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Structure at infinity, zero dynamics and normal forms of systems undergoing sliding motions

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In this article we examine the structure at infinity of non-linear closed-loop systems locally undergoing sliding regimes about a smooth surface defined in state space. By using a locally diffeomorphic state coordinate transformation, associated with the relative degrec of the system, one obtains a normal form exhibiting the basic internal dynamic structure of the controlled system. It is found that the local existence of sliding motions demands a considerably simple local structure at infinity of the original non-linear system. The ideal sliding dynamics in local sliding surface coordinates is shown to coincide precisely with the zero dynamics. The stability properties of this internal behaviour model are studied. Several illustrative examples are presented.

1. Introduction

The structure at infinity of non-linear systems plays a fundamental role in the understanding of non-linear dynamics and has allowed the extension to a non-linear setting of many basic control problems originally defined for linear time-invariant systems. Among such problems one finds local and global feedback stabilization, disturbance decoupling, interaction decoupling, exact linearization and systems invertibility, as well as many other important implications in two-time-scale systems design and non-linear adaptive control (Bynes and Isidori 1984, Isidori 1985, 1987).

Intimately associated with the structure at infinity of a non-linear system is the possibility of expressing the system in special coordinates called 'normal form' coordinates (Isidori 1987). In such a coordinate system, the underlying input–state–output structure of the system is clearly exhibited and its dynamic properties are easily established. At the heart of such a state coordinate transformation is the notion of the 'relative degree' of the system, a fundamental concept that in recent times has allowed a far-reaching understanding of non-linear controlled dynamics.

In this paper we examine the relevance of the relative degree concept, its associated transformation to normal form coordinates and the role of the zero dynamics in non-linear smooth sytems undergoing local sliding regimes (Utkin 1978) on prescribed smooth manifolds locally defined on open sets of $\mathbb{R}^n$.

It is found that a very simple structure at infinity must be exhibited by a non-linear system whose non-linear scalar output function is used as a feedback signal feeding a variable structure controller devised to create a sliding motion. Namely, for a sliding regime to exist locally on the leaf representing the zero level set of the output function, the system must locally have relative degree one. The corresponding $(n-1)$ dimensional zero dynamics precisely portrays the ideal sliding dynamics (Utkin 1978) in local surface coordinates. The stability properties of the ideal sliding dynamics are determined by the nature of the autonomous zero dynamics equations constrained

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to the sliding leaf. Locally, the dimensions of the centre, stable or unstable manifolds associated with the zero dynamics prescribe the minimum or non-minimum phase properties of the ideal sliding dynamics. The problem of inducing sliding regimes on systems which do not exhibit such a simple structure at infinity is also analysed. Several illustrative examples from various application areas are presented.

In § 2 we present all the basic results about relative degree, normal forms and zero dynamics for non-linear single-input single-output (SISO) systems. The exposition follows very closely that of Isidori (1987). The main results about sliding regimes and their relationships with the previously named concepts is also established in this section. Section 3 is devoted to some simple but illustrative examples, and § 4 contains the conclusions and suggestions for further work.

2. Background and main results

2.1. *Relative degree, normal forms and zero dynamics*

Consider non-linear smooth systems of the form

$$dx/dt = f(x) + g(x)u \tag{2.1}$$
$$y = h(x)$$

where $x \in \mathbb{M}$, an open set in $\mathbb{R}^n$, $u: \mathbb{R}^n \to \mathbb{R}$ is a (possibly discontinuous) scalar input function, and f and g represent locally smooth vector fields defined on $\mathbb{M}$. The output function $h: M \to \mathbb{R}$ is a locally smooth scalar function of the state. We frequently refer to (2.1) as the triple (f, g, h). The level set $h^{-1}(0) := \{x \in \mathbb{R}^n : h(x) = 0\}$, defines a smooth $(n-1)$-dimensional, locally regular manifold of constant rank, i.e. locally integrable (Boothby 1975), referred to as the sliding manifold or sliding leaf. The gradient of $h(x)$, denoted dh, is hence assumed to be non-zero in $\mathbb{M}$ except, possibly, on a set of measure zero, $h^{-1}(0)$ is oriented in such a way that dh locally points from the region where $h(x) < 0$, towards that where $h(x) > 0$. All results in this paper are of local nature, restricted to an open neighbourhood $\mathbb{M}$ of $\mathbb{R}^n$ which has non-empty intersection with $h^{-1}(0)$.

