Dynamical Discontinuous Feedback Control of Nonlinear Systems

HEBERTT SIRA-RAMIREZ AND
PABLO LISCHINSKY-ARENAS

Abstract—In this note, a technique is presented for the analysis of discontinuous dynamical feedback regulation of nonlinear systems. A pulse width modulation feedback interconnection scheme, with general duty ratio function, is shown to be easily analyzable in terms of an average model which captures the essential features of the discontinuously feedback controlled system.

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H. Sira-Ramirez is with the Departamento Sistemas de Control, Escuela de Ingeniería de Sistemas, Universidad de Los Andes, Mérida, Venezuela.
P. Lischinsky-Arenas is with the Departamento de Computación, Escuela de Ingeniería de Sistemas, Universidad de Los Andes, Mérida, Venezuela.
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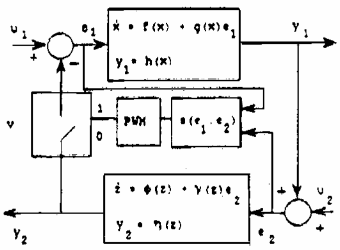

Fig. 1. A PWM discontinuous feedback interconnected system.

I. INTRODUCTION

Discontinuous feedback control of dynamical systems has been traditionally addressed under the assumption of *static (or memoryless) feedback* (see [1] and [2]). Variable structure systems (VSS) and other representatives of discontinuous control schemes, such as pulse width modulation (PWM) and pulse frequency modulation schemes have been restricted to classes of systems in which output signals, output errors, or states, are instantaneously and directly pulsed, usually through a unity feedback loop, into the controlled system. The static character of the proposed discontinuous feedback strategies is also reflected in the synthesis of the command signals for the feedback enabling switch. Typically, in VSS the switch position is regulated by the sliding surface coordinate function, or products of this function with the measured state variables. The more realistic and general situation, within a discontinuous feedback scheme, calls, however, for *dynamical feedback*, or interconnection of the plant and the feedback subsystems constituted by state estimators, controllers, sensors, and actuators whose dynamics cannot be entirely neglected. Similarly, signals commanding the switching can also be generated by means of dynamical subsystems including P-I and P-I-D control schemes excited by output or state errors.

This note addresses, in full generality, the problem of analyzing dynamical discontinuous feedback nonlinear controlled plants. The discontinuous feedback scheme is assumed to be constituted by a dynamical feedback plant, of nonlinear nature, and a controlled switch obeying a PWM type of switching strategy with sufficiently high sampling rate. It is found that the actual closed-loop controlled responses of the system exhibit chattering motions constituting nonideal sliding regimes which converge continuously toward certain *average* manifolds as the sampling frequency is increased. These manifolds are immersed in the regions of the composite state space where the *duty ratio* function is not acting under saturation conditions. In fact, such manifolds are the *integral manifolds* of a suitable *average system* described in the (augmented) state space of the closed-loop system. The average system is simply obtained by an infinite sampling frequency assumption on the PWM process. This note constitutes an extension, to nonstatic discontinuous feedback, of the work in [3]–[5].

Section II presents general results about PWM interconnection of dynamical systems in a feedback arrangement. Section III is devoted to two illustrative examples of discontinuous dynamical feedback schemes for nonlinear systems. Section IV contains the conclusions of the note. The necessary background on PWM control is presented in the Appendix.

II. DEFINITIONS AND MAIN RESULTS

Consider the switched controlled interconnected system shown in Fig. 1. Such a system is described as follows:

$$\begin{aligned}\frac{dx}{dt} &= f(x) + g(x)e_1\\ y_1 &= h(x)\\ e_2 &= y_1 + u_2\\ \frac{dz}{dt} &= \phi(z) + \gamma(z)e_2\\ y_2 &= \eta(z)\\ e_1 &= u_1 - \nu y_2\\ \nu &= \mathrm{PWM}\left[\sigma(t_k)\right]\\ \sigma(t) &= s\left[e_1(t), e_2(t)\right]\end{aligned} \tag{2.1}$$

where f, g, ϕ, and γ are smooth vector fields, x and z are smooth coordinate functions of R^n and R^p, respectively, the functions h and η are, in general, smooth vector functions of their arguments. The signal $\sigma(t)$ is to be generated by either a dynamical system or by a memoryless function. The signals u_1 and u_2 are assumed to be either external reference control inputs or external disturbances. The PWM operator is defined as follows:

