbb{R}^n$ by

$$\frac{dx}{dt} = X(x, u) \tag{2.1}$$

with $u = u(x)$ a scalar, possibly discontinuous, feedback control function and X being a smooth vector field for each given smooth u.

Definition 1

Let u be a given control function, the 'flow' generated by the controlled vector field $X(x, u)$ is the one-parameter group of transformations g_u^t of $\mathbb{R}^n$ such that $g_u^t: x(0) \to x(t)$, where $x(t)$ is a solution at time t of (2.1) for the given u. The vector field $X(x, u)$ is called the 'generating field' of g_u^t.

Example 1

Consider $dx_1/dt = x_2, dx_2/dt = -x_1 + u$. With $u = 0$, the flow g_0^t, generated by the uncontrolled vector field $x_2\, \partial/\partial x_1 - x_1\, \partial/\partial x_2$, is constituted by the group $S_0(2)$ of rigid clockwise rotations in $\mathbb{R}^2$.

Example 2

Consider $dx_1/dt = \lambda_1 x_1$, $dx_2/dt = \lambda_2 x_2 + u$, with $\lambda_1 + \lambda_2 = 0$ and $\lambda_i \neq 0$, for $i = 1, 2$. For $u = 0$, the flow g_0^t in $\mathbb{R}^2$ is represented by the area preserving mapping

$$g_0^t = \mathrm{diag}\,[\exp(\lambda_1 t), \exp(\lambda_2 t)]$$

Definition 2

Let D be a compact subset of $\mathbb{R}^n$. Then, for a given u, the image at time t of D under the flow of X, $g_u^t(D)$ is defined as

$$g_u^t(D) = \{x \in \mathbb{R}^n : x = g_u^t x_0 \text{ for some } x_0 \in D\} \tag{2.2}$$

Example 3

It is easy to see that the flow g_0^t of Example 2 continuously deforms circles centred at the origin into ellipses of equivalent area, i.e. the image, at time t, of a circle D under the flow g_0^t is an ellipse of the same area.

Definition 3

Let u be a fixed control function, the 'divergence' of the vector field $X(x, u)$ is defined as

$$\mathrm{div}\, X(x, u) = \mathrm{Trace}\left[\frac{\partial X}{\partial x}\right] = \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} \tag{2.3}$$

Example 4

In Example 1, the divergence of the uncontrolled vector field is zero (this means the area invariance under the flow map of the evolution of the compact regions in the plane—see Arnold (1978, p. 69), Theorem 2, Arnold (1985, p. 198) Corollary 1). In Example 2, the divergence of the uncontrolled vector field is also zero.

Theorem 1

This is a stronger version of Liouville's Theorem (Arnold 1985, p. 198). Let D be a compact region in $\mathbb{R}^n$ with volume $V(0)$. If $V(t)$ denotes the volume of $g_u^t(D)$ ($:= D(t)$), for some given u, then at any time τ

$$\left(\frac{dV}{dt}\right)\bigg|_{\tau} = \int_{D(\tau)} \operatorname{div} X(x, u) \tag{2.4}$$

Proof

The proof of Theorem 1 was given by Arnold (1978, p. 69).

Let D be a compact region of $\mathbb{R}^n$ whose boundary, denoted by ∂D, is a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, characterized by

$$\partial D = \{x \in \mathbb{R}^n : s(x) = 0\} \tag{2.5}$$

where $s: \mathbb{R}^n \to \mathbb{R}$ is a smooth function with non-zero gradient vector, $\partial s/\partial x$, almost everywhere on ∂D. We assume that ∂D is oriented in such a way that $s(x) < 0$ describes the bounded interior of D, while $s(x) > 0$ is the open unbounded complement of D. The vector field ds denotes the unit outer normal vector field of ∂D, i.e. $\partial s/\partial x = \|\partial s/\partial x\|\, ds$.

Theorem 2: *The Divergence Theorem* (Warner 1971, p. 151)

$$\int_D \operatorname{div} X(x, u) = \int_{\partial D} \langle ds, X(x, u)\rangle \tag{2.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product.

