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Sliding Mode Controlled Relaxation Oscillations

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***Abstract*—The possibility of inducing controlled relaxation oscillations in second order systems is explored from the sliding mode control viewpoint. Applications to the design of triangular and square wave oscillators are presented.**

***Keywords*—relaxation oscillations, sliding mode control, nonlinear systems.**

I. Introduction

Relaxation oscillations have been the object of detailed studies since the fundamental work of Cartan and Cartan [1] and Van der Pol [2] in the 1920's. Non-forced second-order nonlinear differential equations which lead to relaxation oscillations have been shown to posses intriguing and interesting properties. For an extensive treatment of the many aspects related to second-order dynamical systems exhibiting this kind of behavior, the reader is referred to the work of LaSalle [3], Nayfeh [4], Pacheco de Figueiredo [5], and also Guckenheimer and Holmes [6].

In this paper we explore the possibilities of synthesizing a *controlled* relaxation oscillator using the theory of variable structure systems undergoing sliding mode behavior (see Utkin [7]). This theory was used by Sira-Ramirez [8] to obtain harmonic periodic motions in controlled Van der Pol oscillators and it was also used, by the same author, in [9] to study the controlled energy transfer in bilinear networks and dc to dc power supplies.

It is also shown that a relaxation oscillator can be synthesized by a second-order system which periodically exhibits a sliding motion on a suitable "slow manifold" defined in the state space of the system. On such manifold, the sliding mode existence conditions are suddenly lost and the controlled trajectory "falls" through a "fast relaxation manifold" to meet a second portion of the same slow manifold where it sustains a new controlled sliding motion which evolves in the symmetrically opposed direction of the first one. By reasons of symmetry, the sliding mode controlled trajectory subsequently loses again the sliding mode existence properties and quickly reaches the first portion of the sliding manifold. A periodical motion is then obtained, characterized by slow sliding and fast relaxation. By appropriately choosing the parameters defining the sliding surface, it is shown that arbitrary amplitude and frequency characteristics can be imposed

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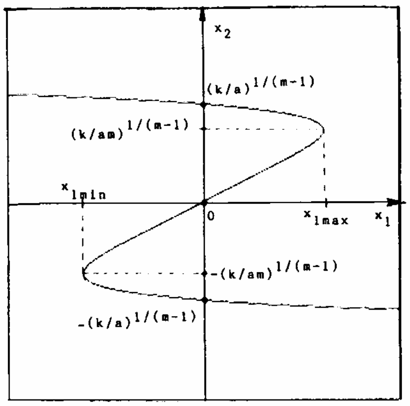

Fig. 1. Sliding surface leading to a relaxation oscillation.

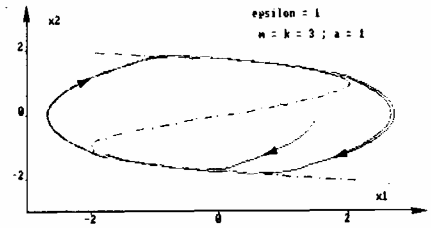

Fig. 2. A sliding mode controlled periodic oscillation.

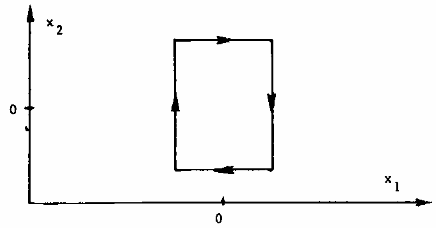

Fig. 3. Phase portrait of an ideal triangular and square wave oscillator.

on the obtained periodic motions which generate either a triangular or a square wave.

Section II contains the main results and presents applications of the sliding mode controlled relaxation oscillator to the synthesis of triangular and square wave oscillators with arbitrarily chosen amplitude and frequency characteristics. The Appendix contains the needed background material about sliding regimes in general nonlinear smooth dynamical systems.

II. Main Results

2.1. A General Sliding Mode controlled Periodic Oscillator

Consider the second-order controlled system, written in state space form:

$$dx_1/dt = x_2;\ \epsilon^2\, dx_2/dt = u;\ u = \text{sign}(y)$$
$$y = h(x) = -x_1 + kx_2 - ax_2^m \tag{2.1}$$

where a, k, and ϵ are positive constants while $m > 1$, is an odd integer. In reference to the notation in the Appendix and [11], the vector fields f and g are given by $f = x_2\partial/\partial x_1$; $g = \epsilon^{-2}\partial/\partial x_2$, while $u^+(x) = +1$ and $u^-(x) = -1$.