$$\mathrm{PWM}\left[\sigma(t_k)\right] = \begin{cases}1 & \text{for } t_k < t \le t_k + \tau\left[\sigma(t_k)\right]T\\ 0 & \text{for } t_k + \tau\left[\sigma(t_k)\right]T < t \le t_k + T\end{cases} \tag{2.2}$$

where T is a fixed (i.e., constant) sampling interval length, known as the *duty cycle*, and $\tau[\sigma(t_k)]$ is a piecewise smooth function, known as the duty *ratio function* which takes values in the closed interval [0, 1]. The duty ratio function represents the fractional length of the sampling interval in which the feedback interconnections are simultaneously enabled, before they are switched off for the rest of the sampling interval. The notation $\sigma(t_k)$ actually stands for $s[e_1(t_k), e_2(t_k)]$ when s is a memoryless operation, or it stands for $s[e_1(t), e_2(t)](t_k)$ when s represents the output of a dynamical subsystem. If during a certain open interval of time the duty ratio function exhibits either the value 0 or 1, the PWM controller is said to be *saturated*, or acting under *saturation conditions*.

The analysis of (2.1) and (2.2) is quite difficult if one uses the discrete-time approximation scheme by which PWM systems have been traditionally analyzed. This is so, even in the case of a linear dynamical plant interconnected to a static feedback system (see, for instance [6, p. 591]). Rather than using this route, we resort to a recent averaging technique, proposed in [3]–[5], used for studying nonlinear discontinuously controlled systems under static (memoryless) feedback. The essential features of this technique which are applicable to system (2.1) are summarized in the Appendix of this note.

Definition 2.1: We define the *average system* of (2.1) as the following dynamical interconnected system;

$$\begin{aligned}\frac{dx}{dt} &= f(x) + g(x)e_1\\ y_1 &= h(x)\\ e_2 &= y_1 + u_2\\ \frac{dz}{dt} &= \phi(z) + \gamma(z)e_2\\ y_2 &= \eta(z)\\ e_1 &= u_1 - wy_2\\ w &= \tau\left[\sigma(t)\right]\\ \sigma(t) &= s\left[e_1(t), e_2(t)\right].\end{aligned} \tag{2.3}$$

The average system (2.3) exhibits exactly the same structure as the original controlled system except for the fact that the feedback enabling switch, previously represented by the function ν, is now substituted by the duty ratio function $\tau[\sigma(t)]$. It will be shown in the Appendix that such a substitution process is justified by letting the sampling frequency, of the pulse modulator, reach an arbitrarily large rate. In other words, the average model (2.3) can be obtained from the original system (2.1) by allowing an infinite sampling frequency assumption on the PWM block. The advantage of the average model lies, precisely, on the smooth character of the controlled response. Such a response, incidentally, entirely coincides with that of the real PWM system in the saturation regions of the PWM operator and it is, moreover, arbitrarily close to the response of the real PWM system in the nonsaturation regions, for large sampling frequencies. The nature of the approximation, on such regions of nonsaturation, is characterized by the existence of a nonideal sliding regime approximating the average responses, or, more precisely, by a *real sliding regime* which continuously converges toward *integral manifolds* of the average model (see [3]-[5]) as the sampling frequency grows to infinity.

The following theorem constitutes an extension of the main result presented in the Appendix.

Theorem 2.1: For identical initial conditions, the responses of system (2.1) entirely coincide with those of the average system (2.3) in the regions of the state space where the duty ratio function acts under saturation conditions. On the regions of nonsaturation [i.e., where the duty ratio function takes values in the open interval (0,1)], the response of the actual PWM controlled system exhibits a nonideal sliding motion which converges toward an integral manifold of the average system (2.3), containing the initial condition prescribed for (2.1), as the sampling frequency grows to infinity.

