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Contributed Paper

A VARIABLE STRUCTURE CONTROL APPROACH TO THE PROBLEM OF SOFT LANDING ON A PLANET*

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Abstract. A discontinuous feedback control policy is proposed for the exponential sliding of terminal descent trajectories of a vertically controlled spacecraft attempting a soft landing maneuver on a planet.

Key Words—Soft landing problem, sliding regimes, nonlinear systems.

1. Introduction

The problem of soft landing on a planet has received some attention in the control systems literature, especially from an optimal control viewpoint. An early contribution using a fuel-optimal control approach, in the case of soft landing on the Moon, was given by Meditch (1964). Further studies, within the same optimal approach, were presented in Flemming and Rishel (1975) and, later on, by Cantoni and Finzi (1980). The minimum fuel problem was shown to yield a bang-bang control solution with a single control switching performed on the basis of sign evaluation of a nonlinear (logarithmic) surface coordinate function, defined in the state space of the system. It was shown in Meditch (1964) that the minimum fuel and the minimum time landing problems are completely equivalent.

In this article, attention is paid to a non-optimal solution for the problem of soft landing on a planet by using the theory of Variable Structure Systems and their associated Sliding Regimes (Utkin, 1978 a). The sliding mode control approach has recently been applied to other classes of aerospace problems in the context of spacecraft reorientation maneuvers (see Vadali, 1986), reorientation and detumbling maneuvers (Dwyer and Sira-Ramirez, 1988), active nutation damping for orbiting satellites (Sira-Ramirez and Dwyer, 1987), flexible spacecraft control (Dwyer et al., 1987), and aircraft dynamics (Calise and Kramer, 1984). Sliding mode control constitutes a simple, robust, feedback control technique with great potential for practical applications. For general background on the subject and thorough surveys, the reader is referred to the many works of Utkin (1978 a; 1977; 1984; 1987), the book by Itkis (1976), the

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tutorial by De Carlo et al. (1988) and recent articles by Sira-Ramirez (1988 a; b; 1989 a; c).

A variable structure control solution to the problem of controlled landing is shown to specify a switching policy which is capable of guaranteeing the soft landing of a thrust-controlled vehicle, by inducing an ideal exponential rate of decay in the height and vertical speed variables. The mathematical idealization, inherent in the solution scheme, is shown to result in the physical inconvenience of total residual fuel exhaustion due to the infinite time duration of the landing maneuver. A hybrid approach is hence proposed in which a sliding mode control strategy is used in combination with an optimal bang-bang policy for the touchdown stage. Switching to the optimal policy is triggered by either critical height or critical residual fuel mass availability. As an alternative, a free fall trajectory may also be proposed for touchdown with a prespecified allowable downward velocity of impact.

Section 2 contains the main results of the article, while Sec. 3 is devoted to the conclusions and suggestions for further research. In order to make the article self-contained, an Appendix is included with some background material about sliding mode control in nonlinear dynamical systems.

2. The Problem of Soft Controlled Landing

2.1 A landing model for a vertically controlled vehicle (Meditch, 1964) Consider the nonlinear model describing the motion, and mass behavior, of a thrust controlled vehicle attempting a vertical regulated landing on the surface of a planet of gravity acceleration constant, g and negligible atmospheric resistance (Meditch, 1964; Flemming and Rishel, 1975; Cantoni and Finzi, 1980):

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = g - \left(\frac{\sigma}{x_3}\right)u, \quad \frac{dx_3}{dt} = -u, \tag{2.1}$$

where x_1 is the position (height) on the vertical axis, positively oriented downwards (i.e., $x_1<0$ for actual positive height), x_2 is the downwards velocity and x_3 represents the combined mass of the vehicle and the residual fuel (See Fig. 1). u is the controlled rate of ejection per unit time, while σ is the relative ejection velocity. The control is restricted to take values on the interval $[0, \alpha]$ with $\alpha>0$. The maximum thrust of the braking engine is thus $\sigma\alpha$. In relation to (A.1) (See Appendix), the vector fields $\boldsymbol{f}$ and $\boldsymbol{g}$ are expressed, in local coordinates, as

$$\boldsymbol{f}(\boldsymbol{x}) = x_2\frac{\partial}{\partial x_1} + g\frac{\partial}{\partial x_2} \quad \text{and} \quad \boldsymbol{g}(\boldsymbol{x}) = -\left(\frac{\sigma}{x_3}\right)\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_2}.$$

It will be assumed that the engine is capable of pulsed thrusting with reasonably high frequency (if only low frequency trusting is technically possible, our results form the basis for a Pulse-Width-Modulated approach thanks to the intimate connections among this discontinuous control technique and sliding regimes. (See Sira-Ramirez, 1988 c, 1989 b)). If no pulsed control is possible at all, then the approach still allows for the specification of either a smooth nonlinear amplitude modulated feedback control action or an equivalent