The regularity assumptions about $h(x)$ induce a local regular foliation (Boothby 1975) of $\mathbb{M}$ into disjoint locally integrable manifolds of the form $h^{-1}(k) := \{x : h(x) = k$, for $k \in K\}$. Such manifolds are called leaves of the foliation. We denote the Lie derivative of a scalar function $\phi(x)$, with respect to a smooth vector field f, by $L_f\phi(x)$. One recursively defines, for any positive integer k

$$L_f^k\phi(x) = L_f[L_f^{k-1}\phi(x)]$$

Definition 1

System (2.1) has, locally around x^0, a zero at infinity of multiplicity r if r is the smallest integer for which $L_g L_f^{r-1} h(x) \neq 0$, for all x in a neighbourhood of x^0.

Remark 1

The integer r is also called the local relative degree of the system at x^0. Assuming that, at some time t^0, $x(t^0) = x^0$, the relative degree of the system is interpreted as the minimum number of times that the output $y(t)$ has to be differentiated at time t^0 in order to have the input $u(t^0)$ appearing explicitly in the derivative. For (SISO), causal, linear, time-invariant systems the relative degree is precisely the difference between the degree of the numerator and denominator polynomials in the system transfer function. If a system has finite relative degree r, then r is not larger than n, the dimension of the state.

Let Ker $dh(x)$ denote the set of vector fields defined at x such that

$$\{\xi(x): L_{\xi(x)}h(x) =: \langle dh, \xi(x)\rangle = 0\}$$

Ker $dh(x)$ is a proper subspace of the tangent space $T_x\mathbb{R}$ of $\mathbb{M}$ at x, and hence, Ker dh is a locally smooth distribution (Isidori 1985) tangent to the leaves of the foliation.

Definition 2 (Byrnes and Isidori 1984)

Let Δ^* be the maximal (f, g)-invariant distribution contained in Ker dh. Let $[\cdot, \cdot]$ stand for the Lie bracket of the involved vector fields. A distribution Δ is said to be locally (f, g)-invariant if $[f_1, \Delta]$ and $[g_1, \Delta]$ are properly locally contained in Δ, where $f_1(x) = f(x) + \alpha(x)g(x)$ and $g_1(x) = \beta(x)g(x)$, for some functions $\alpha(x)$ and $\beta(x)$. $[\phi, \Delta] = \{\psi : \psi = [\phi, \delta], \delta \in \Delta\}$. Then, the system (2.1) is said to have zero dynamics of order v_f, provided $v_f = \dim \Delta^*$ (dimension taken generically).

It is easy to prove that if (2.1) has relative degree r, then $v_f = n - r$ (Byrnes and Isidori 1984).

Proposition 1 (Isidori 1987)

Let (2.1) have local relative degree r around x^0. Set $\phi_i(x) = L_f^{i-1}h(x)$ for $i = 1, 2, \ldots, r$, while the functions $\psi_{r+j}(x), j = 1, 2, \ldots, n-r$, are chosen to be functionally independent of the first r functions, with the only additional requirement that, locally around x^0, $L_g\phi_{r+j}(x) = 0$, for $j = 1, 2, \ldots, n-r$. Define new z coordinates as $z = \Phi(x)$ with $\Phi(x) := \mathrm{col}\,[\phi_1(x), \ldots, \phi_n(x)]$ a local diffeomorphism around x^0. Then system (2.1) is locally expressed, around $z^0 = \Phi(x^0)$, as

$$\left.\begin{aligned} dz_i/dt &= z_{i+1}, \quad i = 1, 2, \ldots, r-1 \\ dz_r/dt &= b(z) + a(z)u \\ dz_{r+j}/dt &= q(z); \quad j = 1, 2, \ldots, n-r \end{aligned}\right\} \qquad (2.2)$$

$$y = z_1$$

where $b(z) = L_f^r h(\Phi^{-1}(z))$ and $a(z) = L_g L_f^{r-1} h(\Phi^{-1}(z))$. $a(z) \neq 0$ locally around z^0, by definition of the relative degree. $q(z)$ is an $n-r$-dimensional vector function of all local coordinates z.

System (2.2) is said to be in normal form coordinates.