Notice that the manifold $k^{-1}(0)$ is represented by a smooth curve that intersects the x_2 axis only three times in the state coordinate plane (see Fig 1). These intersections are located at

$$x_2 = 0,\quad x_2 = +[k/a]^{1/(m-1)};\quad \text{and}\quad x_2 = -[k/a]^{1/(m-1)}. \tag{2.2}$$

The same curve presents a local maximum characterized by

$$x_{1\max} = a(m-1)[k/am]^{m/(m-1)} \text{ at } x_2 = [k/am]^{1/(m-1)} \tag{2.3}$$

and a local minimum, symmetrically opposite to the maximum, specified by

$$x_{1\min} = a(1-m)[k/am]^{m/(m-1)} \text{ at } x_2 = -(k/am)^{1/(m-1)}. \tag{2.4}$$

It is easy to see, from Theorem A.1, and the fact that $L_gh = \epsilon^{-2}[k - amx_2^{m-1}]$, that if a sliding regime locally exists on $h^{-1}(0)$, then it must necessarily exists on the region $|x_2| > (k/am)^{1/(m-1)}$. The equivalent control that would locally sustain the trajectories ideally constrained to $h^{-1}(0)$ is formally obtained as

$$u^{EQ}(x) = -L_fh/L_gh = \epsilon^2 x_2/[k - amx_2^{m-1}], \qquad x \in h^{-1}(0). \tag{2.5}$$

The region of existence of a sliding motion is thus specified, according to Theorem A.2, by the pair of inequalities:

$$-1 < \epsilon^2 x_2/[k - amx_2^{m-1}] < 1. \tag{2.6}$$

Notice that (2.6) is equivalent, for any ϵ, to: $\|k - amx_2^{m-1}\| > \epsilon^2|x_2|$. In other words, the sliding mode existence conditions are lost within a small vicinity of the points where the sliding surface exhibits the local maxima and minima, i.e., around $x_2 = \pm(k/am)^{1/(m-1)}$. Hence, a sliding motion can be sustained on the nearly horizontal branches of the sliding manifold and relaxation occurs at the local extremals where the controlled trajectory "falls", through an arc of a parabola, towards the horizontal branch of the surface. The parameter ϵ governs the velocity of the state trajectories along the "relaxation manifold".

Remark: From Remarks A.1 and A.2 in the Appendix, it is clear that a sliding motion is also possible in the band $|x_2| < (k/am)^{1/(m-1)}$ for the reversed switching logic ($u = -\text{sign}(y)$). For this reason, the necessary and sufficient conditions (2.6) do not fail on the entire band, but only on a small vicinity of the local extremal points. For the fixed switching logic, $u = \text{sign}(y)$, the transversality condition (A.3), necessary for the existence of a sliding regime (Sira-Ramirez [10]), is satisfied everywhere along $y = 0$ except in the band $|x_2| < (k/am)^{1/(m-1)}$, where the controlled motions are constituted by the parabolic arcs. □

Fig. 2 depicts a simulated state space controlled trajectories of system (2.1) for the parameters: $a = 1, \epsilon = 1, k = m = 3$.

2.2. Applications to Synthesis of Triangular and Square Wave Oscillators

Fig. 3 depicts the state portrait of an ideal second-order relaxation oscillator generating triangular, and square, periodic waves. Notice that the time response of the x_1 coordinate is a triangular periodic signal with peak to peak amplitude given by 2 V and frequency $f = S/(4\ V)$ (ideally, the fast trajectories along

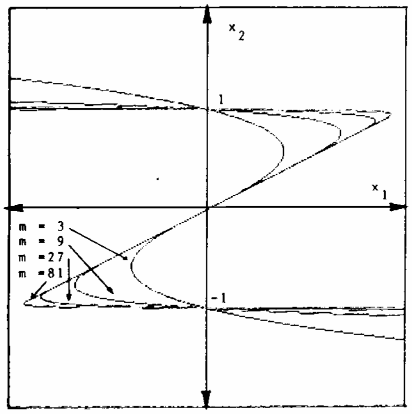

Fig. 4. Dependence on m of the sliding manifold $x_1 = ax_2 + kx_2^m$.

the vertical lines $x_1 = \pm V$, have infinite speed). S represents the amplitude of the square wave or, equivalently, the slope of the ascending branch of the triangular wave. The time response of the x_2 coordinate is then a perfect square wave with peak to peak amplitude $2S$ and frequency also equal to $S/(4\,V)$.