Proof: The first part of the theorem is obvious. Consider the system described by (2.1) in the augmented state space with coordinate functions (x, z)

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} f(x) \\ \phi(z) + \gamma(z)h(x) \end{bmatrix} + \begin{bmatrix} g(x) & 0 \\ 0 & \gamma(z) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} g(x)\eta(z) \\ 0 \end{bmatrix}\nu$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h(x) \\ \eta(z) \end{bmatrix} \tag{2.4}$$

which we shall express as

$$\frac{d}{dt}x_e = f_e(x_e) + G_{1e}(x_e)u_e + g_{2e}(x_e)\nu$$

$$y_e = h_e(x_e) \tag{2.5}$$

with $x_e = \text{col}(x, z)$ and the vector fields: f_e, g_{2e}, the columns of G_{1e}, and the function h_e are trivially defined from the expression (2.4) of the closed-loop system. The system (2.5) is of the same form (A.2), (A.3), with discontinuous input function ν governed by the PWM operator (2.2). The result of Theorem A.1 immediately applies and the result follows. ∎

Remark: Notice that in the event of a prescribed *constant duty* ratio function $0 < \tau < 1$, the sliding motion described by Theorem 2.1 occurs *globally* in the augmented state space of the closed-loop system. This result should be clear since, in such a case, the saturation condition is never reached for the feedback switching device.

III. Illustrative Examples

Example 1 – Dynamic Discontinuous Feedback Control of a dc-to-dc Power Converter: Consider the following bilinear model of a dc-to-dc power converter of the buck-boost type, described in normalized state-space coordinates [7]:

$$\frac{dx_1}{dt} = \omega_0 x_2 - u\omega_0 x_2 + ub$$

$$\frac{dx_2}{dt} = -\omega_0 x_1 - \omega_1 x_2 + u\omega_0 x_1; \qquad y = x_2 \tag{3.1}$$

where x_1 and x_2 are, respectively, the normalized input current and output voltage variables and b is the normalized external input voltage, assumed here to be a negative quantity (so that the average equilibrium voltage for x_2 is positive). The parameters $\omega_0 = 1/\sqrt{LC}$ and $\omega_1 = 1/RC$ represent, respectively, the input circuit natural oscillating frequency and the output circuit time constant. The switch position function is denoted by u, and it takes values in the discrete set $\{0, 1\}$.

If the switch position is changed according to a PWM policy of the following form:

$$u = \begin{cases} 1 & \text{for } t_k < t \le t_k + \tau(t_k)T \\ 0 & \text{for } t_k + \tau(t_k)T < t \le t_k + T \end{cases} \tag{3.2}$$

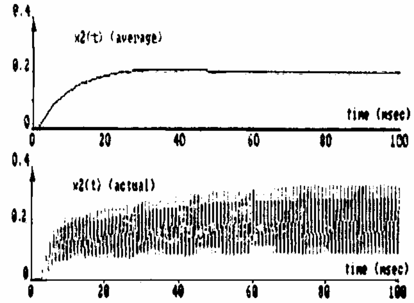

Fig. 2. Actual and average PWM controlled output voltage response of the buck-boost converter.

then, the average PWM model of the converter is simply obtained by substituting, in (3.1), a continuous piecewise smooth duty ratio function $\tau(t)$ in place of the discontinuous control u. The average PWM system is then represented by a bilinear controlled dynamical system with continuous piecewise smooth control input $\tau(\cdot)$ bounded between 0 and 1. The equilibrium point of the average PWM system, for a constant value U of the duty ratio function τ, is obtained as follows:

$$\tau = U;\ X_1(U) = \frac{bU\omega_1}{\left[\omega_0^2(1-U)^2\right]};\ X_2(U) = \frac{-bU}{\left[\omega_0(1-U)\right]}. \tag{3.3}$$

Using the method of *extended linearization* a nonlinear dynamical controller of the proportional integral (P-I) type (see [8]) can be proposed which regulates the average PWM trajectories around the equilibrium point (3.3). Such a nonlinear controller is specified by the following:

$$\frac{d\zeta(t)}{dt} = \left[\frac{\omega_0^2(1-\zeta(t))^3\sqrt{\left(1+\frac{1}{\zeta(t)}\right)}}{4\pi|b|\zeta(t)}\right]e(t)$$

$$\bar{\tau}(t) = \zeta(t) + \left[\frac{0.4\omega_0(1-\zeta(t))^2}{|b|\zeta(t)}\right]e(t)$$