Denote by $\xi := \mathrm{col}\,(z_1, \ldots, z_r)$ and by $\eta := \mathrm{col}\,(z_r, \ldots, z_n)$ and let x^0 be any point on the leaf $h^{-1}(0)$, i.e. $h(x^0) = 0$ and let $z^0 = \Phi(x^0)$. It then follows that the components of the vector ξ are all zero on z^0, and points on the leaf $h^{-1}(0)$ have local normal coordinates $(0, \eta)$. From (2.2), the unique control that locally constrains the evolution of the system to the leaf $h^{-1}(0)$ is expressed as

$$u = -b(0, \eta)/a(0, \eta) \tag{2.3}$$

or, in original coordinates around x^0

$$u = L_f^r h(x)/L_g L_f^{r-1} h(x) \tag{2.4}$$

For any initial point, on $h^{-1}(0)$, the dynamics governing the system, with control action (2.3), stay locally constrained to the leaf $h^{-1}(0)$, and its expression in local normal coordinates is

$$d\eta/dt = q(0, \eta) =: q_0(\eta) \tag{2.5}$$

The dynamics described by (2.5) are referred to as the zero dynamics. The zero dynamics correspond to the dynamics of the 'internal' behaviour of the system when initial conditions and control actions of the form (2.3) or (2.4), constrain the evolution of the state trajectories to maintain locally a zero output value.

Definition 3 (Byrnes and Isidori 1984)

Let $z_0 = (0, \eta_0)$ be an equilibrium point of (2.1) on the leaf $h^{-1}(0)$. Denote by $Q_0(\eta_0)$ the jacobian of the vector function $q_0(\eta)$, with respect to the last $(n-r)$ components of the vector z, evaluated at z_0. System (2.1) is said to have s left half-plane zeros, u right half-plane zeros, and c purely imaginary zeros, whenever the linear approximation to $q_0(\eta_0)$ represented by $Q_0(\eta_0)$ has s eigenvalues with negative real parts, u eigenvalues with positive real parts and c eigenvalues with zero real parts. The associated (generalized) eigenspaces constitute local smooth distributions on the tangent space $\mathbb{M}$. The corresponding local integral manifolds of these distributions are the stable manifolds W^s, the unstable manifold W^u and the centre manifold W^c. Although the stable and unstable manifolds are unique, there may be many centre manifolds (Guckenheimer and Holmes 1983). The dimensions of these tangent subspaces are, respectively, s, u and c, where $s + u + c = v_f$.

The system (2.1) is said to be minimum-phase if and only if locally around z_0 the dimension of the stable manifold is precisely v_f. If the system is minimum-phase and the zero dynamics is globally asymptotically stable, the system is said to be globally minimum-phase (Byrnes and Isidori 1984).

2.3. *Generalities about local sliding regimes*

A variable structure feedback control law is obtained by letting the control function u take one of two possible feedback function values in the set of allowable feedback laws $U = \{u^+(x), u^-(x)\}$, with $u^+(x) > u^-(x)$ locally on $\mathbb{M}$, according to the sign of the scalar output function $h(x)$, as defined by

$$u = \begin{cases} u^+(x) & \text{for } h(x) > 0 \\ u^-(x) & \text{for } h(x) < 0 \end{cases} \tag{2.6}$$

A sliding regime is said to exist locally on $h^{-1}(0)$ if, as a result of the control policy

(2.6), the state trajectories of (2.1) locally satisfy

$$\lim_{s \to +0} L_{f+gu^+} h < 0, \quad \lim_{s \to -0} L_{f+gu^-} h > 0 \tag{2.7}$$

(Utkin 1978, Sira-Ramirez 1988).

Lemma 1

If a sliding regime locally exists on $h^{-1}(0)$, then necessarily the system (f, g, h) locally has relative degree 1.

Proof

If a sliding regime locally exists on the leaf $h^{-1}(0)$ then, subtracting the expressions (2.7) at any point x on the leaf, we have

$$\begin{aligned} L_{f(x)+g(x)u^+(x)}h(x) - L_{f(x)+g(x)u^-(x)}h(x) &= L_{[u^+(x)-u^-(x)]g(x)}h(x) \\ &= [u^+(x) - u^-(x)]L_{g(x)}h(x) < 0 \end{aligned}$$

i.e. $L_g h < 0$ locally around x.