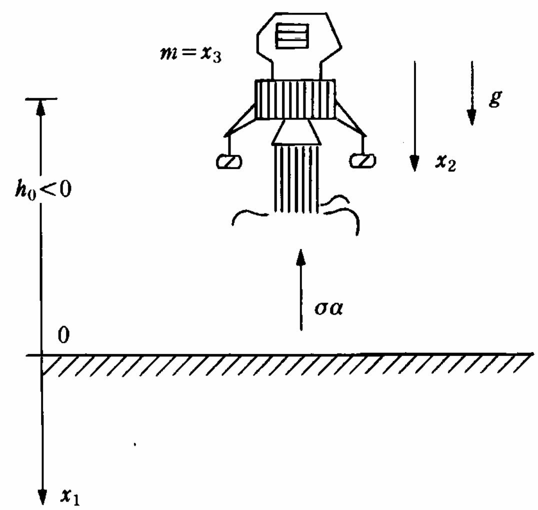


Fig. 1. Vertically controlled descent on the surface of a planet.

smoothed high gain approach which can be easily programmed on the actuator regulating the rate of ejection.

2.2 A variable structure control approach Soft landing on the ground may be seen as a particular case of a hovering maneuver at certain prespecified height. Taking the output as $y=h(\boldsymbol{x})=x_1-K$, a sliding mode approach would take $y=0$ as the sliding surface. However, in this case, $L_g h=0$ and $L_g L_f h=-(\sigma/x_3)\neq 0$ i.e., the system $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ has relative degree 2 (see the Appendix). This indicates that a hovering maneuver, ideally represented by $x_1=K$, cannot be directly accomplished by a sliding mode approach on the basis of height information feedback alone. An auxiliary output must be devised, as indicated in (A.9) of the form,

$$w = k(\boldsymbol{x}) = c_1 h(\boldsymbol{x}) + L_f h(\boldsymbol{x}) = c_1(x_1-K) + x_2. \tag{2.2}$$

Since $L_g k(\boldsymbol{x})=-(\sigma/x_3)<0$, $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{k})$ has relative degree one and a sliding motion locally exists on the manifold $k^{-1}(0)$. On $w=0$, $x_2=-c_1(x_1-K)$, with c_1 chosen as a positive constant. The ideal sliding motion is then governed by $dx_1/dt=-c_1(x_1-K)$, i.e., an asymptotically stable evolution of the position (height) variable may be ideally achieved towards the sliding manifold $x_1=K$, while the velocity x_2 asymptotically approaches zero. This motion exhibits an exponential rate of decay solely prescribed by the design parameter c_1. For a soft landing on the ground we let $K=0$ from now on.

Since, $L_f k(\boldsymbol{x})+[L_g k(\boldsymbol{x})]u=g+c_1x_2-(\sigma/x_3)u$, the equivalent control (see the Appendix) achieving soft landing is given by $u^{\mathrm{EQ}}(\boldsymbol{x})=(x_3/\sigma)[g+c_1x_2]$. Using Theorem A.2, the switching logic,

$$u = \begin{cases} \alpha & \text{for } w > 0, \\ 0 & \text{for } w < 0, \end{cases} \tag{2.3}$$

locally creates a sliding motion on $k^{-1}(0)$ in the region determined by

$$0 < u^{EQ}(x) = (x_3/\sigma)[g+c_1x_2] < \alpha. \tag{2.4}$$

The left hand side inequality is trivially satisfied for a descending maneuver ($x_2>0$). The inequality in the right hand side establishes that the ratio of maximum thrust to remaining mass must exceed gravity plus average controlled acceleration (i.e., $\sigma\alpha/x_3>g+c_1x_2$).

2.3 Reachability of the sliding region from $w<0$ If initially $w<0$ (i.e., the representative point of the state trajectory is below the sliding plane $k^{-1}(0)$), the control action (2.3) specifies a free fall trajectory until reaching of the plane $w=0$. The free fall does not spend any fuel mass and hence $x_3(t)$ remains constant, i.e., it remains equal to the total initial mass of the spacecraft. The point of coordinates x_1^*, x_2^*, x_3^*, on the plane $w=0$, hit by the descending trajectory is obtained by intersecting the parabolic state trajectory of (2.1), computed with $u=0$, with the sliding plane $w=0$:

$$\left.\begin{aligned} x_1^* &= \frac{g}{c_1^2}\left\{1-\sqrt{1-\frac{2c_1^2}{g}\left(h_0-\frac{v_0^2}{2g}\right)}\right\} \\ x_2^* &= -\frac{g}{c_1}\left\{1-\sqrt{1-\frac{2c_1^2}{g}\left(h_0-\frac{v_0^2}{2g}\right)}\right\} \\ x_3^* &= m_0 \end{aligned}\right\}, \tag{2.5}$$

where h_0 (<0) is the initial height v_0 is the initial velocity and m_0 is the total initial mass. Condition (2.4) implies that the sliding mode region is reachable by a free fall trajectory provided the initial data and system parameters satisfy

$$m_0 < \frac{\sigma\alpha}{g+c_1x_2^*} = \frac{\sigma\alpha}{g\left\{\sqrt{1-\frac{2c_1^2}{g}\left(h_0-\frac{v_0^2}{2g}\right)}\right\}}. \tag{2.6}$$

Since the factor $\sqrt{1-(2c_1^2/g)(h_0-v_0^2/2g)}>1$, a necessary but not sufficient condition for the existence of a sliding mode on the region (2.4) is represented by the simpler condition $m_0g<\sigma\alpha$, i.e., the maximum braking force must necessarily exceed the total spacecraft weight; in other words, the maximum trust to initial mass ratio must be greater than the gravitational acceleration.

