V. Conclusions

The conventional dynamic programming procedure has been extended in this note for 2-D discrete systems. Although the generalization is illustrated on the basis of the 2-D Roesser state-space equation, the method is also applicable for other linear and nonlinear equations with constant and variable coefficients. The cost functions may also be nonlinear.

The results of the present contribution may also be considered as a basis for generalization of the 1-D dynamic programming method for optimal estimation problems of 2-D noisy images [18], [25], [26].

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A Geometric Approach to Pulse-Width Modulated Control in Nonlinear Dynamical Systems

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Abstract—**This note demonstrates, under the assumption of high-frequency control switchings, the existence of an ideal equivalence among sliding regimes of variable structure feedback (VSF) control and pulse-width modulated (PWM) control responses in nonlinear dynamical systems. This equivalence constitutes the basis for a geometric approach to PWM control loops design. An illustrative example of energy conversion in a lossless switched-controlled bilinear network is presented.**

I. Introduction

In this note, the most salient features of sliding regimes [1] associated to variable structure feedback (VSF) control, and the specification of PWM control loops [2], [3], are examined from a unified geometric viewpoint. An ideal equivalence is obtained among both approaches under the assumption of high-frequency control switchings. PWM controlled responses are shown to locally sustain sliding motions on an integral manifold associated with a suitably defined ideal average system. As an ideal feedback law, the equivalent control associated with the corresponding ideal sliding motion coincides with the prescribed duty ratio. Conversely, a given discontinuity surface locally qualifies as an integral manifold of a PWM controlled system provided a local sliding motion exists with an associated equivalent control coincident with the duty ratio. This equivalence is exploitable in PWM design problems by replacing the synthesis of duty ratios (as feedback laws) by simpler switching laws leading to the equivalent sliding mode behavior on the suitable manifold.

Section II briefly summarizes a geometric framework for the study of sliding regimes in nonlinear systems of variable structure. Section III analyzes nonlinear systems controlled by means of a PWM feedback loop and obtains an ideal equivalence among PWM control strategies and VSF control options. A simple energy transfer problem is considered for a bilinear switched network.

II. Background Results about Sliding Motions of Variable Structure Systems

Consider the smooth nonlinear system

$$dx/dt = f(x) + g(x)u \tag{2.1}$$

where $x \in X$, an open set of R^n, the scalar control function $u: R^n \to R$ is a (possibly discontinuous) feedback control function, while f and g are smooth, local, vector fields defined on X. Let s denote a smooth real-valued function of x defined by $s: X \to R$. The level set $S_0 = \{x \in R^n : s(x) = 0\} =: s^{-1}(0)$, defines a smooth $n-1$ dimensional, locally regular manifold of constant rank, i.e., locally integrable [4], addressed as the sliding manifold or discontinuity surface. The gradient of $s(x)$, denoted by ds, is hence assumed to be nonzero in X except, possibly, on a set of measure zero. S_0 is oriented in such a way that ds points from the region where $s(x) < 0$ towards that where $s(x) > 0$. Let K be an open set of R containing zero. The regularity assumptions about $s(x)$ induce a local regular foliation of X into disjoint locally integrable manifolds of the form: $S_k = \{x : s(x) = k, \text{ for } k \in K\} =: s^{-1}(k)$. Such manifolds are called leaves of the foliation.

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All results in this note are of local nature, restricted to an open neighborhood X of R^n which has nonempty intersection with S_0.

Without loss of generality, a VSF control law is obtained by letting the control function u take one of two possible values in $U := \{0, 1\}$, according to the sign of $s(x)$, as defined by

$$u = \begin{cases} 1 & \text{for } s(x) > 0 \\ 0 & \text{for } s(x) < 0. \end{cases} \tag{2.2}$$

Let $L_h s$ denote the directional derivative of the scalar function s with respect to the vector field h. $L_h s$ is also denoted as a differential 1-form, $\langle ds, h \rangle$, acting linearly on $T_x R^n$ and taking values in R. (See [5, p. 174].) By Ker ds is meant $\{h : \langle ds, h \rangle = 0\}$. A sliding regime is said to exist locally on S_0, if and only if, as a result of the control policy (2.2), the state trajectories of (2.1) satisfy [1]

$$\lim_{s \to +0} L_{f+g} s = \lim_{s \to +0} \langle ds, f+g \rangle < 0 \tag{2.3}$$

$$\lim_{s \to -0} L_f s = \lim_{s \to -0} \langle ds, f \rangle > 0. \tag{2.4}$$

Lemma 2.1 [6]: If a sliding regime locally exists on S_0, then, necessarily, the transversality condition $L_g s = \langle ds, g \rangle < 0$ is locally satisfied on the manifold S_0.