Definition 4

The flow map g_u^t is locally a 'contraction' on a given compact region D if there exists a small positive scalar ε such that $D \supset g_u^t(D)$ for all $0 < t < \varepsilon$. Conversely, g_u^t is locally an 'expansion' on D if g_u^{-t}, the inverse map of g_u^t, is a contraction on D (i.e. $D \supset g_u^{-t}(D)$, or, equivalently, $g_u^t(D) \supset D$).

Example 5

Consider a disc D of radius r in $\mathbb{R}^2$. The flow g_u^t generated by the vector field

$$[x_2 - x_1(x_1^2 + x_2^2 - u)]\frac{\partial}{\partial x_1} + [-x_1 - x_2(x_1^2 + x_2^2 - u)]\frac{\partial}{\partial x_2}$$

with $u = a^2 = \text{constant} < r^2$, is locally a contraction on D. On the other hand, g_u^t is locally an expansion on D for $u = b^2 = \text{constant} > r^2$ (see Figs 1 (*a*) and 1 (*b*)).

Example 6

The flows of Examples 1 and 2 are neither locally an expansion nor locally a contraction.

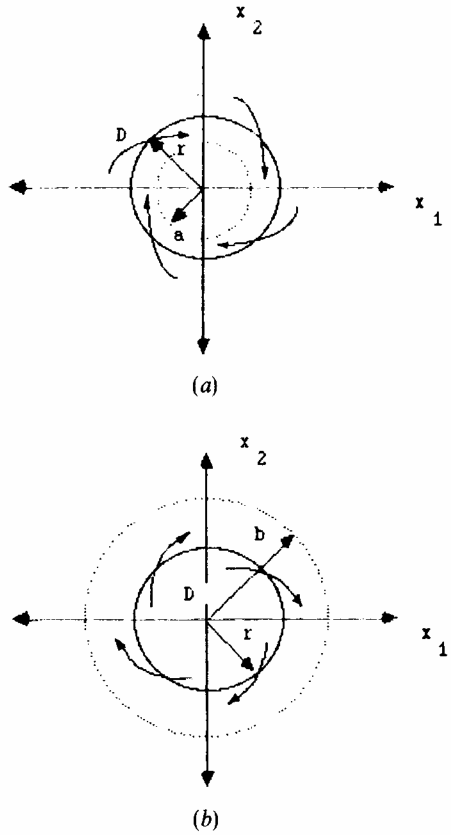

Figure 1. (*a*) Local contraction on D; (*b*) local expansion on D.

Theorem 3

Let $X(x, u)$ be the generating field of g_u^t. Then, g_u^t is locally a contraction (expansion) on D if and only if for all $x \in \partial D$, $\langle ds, X(x, u)\rangle < 0$ (> 0).

Proof

We shall only prove the expansion part in Theorem 3. The contraction part follows by similar arguments.

Let g_u^t be locally an expansion on D, then for each $x \in \partial D$, the inner product

$$\langle ds, g_u^\varepsilon(x) - x\rangle > 0$$

for any arbitrarily small ε. Substituting $g_u^\varepsilon(x)$ by its series expansion about x:

$$g_u^\varepsilon(x) = x + \varepsilon X(x, u) + \text{high-order terms}$$

one finds that $\varepsilon\langle ds, X(x, u)\rangle + o(\varepsilon^2) > 0$, which holds true for an arbitrarily small ε if and only if $\langle ds, X(x, u)\rangle > 0$. To prove the sufficiency, let $\langle ds, X(x, u)\rangle > 0$ for all $x \in \partial D$, but suppose that $g_u^t(D)$ does not entirely contain D, i.e. g_u^t is not an expansion. Then there exists at least one open region of ∂D that has an empty intersection with $g_u^t(D)$ (see Fig. 2). Take any x on such an open region. For a sufficiently small $\varepsilon > 0$,

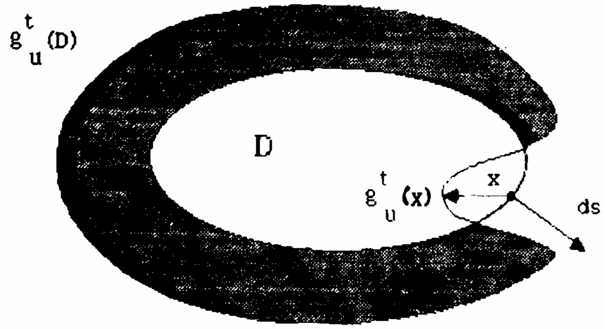

Figure 2. The proof by contradiction of Theorem 3.