$$e(t) = X_2(U) - x_2 \tag{3.4}$$

$$\tau(t) = \sup\{0, \inf\{1, \bar{\tau}(t)\}\}.$$

It is easy to reinterpret system (3.1), and the dynamical feedback loop (3.4), synthesizing the duty ratio function $\tau(t)$, in the framework of the interconnection scheme proposed in (2.1). Fig. 2 shows simulated average and actual PWM controlled responses for the output voltage x_2 of a buck-boost converter with parameter values $C = 20\ \mu$F, $L = 20$ mH, $R = 30\ \Omega$, and $E = b\sqrt{L} = 15$ V. The constant operating value for the duty ratio was set at $U = 0.75$, for which the corresponding state equilibrium point is $X_1(0.75) = -0.8482$, $X_2(0.75) = 0.2012$. The sampling frequency is 1 KHz. To further approximate the actual PWM responses to those of the designed average PWM system, a suitable low-pass filter was placed at the output of the converter.

Remark: The PWM controlled state trajectories exhibit a chattering behavior which can be brought arbitrarily close to an integral manifold of the average PWM controlled system. The analytical expression of such a manifold is usually very difficult to obtain in an explicit manner. Its calculation entitles solving a linear partial differential equation initialized on a smooth submanifold of the augmented state space representing a whole family of arbitrary initial conditions. The crucial advantage of using the average model approach for PWM controller design, over an equivalent variable structure sliding mode control strategy, is that the explicit expression of such a manifold need not be computed at all. This is also illustrated in the next example.

Example 2 – Dynamical Discontinuous Feedback Control for a Single-Axis Spacecraft Reorientation Maneuver: Consider a single axis jet-controlled spacecraft provided with a pair of opposing torque generators of modulated magnitude u. The orientation is defined in terms of the Cayley–Rodrigues attitude parameter x, defined around a known skewed axis (see [9]). The angular velocity about the principal axis is represented by ω and J denotes the corresponding moment of inertia

$$\frac{dx}{dt} = 0.5(1 + x^2)\omega; \qquad \frac{d\omega}{dt} = J^{-1}u$$
$$y = x. \tag{3.5}$$

The system can be driven toward a final desirable attitude x_d by means of the following smooth observer-based stabilizing nonlinear feedback control strategy:

$$u = -4Je_1e_2\left[\tan^{-1}(x) - \tan^{-1}(x_d)\right] - 2J\left[e_1 + e_2\right]\hat{\omega} \tag{3.6}$$

with $-e_1$ and $-e_2$ representing real stable eigenvalues of the closed-loop linearized system and $\hat{\omega}$ is an estimate of the angular velocity, generated by the reduced-order nonlinear asymptotic observer

$$\frac{d\xi}{dt} = -k\xi - 2k^2\tan^{-1}(y) + J^{-1}u; \qquad \hat{\omega} = \xi + 2k\tan^{-1}(y) \tag{3.7}$$

with $k > 0$ being a design constant specifying the exponential decay of the estimation error.

Suppose now that one could pulse, through a switch located in the feedback path of (3.5)–(3.7), the dynamically generated feedback control action u, given in (3.6) and (3.7), by commanding the switch positions by means of a PWM control strategy. We would then have the following model:

$$\frac{dx}{dt} = 0.5(1 + x^2)\omega; \ \frac{d\omega}{dt} = J^{-1}u\nu \tag{3.8}$$

with ν taking values in the discrete set $\{0, 1\}$.

Furthermore, suppose the PWM strategy specifying ν is such that the duty ratio function is constrained to be a constant μ satisfying $0 < \mu < 1$. We would like to recover the qualitative features of the continuous closed-loop system (3.5)–(3.7) in the discontinuously controlled model (3.8).

In such a case, the average PWM model of (3.8) would be obtained by simply replacing ν by μ. Obviously, the obtained smooth closed-loop design (3.5)–(3.7) coincides with the average model of (3.8) if and only if the feedback signal u is replaced by the scaled feedback signal u/μ. The closed-loop PWM controlled system equations, whose average behavior is given by (3.5)–(3.7), are then constituted by (3.7), (3.8), and the following:

$$u = \frac{-4Je_1e_2\left[\tan^{-1}(x) - \tan^{-1}(x_d)\right] - 2J\left[e_1 + e_2\right]\hat{\omega}}{\mu}$$
$$\nu = \begin{cases} 1 & \text{for } t_k < t \le t_k + \mu T \\ 0 & \text{elsewhere.} \end{cases} \tag{3.9}$$

Simulations were run for a spacecraft with $J = 90$ kg^2, $e1 = e2 = 0.2$ s^{-1}, $x_d = 0.15$ rad, $k = 2$ s^{-1}, using a sampling frequency $1/T$ of 0.5 Hz. In Fig. 3, the actual PWM controlled phase response of (3.7)–(3.9) is seen to chatter about the corresponding average PWM controlled phase response of (3.5)–(3.7). A real sliding motion thus takes place in the augmented state space of the average closed-loop interconnected system (3.5)–(3.7) about a certain integral manifold of the average system whose analytic expression is unimportant.