Definition 2.2: Let $\langle ds, g \rangle$ and $\langle ds, f \rangle$ be nonidentically zero on X. S_0 is said to be a local integral manifold for (2.1), with $u(x)$ a given smooth control function, if S_0 is locally integrable and

$$\langle ds, f + gu(x) \rangle = 0, \quad \text{i.e,} \quad f + gu(x) \in \text{Ker } ds \tag{2.5}$$

is pointwise satisfied.

Notice that this definition simply entitles the local pointwise tangency of the smooth controlled vector field $f + gu(x)$ to be manifold S_0.

Theorem 2.3 [6]: A necessary and sufficient condition for the local existence of a sliding mode on S_0 is that there exists, locally on S_0, a smooth control function $u_{EQ}(x)$, which turns S_0 into local integral manifold for (2.1), such that

$$0 < u_{EQ}(x) < 1. \tag{2.6}$$

The above theorem actually provides a definition of the *ideal (average) sliding motion* on the manifold S_0, known as the ideal sliding dynamics. The smooth control function $u_{EQ}(x)$ is called the equivalent control and according to its definition and (2.5) it satisfies $\langle ds, f + gu_{EQ}(x) \rangle = 0$, i.e.,

$$u_{EQ}(x) = -\langle ds, f \rangle / \langle ds, q \rangle = -L_f s / L_g s. \tag{2.7}$$

The transversality condition of Lemma 2.1 is therefore justified on the grounds of existence of the equivalent control. Thus, existence of the equivalent control is also a necessary condition for the existence of a sliding regime [6]. Notice that if $\langle ds, g \rangle = 0$ on an open set of X, then a sliding motion may still exist on a proper integrable submanifold of S_0, provided $\langle ds, f \rangle = 0$ locally in X. Such sliding motions are termed singular [1].

From (2.7) it follows upon formal substitution in (2.1) that the motions starting on S_0, due to the equivalent control (ideal sliding dynamics) are governed by

$$dx/dt = f + gu_{EQ}(x) = [f - g\langle ds, f \rangle / \langle ds, q \rangle]. \tag{2.8}$$

This procedure constitutes the method of the equivalent control [1].

If the output of a variable structure controlled system is taken, in local coordinates, as the surface coordinate function, i.e., $dx/dt = f(x) + g(x)u$; $y = s(x)$ -denoted by (f, g, s)-, the resulting controlled system has a very simple structure at infinity (see [7] and [8]). In particular, the transversality condition of Lemma 2.1 implies the existence of a zero at infinity of multiplicity one.[1] The system (f, g, s) is then said to have relative degree one [18]. For all initial states located on the leaf $s^{-1}(0)$ the equivalent control, $u_{EQ} = -L_f s / L_g s$, zeros the output y in the region of existence of the sliding regime. Hence, the coordinate-free description of the ideal sliding dynamics (2.8) actually corresponds to the zero dynamics [8], associated with the output function $y = s(x)$. This simple structure at infinity is also responsible for locally making the order v_f of the infinite zero dynamics[2] [8] equal to $n - 1$ (i.e., the multiplicity of the finite zeros is $n - 1$). It follows that the maximal (f, g)-invariant distribution, contained in Ker ds, actually coincides with Ker ds itself, i.e., with the distribution tangent to the leaves S_k. It is then easy to see that, for any $x(0) \in S_k (k \neq 0)$, the equivalent control (2.7) would make the controlled system trajectory locally evolve on the leaf S_k. Due to the uniqueness of the equivalent control and (2.6), for arbitrarily small k, where the transversality condition is still valid, the effect of an appropriate control input, $u = 0$ or $u = 1$ according to (2.2), is to pull the state trajectory out of the leaf $s^{-1}(k)$ to make it approach $s^{-1}(0)$.