$\langle ds, g_u^\varepsilon(x) - x\rangle < 0$. Again using $g_u^\varepsilon(x) = x + \varepsilon X(x, u)$ + high-order terms in the inner product, one concludes that $\varepsilon\langle ds, X(x, u)\rangle + o(\varepsilon^2) < 0$, i.e. $\langle ds, X(x, u)\rangle < 0$ on an open region of ∂D. This is a contradiction. □

The proof of the following corollary is an immediate consequence of Theorem 3 and the Divergence Theorem 2 above.

Corollary 1

Let g_u^t be locally a contraction (expansion) on D. Then, $dV/dt|_{t=0} < 0$ (>0), i.e.

$$\int_{\partial D} \langle ds, X(x, u)\rangle = \int_D \operatorname{div} X(x, u) < 0 \quad (>0)$$ □

Example 7

Consider

$$\frac{dx_1}{dt} = -x_1^3 + ux_1(1 + x_1^2), \quad \frac{dx_2}{dt} = (2u - 1)x_2$$

with u taking values in the discrete set $\{0, 1\}$. Let D be a circle of radius r. For $u = 1$, the map $g_{u=1}^t(x_1, x_2) = \operatorname{col}[\exp(t)x_1, \exp(t)x_2]$ is evidently an expansion. Since $\operatorname{div} X(x, 1) = 2$ then

$$\int_D \operatorname{div} X(x, 1) = 2\pi r^2 > 0$$

For $u = 0$, the map

$$g_{u=0}^t(x_1, x_2) = \operatorname{col}[x_1(1 + 2x_1^2)^{-1/2}, \quad \exp(-t)\, x_2]$$

is evidently a contradiction for $t > 0$. In this case, $\operatorname{div} X(x, 0) = -3x_1^2 - 1 < 0$, and hence

$$\int_D \operatorname{div} X(x, 0) = -\pi r^2\left(1 + \frac{3r^2}{4}\right) < 0$$ □

2.2. Conditions for global existence of sliding motions on compact manifolds

Definition 5

A 'variable structure control law' with discontinuity surface ∂D is a specification of a feedback control policy $u(x)$ on (2.1), according to

$$u(x) = \begin{cases} u^+(x) & \text{for } s(x) > 0 \\ u^-(x) & \text{for } s(x) < 0 \end{cases} \quad u^+ \neq u^- \tag{2.7}$$

where one may assume without loss of generality that, pointwise in x, $u^+(x) < u^-(x)$.

Definition 6

A 'global sliding regime' (i.e. one existing everywhere except, possibly, on a set of measure zero) is said to exist on ∂D if and only if at every point $x \in \partial D$, the variable structure control law (2.7) acting on (2.1) is such that

$$\left.\begin{aligned} \lim_{s \to +0} L_{X(x,u^+)} s < 0 \Leftrightarrow \lim_{s \to +0} \langle ds, \quad X(x, u^+) \rangle < 0 \\ \lim_{s \to -0} L_{X(x,u^-)} s > 0 \Leftrightarrow \lim_{s \to -0} \langle ds, \quad X(x, u^-) \rangle > 0 \end{aligned}\right\} \tag{2.8}$$

where L_X denotes the Lie derivative (directional derivative) of the scalar function s with respect to the controlled vector field X.