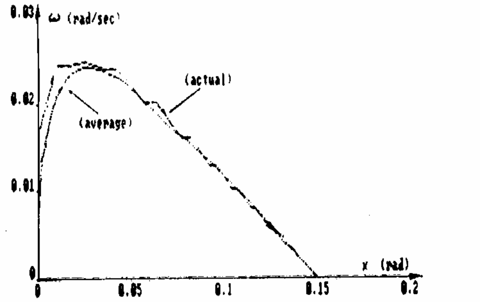

Fig. 3. Actual and average PWM controlled phase trajectory of spacecraft.

IV. Conclusions

An averaging technique has been introduced for the accurate description of discontinuous feedback interconnected systems under a pulse width modulation scheme for the switching element. The averaging process is based on an infinite switching frequency assumption on the feedback enabling device. The proposed average model was shown to entirely capture the main qualitative and quantitative features of the actual finite frequency switched controlled system. The existence of a sliding motion in the augmented state space of the closed-loop pulsed-controlled system, which closely follows the average trajectories, makes the approximation scheme amenable to arbitrary improvement under increased switching frequency specifications for the actual controlled system. This entirely obviates the need for cumbersome approximation schemes, based on traditional discrete-time considerations and the technical difficulties associated with the unbounded character of the PWM operator (see [10]). Such sliding regimes occur in the vicinity of integral manifolds of the average system immersed in regions of the state space where the duty ratio function, associated to the controlled switch, exhibits a nonsaturation condition. In the other regions of the state space, the trajectories of the actual and the average system just coincide for identical initial conditions. Nontrivial illustrative examples were presented.

Appendix

Let $\sigma(t_k)$ denote $\sigma(x(t_k))$, a given scalar function of a state vector x. Consider, then, the nonlinear discontinuously controlled system, described by the following:

$$\frac{dx}{dt} = \begin{cases} f(x) + G_1(x)u + g_2(x) \text{ for } t_k < t \le t_k + \tau\left[\sigma(t_k)\right]T \\ f(x) + G_1(x)u \text{ for } t_k + \tau\left[\sigma(t_k)\right]T < t \le t_k + T \end{cases} \tag{A.1}$$

where the vector field $f(x)$, the columns of $G_1(x)$, and $g_2(x)$ are smooth vector fields defined on R^n. The t_k's represent regularly spaced instants of time where an ideal sampling process takes place. At each of these instants the value of the duty ratio function $\tau[\sigma(x(t_k))] =: \tau[\sigma(t_k)]$ is determined in correspondence with the value of the scalar function $\sigma(x)$, at the sampled value of the state vector $x(t_k)$. The sampling period T is assumed to be sufficiently small, as compared to the time constants associated with the dynamics of the system. Unless otherwise stated, it will be assumed that our considerations are restricted to a region of the state space where the duty ratio function $\tau[\sigma(x)]$ is not saturated, i.e., $\tau[\sigma(x)]$ takes values in the open interval (0,1).

In terms of an ideal switching function ν, taking values in the discrete set $\{0, 1\}$, the aforementioned system can be equivalently represented as the following:

$$\frac{dx}{dt} = \nu\left[f(x) + G_1(x)u + g_2(x)\right] + (1 - \nu)\left[f(x) + G_1(x)u\right],$$

i.e.,

$$\frac{dx}{dt} = f(x) + G_1(x)u + \nu g_2(x) \tag{A.2}$$

with ν obeying a switching policy of the form

$$\nu = \begin{cases} 1 & \text{for } t_k < t \le t_k + \tau\left[\sigma(t_k)\right]T \\ 0 & \text{for } t_k + \tau\left[\sigma(t_k)\right]T < t \le t_k + T. \end{cases} \tag{A.3}$$

The following lemma is a straightforward consequence of the fundamental theorem of calculus.