[1] A nonlinear system (f, g, s) has a zero at infinity of multiplicity v_∞ when such an integer is the first one for which $L_g L_f v_\infty - 1 s \neq 0$.

[2] It is easy to show that $v_\infty + v_f = n$ (see [8]).

III. A Geometric Approach to PWM Control

In a PWM control option for system (2.1), the scalar control u, taking values in $U = \{0, 1\}$, is switched *once* within a duty cycle of fixed small duration Δ. The instants of time at which the switchings occur are determined by the sample value of the state vector at the beginning of each duty cycle. The fraction of the duty cycle on which the control holds the fixed value, say 1, is known as the duty ratio and it is denoted by $D(x(t))$. The duty ratio is usually specified as a smooth function of the state vector x. The duty ratio evidently satisfies $0 < D(x) < 1$.

On a typical duty cycle interval, the control input u is defined as (see Fig. 1)

$$u = \begin{cases} 1 & \text{for } t \leq \tau < t + D(x(t))\Delta \\ 0 & \text{for } t + D(x(t))\Delta \leq \tau < t + \Delta. \end{cases} \tag{3.1}$$

It follows then that, generally

$$x(t+\Delta) = x(t) + \int_t^{t+D(x(t))\Delta} [f(x(\tau)) + g(x(\tau))] \, d\tau + \int_{t+D(x(t))\Delta}^{t+\Delta} f(x(\tau)) \, d\tau.$$

The ideal average model of the PWM controlled system response is obtained by allowing the duty cycle frequency to tend to infinity with the duty cycle length Δ approach zero. In the limit, the above relation yields

$$\lim_{\Delta \to 0} [x(t+\Delta) - x(t)]/\Delta = \lim_{\Delta \to 0} \left[\int_t^{t+\Delta} f(x(\tau)) \, d\tau + \int_t^{t+D(x(t))\Delta} g(x(\tau)) \, d\tau \right] \Big/ \Delta,$$

i.e.,

$$dx/dt = f(x) + g(x)D(x). \tag{3.2}$$

As the duty cycle frequency tends to infinity, the ideal average behavior of the PWM controlled system is represented by the smooth response of the system (2.1) to the smooth control function constituted by the duty ratio $D(x)$. The duty ratio $D(x)$ replaces the discrete function u in (2.1) in the same manner as the equivalent control $u_{EQ}(x)$, of the VSF scheme, replaces u in (2.1) to obtain (2.9).

We refer to (3.2) as the average PWM controlled system.

Lemma 3.1: Let $\Sigma_0 = \{x \in R^n : \sigma(x) = 0\} =: \sigma^{-1}(0)$ be a local integral manifold for the average PWM controlled system (3.4), then

$$0 < D(x) = -\langle d\sigma, f \rangle / \langle d\sigma, g \rangle < 1. \tag{3.3}$$

Proof: The inequalities are obvious from the definition of duty ratio. The expression for $D(x)$ is obtained from the fact that if Σ_0 is an integral manifold of (3.2), then from Definition 2.2 $\langle d\sigma, f + gD(x) \rangle = 0$, locally on Σ_0. From here (3.3) follows immediately. □

Notice that $\langle d\sigma, g \rangle = 0$, on an open set of X, makes $D(x)$ unbounded unless $\langle d\sigma, f \rangle$ is also zero, in which case Σ_0 is an integral manifold of (3.2) for any conceivable $D(x)$. To avoid this, we assume, without loss of

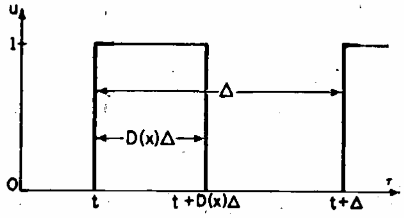

Fig. 1. Typical duty cycle and duty ratio in PWM control.

generality, that $\langle d\sigma, g\rangle < 0$ locally on Σ_0 (the system with output $y = \sigma(x)$ has relative degree one). Notice that if $\langle d\sigma, g\rangle > 0$, and Σ_0 is an integral manifold of (3.2), then, redefining Σ_0 as $\{x \in R^n: \sigma_1(x) = 0\}$ with $\sigma_1(x) = -\sigma(x)$, one obtains $\langle d\sigma_1, g\rangle < 0$ and the assumption would now be valid.