Theorem 4

A sliding motion globally exists on ∂D if and only if $g^t_{u^+}$ is a local contraction on D and $g^t_{u^-}$ is a local expansion on D, i.e. given a sufficiently small positive ε, for all $0 < t < \varepsilon$,

$$D \supset g^t_{u^+}(D) \quad \text{and} \quad g^t_{u^-}(D) \supset D \tag{2.9}$$

Proof

Suppose a sliding regime exists globally on ∂D, then conditions (2.8) hold true. From Theorem 3 the set-inclusions (2.9) are then true. Suppose now that (2.9) holds true. Then, using the results of Theorem 2, one obtains on ∂D,

$$\langle ds, X(x, u^+(x)) \rangle|_{x \in \partial D} = \lim_{s \to +0} \langle ds, X(x, u^+(x)) \rangle < 0$$

On the other hand,

$$\langle ds, X(x, u^-(x)) \rangle|_{x \in \partial D} = \lim_{s \to -0} \langle ds, X(x, u^-(x)) \rangle > 0$$

Hence conditions (2.8) hold true and a sliding motion exists globally on ∂D. □

Example 8

Consider the disc D and the dynamical system of Example 5. A global sliding motion exists on the circumference ∂D when the switching logic

$$u = u^+(x) = a^2 < r^2, \quad \text{for } x_1^2 + x_2^2 - r^2 > 0$$

and

$$u = u^-(x) = b^2 > r^2, \quad \text{for } x_1^2 + x_2^2 - r^2 < 0$$

is used (see Fig. 3).

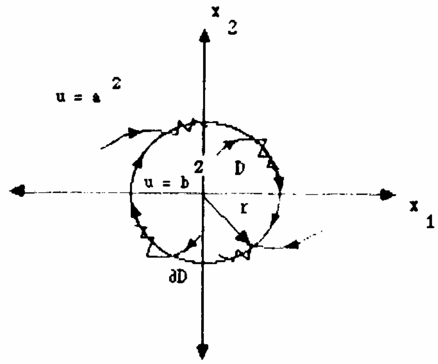

Figure 3. Periodic sliding motions in $\mathbb{R}^2$.

Corollary 2

If a sliding regime exists globally on ∂D then

$$\int_D \operatorname{div} X(x, u^+(x)) < 0 \quad \text{and} \quad \int_D \operatorname{div} X(x, u^-(x)) > 0 \tag{2.10}$$

Proof

Suppose a sliding regime exists globally on ∂D, then from (2.8), for all $x \in \partial D$,

$$\langle ds, X(x, u^+(x))\rangle < 0 \quad \text{and} \quad \langle ds, X(x, x^-(x))\rangle > 0$$

hold valid. Taking the surface integral over ∂D of the inner product and using Theorem 2 on each case, conditions (2.10) follow. □

Example 9

Consider a DC to DC power converter of the Buck–Boost type, shown in Fig. 4 (Sira-Ramirez 1987):

$$\left.\begin{aligned} \frac{dx_1}{dt} &= w_0 x_2 + u(b - w_0 x_2) = X_1(x, u) \\ \frac{dx_2}{dt} &= -w_0 x_1 - w_1 x_2 + u w_0 x_1 = X_2(x, u) \end{aligned}\right\} \tag{2.11}$$

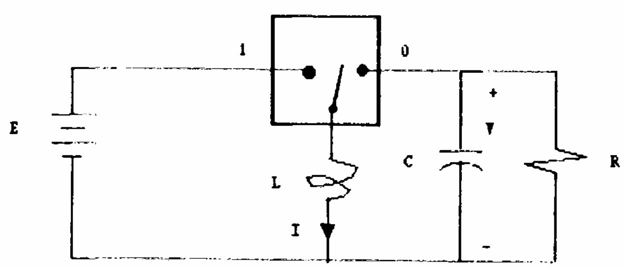

Figure 4. Buck–Boost converter.