Fig. 4 depicts the dependence of the sliding manifold on the parameter m for fixed values of a and k ($a = k = 1$). It is clear that as m grows, the portions transversal to x_2 tend to become horizontal. This is mathematically proved in the next proposition together with the possibilities of emulating the ideal phase portrait.

Proposition 2.1: As m and $1/\epsilon$ tend to infinity, the phase portrait of system (2.1) tends to that of an ideal triangular wave oscillator.

Proof: It should be obvious that when ϵ tends to zero, the relaxation manifold tends to be a vertical line in the phase space. This is clearly seen because the relaxation time, easily computed from (2.1), is given by: $4\epsilon^2(k/a)^{1/(m-1)}$ which tends to zero as ϵ tends to zero, independently of m. To see that the branches transversal to the x_2 axis of the sliding curve tend to become horizontal as m tends to infinity, consider the slope of these branches at the crossings of the x_2-axis ($x_2 = \pm[k/a]^{1/(m-1)}$). This slope is given by

$$dx_1/dx_2 = k(1-m). \tag{2.7}$$

From (2.7), it is clear that the limit: $\lim_{m \to +\infty} dx_1/dx_2 = -\infty$, i.e., the sliding branch of the manifold tends to be parallel to the x_1 axes as m grows. This fact can also be verified by noting that the difference the points x_2 in (2.2) and (2.3) tends to zero as m tends to infinity. □

Triangular Wave Oscillator: From the above proposition, the ideal characteristics of the triangular oscillator, and (2.2)–(2.4), it follows that—for finite but large m—a desired frequency f and a desired amplitude V, of a triangular wave oscillation, may be approximated by the expressions:

$$V = a(k/am)^{m/(m-1)}(m-1)$$
$$s = 4Vf = (k/a)^{1/(m-1)}. \tag{2.8}$$

Solving this set of equations with respect to a and k one obtains

$$a = V(4Vf)^{-m}[m^{m/(m-1)}](m-1)^{-1} \tag{2.9}$$
$$k = a[4Vf]^{m-1} = [4(m-1)f]^{-1}m^{m/(m-1)}. \tag{2.10}$$

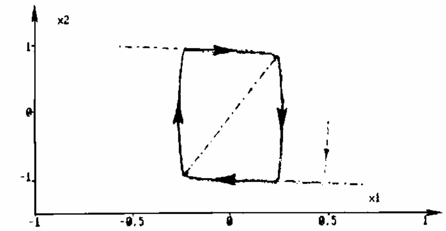

Fig. 5. Phase portrait of sliding mode designed triangular wave oscillator.

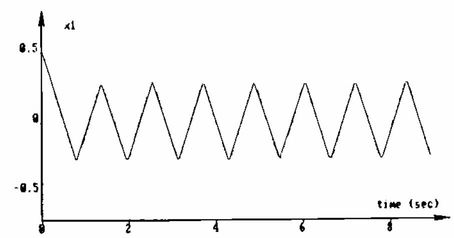

Fig. 6. State trajectory of sliding-mode generated triangular wave oscillation.

The parameters ϵ and m must be chosen to satisfy the following two requirements, respectively: 1) negligible relaxation time of the controlled trajectory, and 2) the required "flatness" of the "slow" portion of the sliding manifold.

Square Wave Oscillator: For a required amplitude s and frequency f of a square wave oscillation, it follows, from (2.9), (2.10), and the ideal relation $s = 4Vf$, that

$$a = (s/4f)s^{-m}[m^{m/(m-1)}](m-1)^{-1} \tag{2.11}$$
$$k = as^{m-1} = [4(m-1)f]^{-1}m^{m/(m-1)}. \tag{2.12}$$

Thus, modulo high frequency chatterings exhibited in the sliding portions of the phase portrait (which can be conveniently eliminated by cascading the chosen output x_1 or x_2 of the proposed oscillator with a low pass filter), a nearly square wave or triangular wave oscillator can be synthesized on the basis of sliding mode induced relaxation oscillations.

Figs. 5 and 6 show the phase portrait and the time response of a computer simulated sliding-mode triangular wave oscillator with required amplitude $V = 0.25$ V, and frequency $f = 1$ Hz. The computed values of the required parameters were $a = k = 0.2947$. The values of m and ϵ were arbitrarily chosen to be $m = 27$ and $\epsilon = 0.05$.