Lemma 3.2: If Σ_0 is a local integral manifold for (3.2), and $\langle d\sigma, g\rangle < 0$, then, in the region of interest, $D(x)$ is unique.

Proof: Suppose $D_1(x) \neq D(x)$ are duty ratios for which Σ_0 is a local integral manifold of (2.1). It follows from Definition 2.3 that, locally on Σ_0,: $\langle d\sigma, f + D(x)g\rangle = Y\langle d\sigma, f + D_1(x)g\rangle = 0$. From this equality it follows that $\langle d\sigma, (D(x) - D_1(x))g\rangle = (D(x) - D_1(x))\langle d\sigma, g\rangle = 0$. Since by hypothesis $\langle d\sigma, g\rangle < 0$ then, necessarily, $D(x) = D_1(x)$ locally on Σ_0. This is a contradiction. □

Theorem 3.3: Suppose the transversality condition $\langle d\sigma, g\rangle < 0$ holds locally true on Σ_0, then a necessary and sufficient condition for Σ_0 to be a local integral manifold of (3.2) is that locally on Σ_0

$$\langle d\sigma, f+g\rangle < 0 \quad \text{and} \quad \langle d\sigma, f\rangle > 0. \tag{3.4}$$

Proof: Let Σ_0 be a local integral manifold for (3.2), then using the hypothesis that $\langle d\sigma, g\rangle < 0$, it follows from the right-hand side of (3.3) that $-\langle d\sigma, f\rangle > \langle d\sigma, g\rangle$ and therefore $\langle d\sigma, f + g\rangle < 0$. On the other hand, using the first inequality of (3.3), it follows that $-\langle d\sigma, f\rangle < 0$, i.e., $\langle d\sigma, f\rangle > 0$.

To prove sufficiency, suppose (3.4) holds true locally on Σ_0. Then, there exists strictly positive smooth functions $a(x)$ and $b(x)$ such that on the region of interest $a(x)\langle d\sigma, f + g\rangle + b(x)\langle d\sigma, f\rangle = 0$. Rearranging the above expression $\langle d\sigma, f + [a(x)/(a(x) + b(x))]g\rangle = 0$, i.e., there exists a smooth control function $0 < D(x) = a(x)/[a(x) + b(x)] < 1$ such that, locally on Σ_0, $\langle d\sigma, f + D(x)g\rangle = 0$. In other words, in X, Σ_0 is a local integral manifold of (3.2). □

Theorem 3.4: A sliding regime of (2.1) locally exists on an integrable manifold Σ_0 if and only if Σ_0 is a locally integral manifold of an average PWM controlled system whose duty ratio coincides with the equivalent control.

Proof: Suppose Σ_0 is an integral manifold for the average PWM controlled system (3.2), then Theorem 3.3 applies and (3.4) holds true. It follows that locally on Σ_0: $\langle d\sigma, f + g\rangle = \lim_{\delta \to +0}\langle d\sigma, f + g\rangle < 0$ and $\langle d\sigma, f\rangle = \lim_{\sigma \to -0}\langle d\sigma, f\rangle > 0$, i.e., the variable structure control law $u = 1$ for $\sigma(x) > 0$ and $u = 0$ for $\sigma(x) < 0$ applied on system (2.1) creates a sliding mode locally on Σ_0. Then, necessarily the transversality condition $\langle d\sigma, g\rangle < 0$ holds, according to Lemma 2.1 The corresponding equivalent control $u_{EQ}(x)$ satisfies $\langle d\sigma, f + gu_{EQ}(x)\rangle = 0$ and because, by hypothesis, Σ_0 is an integral manifold of (3.4), $\langle ds, f + gD(x)\rangle = 0$ also holds locally. It follows that $(u_{EQ}(x) - D(x))\langle d\sigma, g\rangle = 0$, i.e., $u_{EQ}(x) = D(x)$.