Remark 2

The above lemma establishes that if the output of the variable structure control system is taken in local coordinates as the sliding surface coordinate function $h(x)$, the resulting controlled triple (f, g, h), under local sliding regime conditions on the leaf $h^{-1}(0)$, has a very simple 'structure at infinity' and it only exhibits one zero at infinity. The finite zero dynamics are necessarily of order $v_f = n - 1$. The condition $L_g h < 0$ is also known as the transversality condition (Sira-Ramirez 1988).

For all initial states located on the sliding leaf $h^{-1}(0)$ the unique control locally zeroing the output $y = h(x)$ in the region of existence of the sliding motions is known as the equivalent control (Utkin 1978), $u_{EQ} = -L_f h/L_g h$. When the initial conditions of (2.1) are set on the sliding leaf and the equivalent control is formally used, the resulting dynamics are the ideal sliding dynamics. A description of such dynamics is

$$dx/dt = f(x) - g(x)[L_f h/L_g h] \tag{2.8}$$

which, by definition, evidently corresponds to a coordinate-free description of the finite zero dynamics associated with the output function $y = h(x)$. On the other hand, the zero dynamics (2.5) represents the description in local normal form coordinates of the $(n-1)$-dimensional ideal sliding dynamics taking place on the sliding manifold.

The simple nature of the structure at infinity results in the fact that Δ^*, the maximal (f, g)-invariant distribution contained in $\operatorname{Ker} dh$, actually coincides with $\operatorname{Ker} dh$ itself, i.e. with the distribution tangent to the leaves $h^{-1}(k)$. It is then easy to see that for any initial state $x \in h^{-1}(k)$ sufficiently close to the sliding leaf $h^{-1}(0)$ (i.e. for k a non-zero arbitrarily small constant) the equivalent control would also make the controlled system trajectory locally evolve on the leaf $h^{-1}(k)$. Owing to the uniqueness of the equivalent control and the nature of the transversality condition, the effect of an appropriate control input, according to (2.6), is to pull this ideal trajectory out of the leaf $h^{-1}(k)$ to make it approach the sliding manifold $h^{-1}(0)$.

Theorem 1

A necessary and sufficient condition for the local existence of a sliding regime on the leaf $h^{-1}(0)$ is that locally where $L_g h(x) < 0$ is valid

$$u^-(x) < -L_f h(x)/L_g h(x) < u^+(x) \tag{2.9}$$

Proof

For the proof, see Sira-Ramirez (1988, 1989).

2.4. *Sliding regimes in variable structure systems with relative degree higher than one*

If for the proposed output function $y = h(x)$ the system locally exhibits relative degree r higher than 1, then an alternative to create a sliding motion which eventually reaches $h^{-1}(0)$ is to use the auxiliary output function (see Isidori 1987 for the original ideas related to local feedback stabilization).

$$w = k(x) = L_f^{r-1} h(x) + c_{r-2} L_f^{r-2} h(x) + \dots + c_1 L_f h(x) + c_0 h(x) \tag{2.10}$$

or, in normal form coordinates

$$w = z_r + c_{r-2} z_{r-1} + \dots + c_1 z_2 + c_0 z_1 \tag{2.11}$$

Evidently, $L_g k(x) = L_g L_f^{r-1} h(x) \neq 0$, i.e. the system (f, g, w) has relative degree one, and a local sliding motion may now be created on $k^{-1}(0)$. If such a motion exists, then ideally $w = 0$ and $z_r = -c_{r-2} z_{r-1} - \dots - c_1 z_2 - c_0 z_1$. Under sliding mode conditions on $k^{-1}(0)$, the system

$$\left.\begin{aligned} dz_i/dt &= z_{i+1}, \quad i = 1, 2, \dots, r-2 \\ dz_{r-1} &= z_r = -c_{r-2} z_{r-1} - \dots - c_1 z_2 - c_0 z_1 \\ dz_{r+j}/dt &= q(z_1, z_2, \dots, z_{r-1}, -(c_{r-2} z_{r-1} + \dots + c_1 z_2 + c_0 z_1), \eta) \\ j &= 1, 2, \dots, n-r \\ y &= z_1 \\ w &= 0 \end{aligned}\right\} \tag{2.12}$$

clearly corresponds to the ideal sliding dynamics (zero dynamics) associated with the new sliding surface $k^{-1}(0)$. It is easy to see that by suitable choice of the parameters $c_0, c_1, \dots, c_{r-2}$, an asymptotically stable motion towards zero is obtained for the first $(r-1)$ coordinates, z_1 through z_{r-1} (and hence for z_r too). Thus, while a sliding motion is taking place on $k^{-1}(0)$, the original output y and its first $(r-1)$ derivatives asymptotically approach zero (i.e. the state vector of the original system apporaches $h^{-1}(0)$).