III. Conclusions and Suggestions for Further Research

Relaxation oscillations were shown to be entirely possible in simple controlled second order variable structure dynamical systems undergoing sliding motions on appropriate nonlinear manifolds. The basic feature of such manifolds is that they do not guarantee global existence conditions for the sliding regime and allow the relaxed trajectory to periodically, and symmetrically, meet a different portion of the same manifold where the sliding conditions are recovered. The result allows for the conceptually feasible design, demonstrated only through computer simulations, of a simple class of oscillators with arbitrarily specified output wave amplitude and frequency characteristics.

The result is related, in higher dimensional cases, to the possibilities of sliding mode induced catastrophes in a number of intellectually stimulating examples. This research topic may be further pursued in the future.

Appendix

Consider the smooth (C^∞) nonlinear dynamical system

$$dx/dt = f(x) + g(x)u$$
$$y = h(x) \qquad \text{(A.1)}$$

with f and g locally defined smooth vector fields and h a C^∞ function. Equation (A.1) is referred as the triple (f, g, h). The set $h^{-1}(0) = \{x : y = h(x) = 0\}$ is assumed to be a *locally regular integrable manifold* (Boothby [11]). $L_\phi \eta$ denotes the Lie derivative (directional derivative [11]) of the smooth function η in the direction of the vector field ϕ.

Definition A.1 [7]: A variable structure feedback control law:

$$u = \begin{cases} u^+(x), & \text{for } y > 0 \\ u^-(x), & \text{for } y < 0 \end{cases} \qquad \text{(A.2)}$$

is said to locally create a *sliding regime* on $h^{-1}(0)$ if and only if

$$\lim_{y \to +0} L_{f+gu^+} h < 0 \quad \text{and} \quad \lim_{y \to -0} L_{f+gu^-} h > 0 \qquad \text{(A.3)}$$

where $u^+(x)$ and $u^-(x)$ are given smooth scalar feedback control functions. Without loss of generality we assume that, locally, $u^+(x) > u^-(x)$.

Theorem A.1 [10]: If a sliding regime locally exists on $h^{-1}(0)$ then, necessarily, the *transversality condition*:

$$L_g h < 0 \qquad \text{(A.4)}$$

is locally valid on $h^{-1}(0)$.

Remark A.1: Notice that by changing $h(x)$ by $-h(x)$, the sliding manifold is not altered. However, in such case (A.4) would be necessarily of the form $L_g h > 0$ and to locally create a sliding regime, the switching law (A.3) should be reversed. For this reason, one may say, in general, that the transversality condition is represented by $L_g h \neq 0$. In such a case, however, the inequality sign of (A.4) would be intimately related to the adopted switching logic. Therefore, we shall assume that the feedback law (A.2) is fixed from the outset and that it can not be reversed. Hence, if a sliding motion is known to exist, then, as a function of x, the expression $L_g h < 0$ represents an open set containing the region of existence of such sliding motions. □

The sliding mode controlled trajectories locally evolve around $h^{-1}(0)$ in a chattering fashion. Ideally one may assume that the controlled state trajectories adopt $h^{-1}(0)$ as a local integral manifold. One may then formally assume that the constrained trajectories are being controlled by a smooth scalar feedback control function $u^{\text{EQ}}(x)$, known as the *equivalent control* [7]. The equivalent control satisfies, on $y = 0$:

$$L_{f+gu^{\text{EQ}}} h = 0 \text{ i.e., } u^{\text{EQ}}(x) = -L_f h / L_g h, \qquad x \in h^{-1}(0). \qquad \text{(A.5)}$$

The region of existence of a sliding regime on the manifold $h^{-1}(0)$ is specified by the following theorem. (See also [12] for a more general case.)

Theorem A.2 [10]: A sliding regime locally exists on $h^{-1}(0)$ if and only if locally on $h^{-1}(0)$:

$$u^-(x) < u^{\text{EQ}}(x) = -L_f h / L_g h < u^+(x). \qquad \text{(A.6)}$$

Remark A.2: Condition (A.6) is somewhat independent of the switching logic (A.2), i.e., it does not distinguish whether $u = u^+(x)$ is being used for $y > 0$ or for $y < 0$. It only asserts that as long as the equivalent control is locally intermediate among the extreme feedback laws u^+ and u^-, a sliding regime may be created on such region. If the switching logic is fixed at the outset —as we have assumed—then the sliding region is included in the set represented by (A.6) but a larger portion is also included, namely; the one where the *reversed* switching logic also creates a sliding regime. For this reason, the combination of the transversality condition (A.3) and (A.6) is the appropriate way to establish the region of existence of a sliding mode, for a fixed switching logic, in a nonlinear dynamical system. □

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