where $x_1 = \sqrt{L}I$, $x_2 = \sqrt{C}V$, $b = E/\sqrt{L}$, $w_0 = 1/\sqrt{LC}$, $w_1 = 1/RC$, and u denotes the switch position function (acting as a control input) that takes values in the discrete set $U = \{0, 1\}$. We wish to know whether or not, using a suitable switching policy, harmonic motions are possible for the Buck–Boost converter responses (DC to AC conversion), i.e. D is taken as a disc of radius r, centred at the origin, and ∂D is the bounding circumference. An evaluation of the necessary conditions (2.10) leads to

$$\int_D \operatorname{div} X(x, 1) = -\pi r^2 w_1 < 0 \quad \text{and} \quad \int_D \operatorname{div} X(x, 0) = -\pi r^2 w_1 < 0 \qquad (2.12)$$

which readily reveals that a global sliding motion does not exist on ∂D for the available control inputs in the discrete set U. As a matter of fact, a sliding motion does not exist on any non-trivial circumference in $\mathbb{R}^2$.

The fact that (2.10) is only a necessary, but not sufficient, condition for the existence of a global sliding regime on the boundary ∂D of a compact manifold is illustrated by the following example in $\mathbb{R}^2$.

Example 10

Consider the controlled system

$$\frac{dx_1}{dt} = (2u - 1)x_1$$

$$\frac{dx_2}{dt} = -x_2 - u(1 - 2x_2)$$

$$u \in \{0, 1\}$$

Let D be a circle of radius r, centred at the origin, with $0 < r < 1$. It is easy to verify that for $u = 0$, the flow $g^t_{u=0}(x_1, x_2) = \operatorname{col}[\exp(-t)x_1, x_2]$, while for $u = 1$, the flow map $g^t_{u=1}(x_1, x_2) = \operatorname{col}[\exp(t)x_1, \exp(t)(x_2 - 1)]$. Moreover,

$$g^t_{u=0}(D) = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant \exp(-2t)r^2\}$$

and

$$g^t_{u=1}(D) = \{x \in \mathbb{R}^2 : x_1^2 + (x_2 + \exp(2t))^2 \leqslant \exp(2t)r^2\}$$

Hence

$$\frac{d}{dt}(\text{area } g^t_{u=0}(D))|_{t=0} = \frac{d}{dt}(2\pi \exp(-2t)r^2)|_{t=0} = -4\pi r^2 < 0$$

and therefore

$$\int_D \operatorname{div} X(x, 0) < 0$$

On the other hand

$$\frac{d}{dt}(\text{area } g^t_{u=1}(D))|_{t=0} = \frac{d}{dt}(2\pi \exp(2t)r^2)|_{t=0} = 4\pi r^2 > 0$$

and hence

$$\int_D \operatorname{div} X(x, 1) > 0$$

Thus, conditions (2.12) hold valid. However, a sliding motion does not globally exist on the circumference ∂D bounding D, as can be easily inferred from Figs 5 (*a*), 5 (*b*) and 5 (*c*).