The corresponding equivalent control is now given, in original coordinates, as

$$\begin{aligned} u_{EQ}(x) &= -L_f k(x)/L_g k(x) \\ &= -[L_f^r h(x) + c_{r-2} L_f^{r-1} h(x) + \dots \\ &\qquad + c_1 L_f^2 h(x) + c_0 L_f h(x)]/L_g L_f^{r-1} h(x) \end{aligned} \tag{2.13}$$

Note that when $h^{-1}(0)$ is reached by the sliding controlled trajectory, the equivalent control locally becomes $u_{EQ}(x) = -[L_f^r h(x)]/L_g L_f^{r-1} h(x)$.

The use of the auxiliary output $w = k(x)$ implies the possibility of either being able to completely measure the original state variables and proceed to use (2.10), or else being able to generate $(r-1)$ derivatives of the original output function y when the state is not available. This last possibility is usually accomplished by means of a high gain 'post-processor' (Isidori 1987) fed by the output signal $y(t)$. The transfer function of such a post-processor is given by

$$\frac{-Kn(s)}{(1+Ts)^{r-1}} \tag{2.14}$$

with T sufficiently small, K sufficiently large, with locally the same sign of $L_g L_f^{r-1} h(x)$, and $n(s)$ a stable polynomial built as

$$n(s) = s^{r-1} + c_{r-2}s^{r-2} + \dots + c_1 s + c_0$$

3. Some illustrative examples

Example 1

Consider the linear time-invariant system described by $dx/dt = Ax + bu$, $y = cx$, i.e. $f(x) = Ax$, $g(x) = b$ and $h(x) = cx$. Hence, $L_g h = cb$, $L_f h = cAx$. If a sliding motion exists on $y = 0$, then necessarily $L_g h = cb \neq 0$, which is a well-known necessary condition (Utkin 1978). Assume, without loss of generality, that the pair (A, b) is originally set in controllable canonical form. Choosing normal form coordinates

$$z_1 = y = c_0 x_1 + c_1 x_2 + \dots + c_{n-2} x_{n-1} + x_n$$

$$z_2 = x_1$$

$$z_3 = x_2, \dots, z_n = x_{n-1}$$

The $(n-1)$-dimensional zero dynamics are expressed as

$$dz_i/dt = z_{i+1}, \quad i = 2, 3, \dots, n-1$$
$$dz_n/dt = -c_0 z_2 - c_1 z_3 - \dots - c_{n-2} z_n$$

The ideal sliding dynamics are independent of the coefficients of A and are entirely governed by the chosen output coefficients $c_0, c_1, \dots, c_{n-2}$. These coefficients are also the coefficients of the numerator polynomial in the transfer function of the original system. In other words, the ideal sliding dynamics are governed by the finite transmission zeros of the system (A, b, c). If the system is minimum-phase, all the zeros lie in the left half of the complex plane and the ideal sliding dynamics is asymptotically stable to the origin.

Example 2

Consider the controlled Van der Pol oscillator

$$dx/dt = \begin{bmatrix} x_2 \\ 2\omega\xi(1-\mu x_1^2)x_2 - \omega^2 x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

with output function

$$y = h(x) = (x_1^2 + x_2^2)^{1/2} - r$$

$$L_g h(x) = x_2(x_1^2 + x_2^2)^{-1/2}, \; L_f h(x) = [(1-\omega^2)x_1 x_2 + 2\omega\xi(1-\mu x_1^2)x_2^2](x_1^2 + x_2^2)^{-1/2}$$