Lemma A.1: Let f be a smooth vector field and let

$$I_f(t) := \int_0^t f(x(s))\,ds.$$

Then, for any continuous piecewise smooth, strictly positive, function $\mu(x)$

$$\lim_{T\to 0,\, t_k \to t}\left[\frac{I_f(t_k + \mu[x(t_k)]T) - I_f(t_k)}{T}\right] = \mu[x(t)]f(x(t)). \quad (A.4)$$

The next theorem determines the nature of the infinite-frequency average dynamics of (A.2), (A.3) under nonsaturation conditions.

Theorem A.1: Consider a region where the PWM controller is not saturated. Then, as the sampling frequency $1/T$ tends to infinity in system (A.2), (A.3), the discontinuously controlled system coincides with *Filippov's average model*

$$\begin{aligned}\frac{dx}{dt} &= \mu(x)[f(x) + G_1(x)u + g_2(x)] \\ &\quad + (1 - \mu(x))[f(x) + G_1(x)u] \\ &= f(x) + G_1(x)u + \mu(x)g_2(x) = f_{av}(x,u) \quad (A.5)\end{aligned}$$

with a corresponding convex combination function, $\mu(x)$, exactly represented by the duty ratio function $\tau(x)$. Moreover, in such a region, a nonideal sliding regime is exhibited by the actual PWM controlled system (A.2), (A.3) which converges toward an integral manifold M of (A.5) as the sampling frequency tends to infinity.

Proof: Let $f_1(x,u) := f(x) + G_1(x)u + g_2(x)$ and $f_2(x,u) := f(x) + G_1(x)u$, and, as before, let $\sigma(t_k)$ denote $\sigma(x(t_k))$. From (A.2) and (A.3), the state x at time $t_k + T$ is exactly computed as the following:

$$\begin{aligned}x(t_k + T) &= x(t_k) + \int_{t_k}^{t_k+\tau[\sigma(t_k)]T} f_1(x(s), u(s))\,ds \\ &\quad + \int_{t_k+\tau[\sigma(t_k)]T}^{t_k+T} f_2(x(s), u(s))\,ds \\ &= x(t_k) + \int_{t_k}^{t_k+\tau[\sigma(t_k)]T} f_1(x(s), u(s))\,ds \\ &\quad + \int_{t_k}^{t_k+T} f_2(x(s), u(s))\,ds \\ &\quad - \int_{t_k}^{t_k+\tau[\sigma(t_k)]T} f_2(x(s), u(s))\,ds\end{aligned}$$

assuming that $\tau(\sigma(x))$ is neither 0 or 1 in the region of interest, and using the result of Lemma A.1, one has the following:

$$\begin{aligned}\frac{dx(t)}{dt} &= \lim_{T\to 0,\, t_k\to t}\left[\frac{x(t_k + T) - x(t_k)}{T}\right] \\ &= \lim_{T\to 0,\, t_k\to t}\frac{1}{T}\left[x(t_k) + \int_{t_k}^{t_k+\tau[\sigma(t_k)]T} f_1(x(s), u(s))\,ds\right. \\ &\quad + \int_{t_k}^{t_k+T} f_2(x(s), u(s))\,ds \\ &\quad \left. - \int_{t_k}^{t_k+\tau[\sigma(t_k)]T} f_2(x(s), u(s))\,ds\right] \\ &= \tau[\sigma(t)]f_1(x(t), u(t)) \\ &\quad + (1 - \tau[\sigma(t)])f_2(x(t), u(t))\end{aligned}$$

or

$$\begin{aligned}\frac{dx}{dt} &= \tau[\sigma(x)]f_1(x,u) + (1 - \tau[\sigma(x)])f_2(x,u) \\ &= f(x) + G_1(x)u + \tau(x)g_2(x) = f_{av}(x,u), \quad (A.6)\end{aligned}$$

i.e., the infinite frequency model of (A.2), (A.3) coincides with Filippov's geometric average model in which the *convex combination function* $\mu(x)$ defining the average controlled vector field $f_{av}(x,u)$, is precisely taken as the duty ratio function $\tau[\sigma(x)]$. It is clear that for a given reference control input u, on an integral manifold of (A.6) described by, say, $M = \{x \in R^n : m(x) = 0\}$, the controlled vector field of (A.6) is pointwise orthogonal to the gradient of $m(x)$, i.e.,

$$\frac{\partial m}{\partial x}[f(x) + G_1(x)u + \tau(x)g_2(x)] = 0 \qquad \text{on } m(x) = 0. \quad (A.7)$$