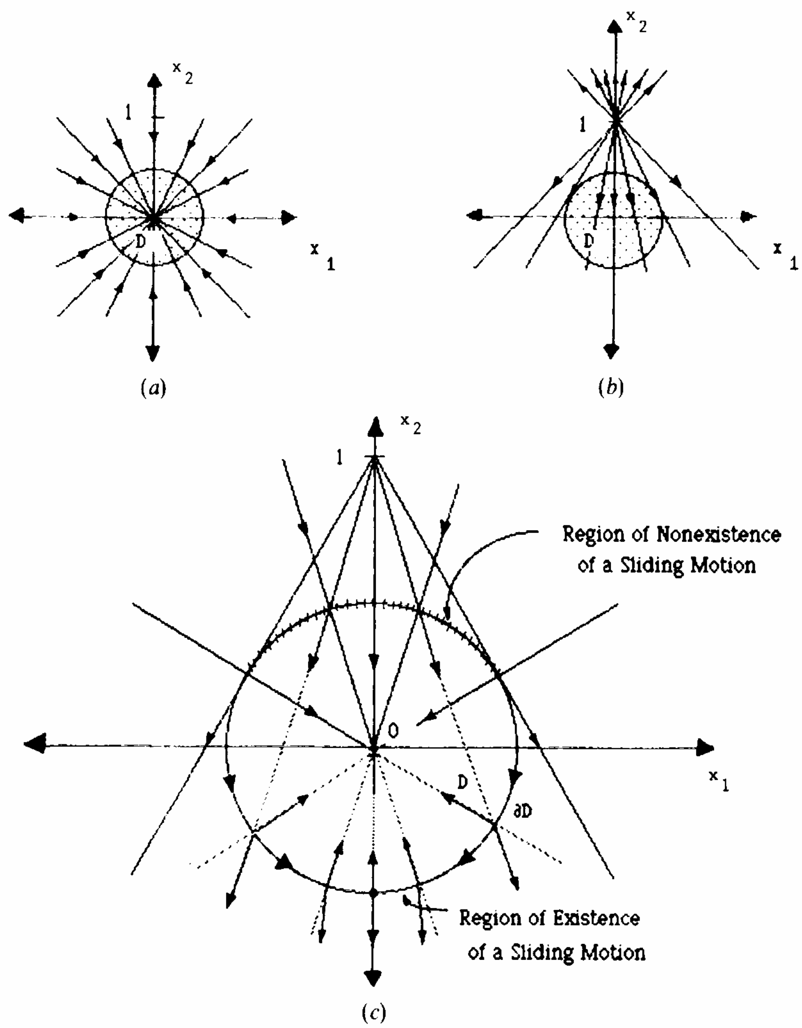

Figure 5. (*a*) Controlled vector field for $u = 1$; (*b*) controlled vector field for $u = 0$; (*c*) region of non-existence of sliding motion.

2.3. *Characterization of ideal sliding dynamics and equivalent control*

Definition 7 (Hale 1969, p. 266)

Let $s(x) = 0$ be a smooth manifold in $\mathbb{R}^n$. We say that $s(x) = 0$ is a 'global integral manifold' for the controlled system (2.1) if, for certain smooth control function $u(x)$, the state trajectories that start anywhere on $s(x)=0$ remain on $s(x)=0$ for all the time, i.e. for each $x \in \partial D$, $g_u^t(x) \in \partial D$ for all $t>0$.

Theorem 5

The compact manifold ∂D is an integral manifold of (2.1), for a given smooth u, if and only if $g_u^t(\partial D)=\partial D$ for all $t>0$.

Proof

If ∂D is an integral manifold of (2.1) for some smooth u, then, by definition of integral manifold for each $x \in \partial D$ and all t, $g_u^t(x) \in \partial D$, i.e. $\partial D \supset g_u^t(\partial D)$ for all t. Suppose now that $g_u^t(\partial D)$ does not contain ∂D for some t, then there exist open sets in ∂D that have an empty intersection with $g_u^t(\partial D)$. Taking any x on such an open set, one concludes that $g_u^t(x) \notin \partial D$, i.e. ∂D is not a global integral manifold for (2.1). This is a contradition. Hence, $g_u^t(D) \supset \partial D$ for all t. From the double inclusion shown, it follows that $g_u^t(\partial D)=\partial D$. The sufficiency is obvious. □

If a global sliding motion exists on ∂D then the average trajectories of (2.1) can be defined as ideally constrained to ∂D under the action of a certain smooth control function known as the equivalent control (Utkin 1978), and denoted by $u^{\mathrm{EQ}}(x)$ with $x \in \partial D$. The equivalent control associated to a sliding regime is thus defined as a smooth state feedback control function, $u^{\mathrm{EQ}}(x)$, for which the global sliding manifold ∂D becomes an integral manifold of (2.1). The tangency of the average trajectories to ∂D is characterized by the following manifold invariance condition, satisfied by the ideally smoothly controlled vector field $X(x, u^{\mathrm{EQ}}(x))$:

$$\left.\begin{aligned} &L_{X(x,u^{\mathrm{EQ}}(x))}s=0 \quad \text{on } s=0 \\ \text{i.e.} \\ &\langle ds, X(x,u^{\mathrm{EQ}}(x))\rangle|_{s=0}=0 \end{aligned}\right\} \tag{2.13}$$

If an equivalent control satisfying (2.13) is known, the ideal sliding dynamics is obtained by formally substituting u by $u^{\mathrm{EQ}}(x)$ in (2.1). One obtains

$$\frac{dx}{dt}=X(x,u^{\mathrm{EQ}}(x)), \quad x \in \partial D \tag{2.14}$$

as the idealized description of the average trajectories of the variable structure controlled system on ∂D. This is the basis of the method of the equivalent control (Utkin 1978, Chapter 2).