The system has relative degree 1 everywhere on the plane except on the line $x_2 = 0$, i.e. a sliding motion may exist everywhere on the circle of radius r.—The oscillator ideally behaves as a perfect sinusoidal oscillator (Sira-Ramirez 1987 *a*)—except on arbitrarily small neighbourhoods of the intersection points of the circle with the line $x_0 = 0$ (the equivalent control becomes unbounded at these points). Defining the locally diffeomorphic state coordinate transformation

$$\phi_1(x) = z_1 = (x_1^2 + x_2^2)^{1/2} - r$$
$$\phi_2(x) = z_2 = x_1$$

(note that the transformation is everywhere a diffeomorphism except on the line $x_2 = 0$), one obtains the normal form description of the system:

$$dz_1/dt = (1 - \omega^2)(z_1 + r) \cos z_2 \sin z_2 + 2\omega\xi[1 - \mu(z_1 + r)^2 \cos^2 z_2] \times (z_1 + r) \sin^2 z_2 + u \sin z_2$$
$$dz_2/dt = (z_1 + r) \sin z_2$$
$$y = z_1$$

The ideal sliding dynamics are obtained by using

$$u_{EQ} = -L_f h(\Phi^{-1}(z))/L_g h(\Phi^{-1}(z)) = -(1 - \omega^2) r \cos z_2 - 2\omega\xi[1 - \mu r^2 \cos^2 z_2] r \sin z_2$$

on points located on $z_1 = 0$ and it evidently results in

$$dz_2/dt = r \sin z_2$$

In original coordinates one obtains that the ideal sliding motions are described by

$$dx_1/dt = x_2$$
$$dx_2/dt = -x_1$$

and there are no equilibrium points on the sliding manifold.

Example 3

Consider the kinematic and dynamic model of a single-axis externally controlled spacecraft whose orientation is given in terms of the Cayley–Rodrigues representation of the attitude parameter ξ. The angular velocity is represented by ω.

$$d\xi/dt = 0{\cdot}5(1 + \xi^2)\omega, \quad d\omega/dt = I^{-1}u$$

where I represents the moment of inertia and u is the applied torque, usually constrained to the compact set $[-\tau_{max}, \tau_{max}]$. Let the output function be given by $y = h(\xi, \omega) = \omega - 2\lambda(1 + \xi^2)^{-1}\xi$, with $\lambda < 0$. In this case one finds

$$L_f h = -2\lambda^2 \xi(1 - \xi^2)(1 + \xi^2)^{-2}$$
$$L_g h = I^{-1}$$

The system globally has relative degree one and a sliding motion may exist on $y = 0$ by use of the appropriate discontinuous control laws (Dwyer and Sira-

Ramirez 1988). The equivalent control is obtained as $u_{EQ} = -L_f h/L_g h = 2\lambda^2 I\xi(1-\xi^2)(1+\xi^2)^{-2}$. From Theorem 1, it follows that a sliding motion exists on the region $|\xi(1-\xi^2)|(1+\xi^2)^{-2} < \tau_{max}/2\lambda^2 I$. By proper choice of τ_{max} and λ one can easily make the sliding surface bounded away from the forbidden region

$$|\xi(1-\xi^2)|(1+\xi^2)^{-2} > \tau_{max}/2\lambda^2 I$$

Evidently, the ideal sliding dynamics is governed by the asymptotically stable linear equation $d\xi/dt = \lambda\xi$. A global diffeomorphic state coordinate transformation given by

$$z_1 = \phi_1(\xi, \omega) = \omega - 2\lambda(1+\xi^2)^{-1}\xi$$

$$z_2 = \phi_2(\xi, \omega) = \xi$$

transforms the system into normal form.

$$dz_1/dt = -\lambda z_1(1-z_2^2)(1+z_2^2)^{-1} - 2\lambda^2 z_2(1-z_2^2)(1+z_2^2)^{-2} + I^{-1}u$$

$$dz_2/dt = 0{\cdot}5 z_1(1+z_2^2) + \lambda z_2$$

$$y = z_1$$

The equivalent control, in local coordinates is given by $u_{EQ} = -L_f h/L_g h = 2\lambda^2 I z_2(1-z_2^2)(1+z_2^2)^{-2}$ and the ideal sliding dynamics is simply obtained as the linear asymptotically stable dynamics $dz_2/dt = \lambda z_2$.