The duty ratio function admits, then, a geometrically based definition as follows:

$$\tau(x) = -\frac{\dfrac{\partial m}{\partial x}[f(x) + G_1(x)u]}{\dfrac{\partial m}{\partial x}[g_2(x)]} \qquad \text{on } m(x) = 0. \quad (A.8)$$

From known results about the relation between Filippov's average dynamics and sliding regimes [1], and the assumption that the duty ratio function is locally bounded in the open interval (0,1), it follows that an ideal *sliding regime* exists locally on the manifold M for the variable structure system (A.2), (A.3). The equivalent control $\nu^{EQ}(x)$, associated with such a sliding mode, is simply obtained from the *invariance conditions* [1], [11] of the ideal sliding mode taking place on the integral manifold $M = \{x : m(x) = 0\}$ of the average system, i.e., from the conditions

$$dm/dt = 0 \qquad \text{on } m = 0.$$

In local coordinates, one obtains the following:

$$\begin{aligned}\frac{dm}{dt} &= \frac{\partial m}{\partial x}[\nu^{EQ}(x)f_1(x,u) + (1 - \nu^{EQ}(x))f_2(x,u)] \\ &= \nu^{EQ}(x)\frac{\partial m}{\partial x}[f_1(x,u)] + (1 - \nu^{EQ}(x))\frac{\partial m}{\partial x}[f_2(x,u)] \\ &= 0.\end{aligned}$$

The corresponding equivalent control $\nu^{EQ}(x)$ is then obtained as follows:

$$\nu^{EQ}(x) = -\frac{\dfrac{\partial m}{\partial x}[f_2(x,u)]}{\dfrac{\partial m}{\partial x}[f_1(x,u) - f_2(x,u)]}$$

i.e.,

$$\nu^{EQ}(x) = \frac{\dfrac{\partial m}{\partial x}[f(x) + G_1(x)u]}{\dfrac{\partial m}{\partial x}[g_2(x)]}. \quad (A.9)$$

It follows from (A.8), (A.9) and the uniqueness of the equivalent control [11], that

$$\nu_{EQ}(x) = \tau(x). \quad (A.10)$$

i.e., the *equivalent control* of the sliding motion associated with (A.2) and (A.3) is then precisely constituted by the *duty ratio* associated to the proposed PWM control scheme. The corresponding ideal sliding dynamics is then represented by the following:

$$\begin{aligned}\frac{dx}{dt} &= \nu^{EQ}(x)f_1(x,u) + (1 - \nu^{EQ}(x))f_2(x,u) \\ &= \tau(x)f_1(x,u) + [1 - \tau(x)]f_2(x,u) \\ &= f(x) + G_1(x)u + g_2(x)\tau(x)\end{aligned}$$

which is just the average PWM model (A.6).

It was shown in [11] that the region of existence of a sliding motion is determined by the region on M where $\tau(\sigma(x))$ satisfies the following conditions:

$$0 < \tau\left[\sigma(x)\right] = \nu^{EQ}(x) < 1.$$

By definition of the duty ratio, the aforementioned conditions are evidently satisfied along the integral manifold M, in all regions of the state space where the PWM controller is not saturated.

To show continuity of solutions with respect to the sampling frequency one rewrites the controlled equation (A.2), (A.3) and the average system (A.5) as the following *integral equations* for any sampling interval:

$$x(t_k + T) = x(t_k) + \int_{t_k}^{t_k+T}\left[f(x(s)) + G_1(x(s))u(s)\right]ds + \int_{t_k}^{t_k+\tau(t_k)T} g_2(x(s))\,ds \quad \text{(A.11)}$$

$$x(t_k + T) = x(t_k) + \int_{t_k}^{t_k+T}\left[f(x(s)) + G_1(x(s))u(s)\right]ds + \int_{t_k}^{t_k+T} g_2(x(s))\tau(s)\,ds. \quad \text{(A.12)}$$