By definition of the equivalent control and Theorem 5, it follows that for all t,

$$g_{u^{\mathrm{EQ}}}^t(\partial D)=\partial D \tag{2.15}$$

In general, for controlled vector fields of the form $X(x,u)$, (2.13) or (2.15) do not uniquely define the equivalent control (this topic is considered at length by Utkin (1978, pp. 64–66)) except in some special cases (typically when the controlled vector field is of the linear-in-the-control form: $X(x,u)=f(x)+ug(x)$, provided the transversality condition $\langle ds, g\rangle \neq 0$ is satisfied: see Sira-Ramirez (1988)).

Remark

It has been shown for the linear-in-the-control case that a necessary and sufficient condition for the existence of a sliding regime on $s(x)=0$—see Utkin (1978, p. 119) and Sira-Ramirez (1988)—is the existence and uniqueness of the equivalent control, pointwise bounded within the extreme feedback laws, i.e. $u^+(x) < u^{EQ}(x) < u^-(x)$. In the general case, the same condition holds for isolated solutions of (2.13) if they exist.

The invariance conditions (2.13) and (2.15) are easily seen to imply an invariance condition on the evolution of the volume of $g^t_{u^{EQ}(x)}(D)$ for all t:

$$\left.\frac{dV}{dt}\right|_{t=\tau} = \int_{D(\tau)} \operatorname{div} X(x, u^{EQ}(x)) = 0 \qquad (2.16)$$

The existence of a smooth feedback control $u^{EQ}(x)$, such that the volume invariance condition (2.16) is satisfied, constitutes only a necessary, but not sufficient, condition for the existence of an equivalent control associated to a sliding regime on ∂D. This is established in the next corollary.

Corollary 3

If an equivalent control exists globally on ∂D then (2.16) holds true, i.e. the volume of D remains constant under $g^t_{u^{EQ}}$.

Proof

From the definition of equivalent control, $\langle ds, X(x, u^{EQ}(x))\rangle = 0$ at all times. From Theorem 1, the Divergence Theorem 2, Corollary 3, and the fact that, for any $\tau \geqslant 0$, the boundary of $g^\tau_u(D)$ equals the image of the boundary of D under g^τ_u, i.e. $\partial[g^\tau_u(D)] = g^\tau_u(\partial D)$, it then follows that for any $\tau \geqslant 0$,

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{t=\tau} &= \int_{g^\tau_{u^{EQ}}(D)} \operatorname{div} X(x, u^{EQ}(x)) \\ &= \int_{\partial[g^\tau_{u^{EQ}}(D)]} \langle ds, X(x, u^{EQ}(x))\rangle \\ &= \int_{g^\tau_{u^{EQ}}(\partial D)} \langle ds, X(x, u^{EQ}(x))\rangle \\ &= \int_{\partial D} \langle ds, X(x, u^{EQ}(x))\rangle = 0 \end{aligned} \qquad (2.17)$$

□

The equivalent control forces the flow map $g^t_{u^{EQ}}(x)$ to preserve the volume of the region D. A sufficient condition to make (2.17) valid is that the subintegral quantity becomes zero, i.e. $\operatorname{div} X(x, u^{EQ}(x)) = 0$. This leads to a first-order quasi-linear partial differential equation of the form

$$\sum_{i=1}^{n} \left\{\frac{\partial X_i(x, u^{EQ})}{\partial x_i} + \left[\frac{\partial X_i(x, u^{EQ})}{\partial u^{EQ}}\right]\left(\frac{\partial u^{EQ}}{\partial x_i}\right)\right\} = 0 \qquad (2.18)$$

from where an equivalent control candidate, $u^{EQ}(x)$, may be found.