Example 4

Recently, sliding regimes have been proposed as a means of explaining and designing classical and modern analogue signal encoding circuits of the 'delay modulation' type (Steele 1975, Sira-Ramirez 1987 b). One such circuit is constituted by the double integration delta modulation system whose encoder portion is described by

$$dx_1/dt = x_2, \quad dx_2/dt = u, \quad u = V \operatorname{sign} y$$

$$y = h(x_1, x_2) = [a(t) - x_1]$$

where $a(t)$ in the analogue signal to be encoded by the circuit, V is the quantization voltage, and the arrangement of cascaded integrators is called the local decoder. The fast switching sequence u (assumed to be detectable by a suitable device) is transmitted over a transmission channel and remotely decoded by two cascaded integrators. In this case $L_g h = 0$ and $L_g L_f h = -1$. The relative degree of the system is 2 and a sliding regime does not exist on $y = 0$ (i.e. x_1 does not follow $a(t)$ in a sliding mode fashion). An auxiliary output may be devised:

$$w = L_f h(x) + c_0 h(x) = -x_2 + c_0[a(t) - x_1]$$

and we now have $L_f w = L_f^2 h(x) + c_0 L_f h(x) = -c_0 x_2$, $L_g w = -1 \neq 0$, i.e. the new system has relative degree one. Under ideal sliding conditions on $w = 0$, one has $x_2 = c_0[a(t) - x_1]$ and the associated zero dynamics are $dx_1/dt = -c_0[x_1 - a(t)]$. Choosing c_0 as a positive constant, the ideal sliding dynamics on $w = 0$ drives x_1 to follow asymptotically $a(t)$. For the existence of a sliding regime on $w = 0$, the corresponding equivalent control $u_{EQ} = L_f w/L_g w = c_0 x_2 = c_0^2[a(t) - x_1]$ must satisfy

$$-V/c_0^2 < [a(t) - x_1] < V/c_0^2$$

which is a modified tracking error overload condition (Steele 1975).

4. Conclusions and suggestions for further research

The relative degree of the system has been shown to play a fundamental role in the investigation of the local existence of sliding regimes on regular smooth manifolds defined in state space by zero level sets of non-linear scalar output equations. The local existence of a sliding regime demands the existence of just one local zero at infinity for the non-linear controlled system. The transformation of the system to normal form coordinates clearly shows the minimum, or non-minimum, phase properties of the zero dynamics. This is intimately related to the qualitative properties of the ideal sliding dynamics. When the system locally exhibits relative degree higher than one, a method based on the ideas of Isidori (1987) was established to accomplish the zeroing of the original output function via the creation of a sliding regime on an auxiliary sliding surface. This was defined on the basis of an extra output function obtained by suitable 'original output post-processing' (Isidori 1987). This new sliding motion was shown to be always capable of asymptotically converging towards the leaf corresponding to the zero level set of the original output function. Several illustrative examples were presented.

The concept of relative degree has been also extended to multivariable non-linear systems by Isidori (1987). An area for further research is the investigation of the role of this important extended concept in determining the existence of sliding regimes in multi-input non-linear variable structure systems.

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REFERENCES

BOOTHBY, W. R., 1975, *An Introduction to Differentiable Manifolds and Riemannian Geometry* (New York: Academic Press).

BYRNES, C. I., and ISIDORI, A., 1984, A frequency domain philosophy for nonlinear systems, with applications to stabilization and to adaptive control. *Proc. 23rd I.E.E.E. Conf. on Decision and Control*, Las Vegas.

DWYER, T. A. W., III, and SIRA-RAMIREZ, H., 1988, *A.I.A.A. J. Guidance, Control, Dynamics*, **11**, 262.

GUCKENHEIMER, J., and HOLMES, P., 1983, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (New York: Springer-Verlag).

ISIDORI, A., 1985, *Nonlinear Control Systems: An introduction*, Notes on Information and Control Sciences, Vol. 72 (New York: Springer Verlag); 1987, *Lectures on Nonlinear Control*, Notes for a course at the Carl Cranz Gesellshaft.

SIRA-RAMIREZ, H., 1987 a, *I.E.E.E. Trans. Circuits Syst.*, **34**, 103; 1987 b, Analog signal encoding in delta modulation circuits using the theory of variable structures systems. *Proc. XXV Annual Allerton Conference on Communications, Control and Computing*, University of Illinois, Monticello; 1988, *Int. J. Control*, **48**, 1359; 1989, *I.E.E.E. Trans. autom. Control*, **34**, 1186.

STEELE, R., 1975, *Delta Modulation Systems* (London: Pentech).

UTKIN, V., 1978, *Sliding Modes and their Application in Variable Structure Systems* (Moscow: MIR).