It is quite easy to see, using a Taylor series expansion of the last integral terms in (A.11), (A.12) about t_k that (A.11) is a *regular perturbation* of (A.12) in terms which are, *at least*, second order in T [i.e., $o(T^2)$]. The theory of regular perturbations of integral equations (see [12, pp. 273–285]) guarantees continuity of the solutions of (A.11) with respect to (A.12) as T goes to zero. □

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Generalization of Strong Kharitonov Theorems to the Left Sector

Y. C. SOH AND Y. K. FOO

***Abstract*—In this note, we examine the zero locations of interval polynomials. In particular, we shall show that a family of interval polynomials will have only zeros in a certain class of left sector if and only if a finite number of specially chosen vertex polynomials have only zeros in the left sector. This finite number of vertex polynomials is dependent on the damping margins of the left sector but is independent of the orders of the polynomials.**

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The authors are with the School of Electrical Engineering, Nanyang Technical Institute, Singapore 2263, Republic of Singapore.

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I. Introduction

Stability analysis of polynomials are fundamental to the analysis of many control problems. For example, the stability or instability of a given system will be determined by the roots of its characteristic polynomial. If the coefficients of the polynomial are known exactly, then its stability can be readily checked by well-known methods. The difficulty arises when the coefficients are not known exactly. This inevitably will be the case for any real system since approximations, linearizations, and simplifications are often used in the process of deriving a tractable model for the system. Thus, the real problem is to determine the stability of a family of polynomials. One such family of polynomials is the interval polynomials, i.e.,

$$p(s) = t_n^n + t_{n-1}s^{n-1} + \cdots + t_0 \qquad (1.1)$$

where $\alpha_i \le t_i \le \beta_i$, $i = 0, 1, \cdots, n$. It is, of course, not practical to check the stability of each and every member of (1.1). Thus, our problem is to determine if the stability of (1.1) can be determined from a simple subset of (1.1). If we are only interested in the Hurwitz property, then we have the seminal result due to Kharitonov [1] which states that the stability of four vertex polynomials is both necessary and sufficient for the stability of the whole family of interval polynomials. The importance of the Kharitonov result lies with the great reduction in computation cost associated with checking the stability of interval polynomials. Thus, there is a motivation to extend the Kharitonov result to a more general class of stability regions where the Kharitonov-like theorem holds [2]–[4]. A Kharitonov region is defined as a stability region where the zero locations of the entire family of interval polynomials within that region can be inferred from all its vertex polynomials. Note, however, that the concept of a Kharitonov region is only applicable to interval polynomials. For more general polytopes of polynomials, we would have to check all the edge polynomials [5]–[6].

The cost associated with checking all the vertex polynomials can still be great since, in general, we have to check 2^{n+1} polynomials. In this note, we shall show that in order to determine the stability of interval polynomials with respect to the left sector, it is necessary and sufficient to check a finite subset of the vertex polynomials. The exact number of vertex polynomials to be checked will depend on the damping margins, but it will be independent of the orders of the polynomials.

II. Preliminaries

Let S^n denote the family of real interval polynomials defined by

$$p(s) = t_n s^n + t_{n-1}s^{n-1} + \cdots + t_0 \qquad (2.1)$$

where $0 < \alpha_i \le t_i \le \beta_i$, $i = 0, 1, \cdots, n$. Let F_ϕ^n denote the set of all nth-order polynomials which have only zeros in the left sector as shown in Fig. 1. If $p(s) \in S^n$ belongs to F_ϕ^n, then from the Argument Principle, the phase advance of $p(z)$ as z traverses the contour of the left sector once in the counterclockwise direction must be equal to $2n\pi$ rad. However, the polar plot of $p(z)$, as z traverses the upper half of the left sector, will be the mirror image of the polar plot of $p(z)$ as z traverses the lower half of the left sector. Thus, in order to ensure that an nth-order real polynomial $p(s)$ has only zeros within the left sector, it is necessary and sufficient to check if $p(z)$ has gone through a net phase advance of $n\pi$ rad as z traverses the upper half of the left sector.

The upper contour of the left sector is given by the following two segments:

$$s_1:\ e^{j\phi}x;\ x \in [0, r),\ r \to \infty,\ \frac{\pi}{2} \le \phi < \pi.$$

$$s_2:\ \lim_{r\to\infty} re^{j\gamma};\ \phi \le \gamma < \pi.$$

For any real polynomial $p(s) \in S^n$, its polar plot on segment s_2 is given

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