Example 11

In Example 5 an equivalent control may be obtained, using (2.18), as a solution of

$$\operatorname{div} X(x, u^{EQ}(x)) = -4(x_1^2 + x_2^2) + 2u^{EQ} + \frac{x_1\, \partial u^{EQ}}{\partial x_1} + \frac{x_2\, \partial u^{EQ}}{\partial x_2} = 0$$

It is easily verified that $u^{EQ}(x) = x_1^2 + x_2^2$ is such a solution, and hence

$$u^{EQ}(x)|_{x \in \partial D} = x_1^2 + x_2^2 = r^2 \qquad \square$$

The following example confirms the fact that, in general, the qualification of the sliding manifold as an invariant manifold for some appropriate smooth feedback control is merely a necessary, but not sufficient, condition for the existence of a sliding regime on such a manifold—Sira-Ramirez (1988). In other words, condition (2.16) may be satisfied by a smooth feedback control $u(x)$ defined on ∂D, without a sliding regime necesssarily existing on such a manifold.

Example 12

Consider the problem of finding a control $u(x)$ that satisfies (2.16) for the case of Example 9. This control must cause smooth oscillatory responses of an harmonic nature for (2.11). Consider therefore the sliding surface candidate $\partial D = \{x \in \mathbb{R}^2 : s(x) = x_1^2 + x_2^2 - r^2 = 0\}$. Then

$$\int_D \operatorname{div} X(x, u(x)) = \int_D \left\{ (bw_0^{-1} - x_2)\left[\frac{\partial u}{\partial x_1}\right] + x_1\left[\frac{\partial u}{\partial x_2}\right] - w_1 w_0^{-1} \right\} dx_1\, dx_2 = 0 \tag{2.19}$$

A sufficient condition for (2.19) to be valid is that the sub-integral quantity becomes zero. Hence, the following partial differential equation is satisfied by $u(x)$:

$$(bw_0^{-1} - x_2)\left[\frac{\partial u}{\partial x_1}\right] + x_1\left[\frac{\partial u}{\partial x_2}\right] = \frac{w_1}{w_0} \tag{2.20}$$

Equation (2.20) has the following solution:

$$u(x_1, x_2) = \left[\frac{w_1}{w_0}\right] \tan^{-1}\left[x_1 (bw_0^{-1} - x_2)^{-1}\right]$$

A smooth feedback control $u(x)$ thus exists, which satisfies the invariance condition (2.16). However, as shown in (2.12), a sliding motion does not exist globally on ∂D. Hence, the smooth control $u(x)$ found above does not qualify as an equivalent control associated to a sliding regime on ∂D.

A corollary to Liouville's theorem (Arnold 1978, p. 69, and Arnold 1985, p. 198) explicitly states that the flow maps generated by hamiltonian vector fields preserve the volume of compact regions. The ideal sliding dynamics $dx/dt = X(x, u^{EQ}(x))$ enjoys the same property, as established by Corollary 3, when applied to the particular region D bounded by the compact sliding manifold ∂D. Notice, however, that this does not mean that the ideal sliding dynamics of VSS undergoing sliding motions on compact manifolds are represented by hamiltonian systems. The corollary to Liouville's theorem is a necessary, but not sufficient, condition for a system to be hamiltonian.

3. Conclusions

A general geometric characterization for the existence of sliding regimes on compact manifolds for non-linear variable structure feedback systems is given. The characterization involves set-theoretic inclusion conditions generated by the control-dependent flow map when applied to the compact region contained by the sliding manifold. Sign conditions on the volume integral of the divergence of the available feedback controlled vector fields are derived as necessary, but not sufficient, conditions for the existence of a sliding motion. The invariance condition, or ideal sliding condition, is characterized in terms of a volume-preserving evolution generated by the flow map associated with the ideal sliding dynamics. An application of the general results to periodic sliding motions in $\mathbb{R}^2$ is illustrated using several simple examples. An extension of the obtained results to the case of non-compact manifolds is by no means trivial, and constitutes an area for further research. The results presented here are also applicable to the characterization of quasi-sliding motions on compact manifolds for non-linear discrete-time systems.

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