Suppose now that a sliding motion locally exists on Σ_0, then (2.3) and (2.4) hold true locally on Σ_0. Therefore, the hypothesis of Theorem 3.3 is also valid. Hence, Σ_0 qualifies as a local integral manifold on the average PWM controlled system (3.2) for some $D(x)$. Notice that from Theorem 2.3, $0 < u_{EQ}(x) < 1$ is satisfied in the region of interest. By definition, the equivalent control $u_{EQ}(x)$ also turns Σ_0 into a local integral manifold in the region of existence of a sliding regime. By virtue of the uniqueness of the duty ratio of Lemma 3.2, the duty ratio $D(x)$ coincides with the equivalent control $u_{EQ}(x)$ as a smooth feedback function of the state vector. □

Due to the above equivalence, the same remarks about the structure at infinity made for the case of ideal sliding dynamics, in VSF systems also apply to average PWM controlled systems of the form (f, g, σ), i.e., with output function y taken as the integral manifold coordinate function $\sigma(x)$.

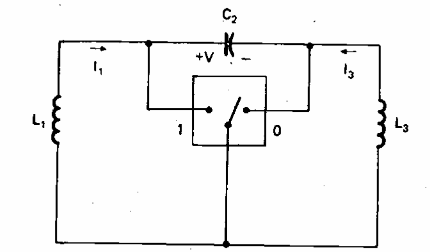

Fig. 2. Switched controlled bilinear lossless network.

Example 3.5: Consider the lossless circuit shown in Fig. 2, also treated by Wood [10] in a different context. In this circuit, energy stored in the inductor L_1 can be transferred to L_3 using appropriate switchings of the capacitor branch. If u denotes the switch position function, taking values in the discrete set $\{0, 1\}$, the equations describing this network are

$$\dot{x} = \begin{vmatrix} 0 & -w_1 & 0 \\ w_1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} x + u \begin{vmatrix} 0 & w_1 & 0 \\ -w_1 & 0 & -w_2 \\ 0 & w_2 & 0 \end{vmatrix} x$$

with $x_i = I_i\sqrt{L_i}$ for $i = 1, 3$ and $x_2 = V_2\sqrt{C_2}$, $w_1 = 1/\sqrt{L_1C_2}$, $w_2 = 1/\sqrt{L_3C_2}$.

It is easy to show that the total stored energy $E = 0.5x^Tx$ is an invariant quantity. For unit norm initial conditions, the state of the system evolves on S^2 (the unit sphere in R^3). The vector fields corresponding to (2.1) are given, in differential operator notation [4], by $f = -w_1x_2\partial/\partial x_1 + w_1x_1\partial/\partial x_2$ and $g = w_1x_2\partial/\partial x_1 - (w_1x_1 + w_2x_3)\partial/\partial x_2 + w_2x_2\partial/\partial x_3$.

For $u = 0$ and $u = 1$ the family of trajectories is characterized by $x_3 =$ constant and $x_1 =$ constant, respectively, on the unit sphere (see Fig. 3). An energy transfer from L_1 to L_3 is accomplished by making the state trajectory evolve from the initial state $(1, 0, 0)^T$ to the final state $(0, 0, 1)^T$. This can be done by switching just once on the point (0, 1, 0) as depicted in Fig. 4. An energy conversion can also take place, however, by means of a PWM control design while keeping the capacitor voltage ideally constant. Using the equivalence between PWM and VSF control of Theorem 3.4, a sliding motion created on the submanifold $S_0 = \{x \in S^2: s = x_2 - K = 0, 0 < K < 1, K = \text{constant}\}$, by means of the variable structure control law $u = 0.5(1 + \text{sign } s)$, achieves the energy transfer as depicted in Fig. 4. This is accomplished provided the switch is set fixed at $u = 1$ once x_1 becomes 0. In this case the necessary duty ratio (equivalent control) is obtained from the local integral manifold condition of the proposed switching line on the sphere $\langle ds, f + gD(x)\rangle = 0$, or equivalently, $f + gD(x) \in \text{Ker } ds = \text{span }\{-x_3\partial/\partial x_1 + x_1\partial/\partial x_3\}$. Using the expressions for the vector fields f and g, given above, the nonlinear duty ratio is $D(x) = w_1x_1/(w_1x_1 + w_2x_3)$. Notice that it is not necessary to synthesize such a nonlinear feedback law since using the sliding mode equivalence such feedback action is automatically synthesized, on the average, by performing fast switchings about the appropriately identified sliding line S_0. The control decisions are taken on the basis of the sign of s (i.e., one bit of data). The local character of the sliding motions results from the fact that the transversality condition $\langle ds, g\rangle = -(w_1x_1 + w_2x_3) < 0$ is not globally satisfied along S_0 on S^2. The sliding motion only exists on the first orthant. Hence, the duty ratio locally satisfies $0 < D(x) < 1$. The dual circuit to this example is treated, via sliding regimes, in [11] without reference to PWM control.

IV. Conclusions

A coordinate free methodology in sliding regime analysis and design is not only conceptually beneficial for the understanding of known features about sliding modes and their idealized features. It also provides the right mathematical tools for the investigation of new connections with areas such as PWM control. The equivalence established in this note constitutes

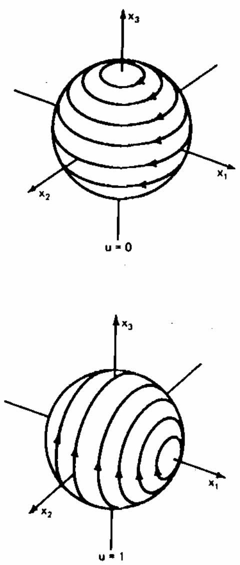

Fig. 3. Controlled trajectories on the sphere.

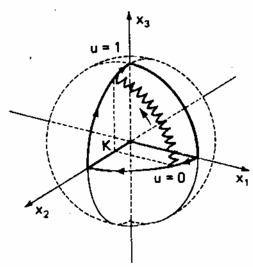

Fig. 4. Energy transfer via PWM (sliding mode) control.

a step towards the systematic treatment of PWM control design via sliding regimes. The advantage of such an equivalence results in the automatic synthesis of prescribed feedback duty ratios by means of on-the-average equivalent variable structure feedback strategies defined on an appropriate sliding surface. Additional benefits are also drawn from hardware simplicity, characteristic of the equivalent sliding mode approach. From a purely theoretical viewpoint, this unified treatment also reveals interesting connections with the frequency domain package for nonlinear systems and the inherent simplicity of the structure at infinity shared by VSF and PWM controlled systems.

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Linear Stable Unity-Feedback System: Necessary and Sufficient Conditions for Stability Under Nonlinear Plant Perturbations

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***Abstract*—We consider a linear (not necessarily time-invariant) stable unity-feedback system, where the plant and the compensator have normalized right-coprime factorizations; we study two cases of *nonlinear* plant perturbations *(additive* and *feedback)*, with four subcases resulting from: 1) allowing exogenous input to ΔP or not; 2) allowing the observation of the output of ΔP or not. The plant perturbation ΔP is *not* required to be stable. Using the factorization approach, we obtain *necessary* and *sufficient* conditions for all cases in terms of two pairs of nonlinear pseudostate maps. Simple physical considerations explain the form of these necessary and sufficient conditions. Finally, we obtain the characterization of all perturbations ΔP for which the perturbed system remains stable.**

INTRODUCTION

Robust stability of feedback systems under *unstructured* perturbations of the plant model has been studied extensively. In the nonlinear case, the small gain theorem [23], [6] gives a *sufficiency* condition for robust stability of a stable system under nonlinear *stable* additive perturbations. *Sufficient* robust stability conditions were also obtained in [1], [5], [8], [10], [15], [16], and [18]. In the linear time-invariant case, *necessary and sufficient* conditions for robust stability for a *certain* class of possibly *unstable* plant perturbations have been obtained in [9] and references therein, [3]; for a *general* class of possibly unstable perturbations, the factorization approach yields *necessary and sufficient* conditions for robust stability of the feedback system under *fractional* perturbations of the subsystems [4]. Furthermore, necessary and sufficient conditions for the *existence* of a controller for plants with additive or multiplicative uncertainty are given in [19].

For linear time-invariant stable unity-feedback systems with *nonlinear additive* plant perturbations, necessary and sufficient conditions have been obtained in two cases: i) the additive perturbation has an independent

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