Figure 2 shows a free fall trajectory, in state space, which reaches the region of existence of the sliding regime, on the plane $w=0$, after a free fall descent with zero initial downwards velocity.

2.4 Reachability of the sliding region from $w>0$ If $w>0$ initially, (i.e., the representative point is *above* the sliding plane) a full thrust action is necessary to reach $k^{-1}(0)$. However, certain restrictions must be satisfied on the initial data in order to guarantee reachability of the region of existence of a sliding regime, (2.4), within physically meaningful conditions.

The full-thrust trajectory is easily computed by integration of the system equations (2.1), with $u=\alpha$. In this case, the parametric representation of the

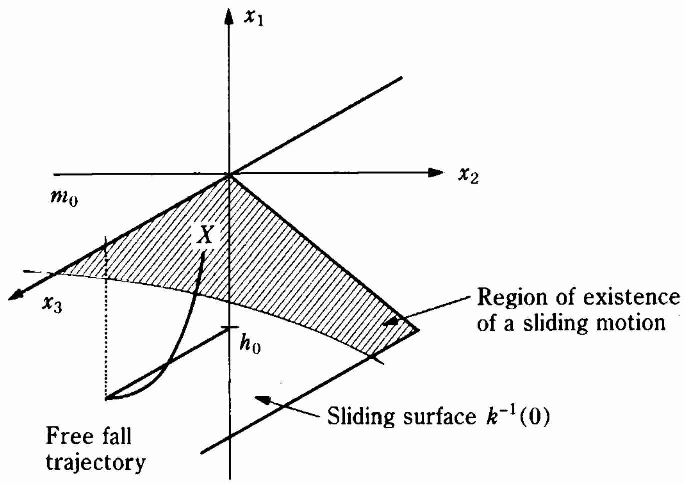


Fig. 2. Free fall trajectory reaching the region of sliding mode existence on the sliding plane.

state trajectory, with initial conditions: h_0, v_0 and m_0 turns out to be

$$\left.\begin{aligned} x_1(t) &= h_0 + (v_0-\sigma)t + \left(\frac{g}{2}\right)t^2 - \left(\frac{\sigma}{\alpha}\right)[m_0-\alpha t]\log\left[1-\frac{\alpha}{m_0}t\right] \\ x_2(t) &= v_0 + gt + \sigma\log\left[1-\frac{\alpha}{m_0}t\right] \\ x_3(t) &= -\alpha t + m_0 \end{aligned}\right\}. \quad (2.7)$$

If one considers the set of trajectories parametrized by the initial data (h_0, v_0, m_0), for which, at some time t_1, $x(t_1)=x_2(t_1)=0$, one obtains from (2.7) a transcendental surface equation, independent of x_3, which is, precisely the minimum-time switching surface (Meditch, 1964).

The specification of this minimum-time switching surface can be viewed as the problem of specifying an integral manifold of (2.1), with $u=\alpha$, whose intersection with the sliding plane is represented by the line $x_1=x_2=0$. This integral manifold is henceforth denoted by $s(\boldsymbol{x})=0$. It is easy to see, that the surface coordinate function s must satisfy the following partial differential equation,

$$\frac{\partial s}{\partial x_1}x_2 + \frac{\partial s}{\partial x_2}\left(g-\frac{\sigma\alpha}{x_3}\right) - \left(\frac{\partial s}{\partial x_3}\right)\alpha = 0, \quad (2.8)$$

with the boundary condition represented by the line $x_1=x_2=0$.

The characteristics (Arnol'd, 1983, Chapter 2) of (2.8), which in fact correspond to the trajectories of (2.1) with $u=\alpha$, may also be represented, aside from (2.7), by the intersection of its projecting cylinders,

$$\left.\begin{aligned} x_1 &= h_0 + \frac{(\sigma - v_0)}{\alpha}[x_3 - m_0] + \left(\frac{g}{2\alpha^2}\right)[x_3 - m_0]^2 \\ &\quad - \left(\frac{\sigma}{\alpha}\right)x_3 \log\left[\frac{x_3}{m_0}\right] \\ x_2 &= v_0 - \frac{g}{\alpha}[x_3 - m_0] + \sigma \log\left[\frac{x_3}{m_0}\right] \end{aligned}\right\}. \tag{2.9}$$

An explicit, exact, expression for the manifold of all such integral curves is not possible to obtain. Hence, one usually resorts to an approximation by letting, within a 2.23[%] error (See Meditch, 1964),

$$\log\left[\frac{x_3}{m_0}\right] = \log\left[1 - \frac{\alpha}{m_0}t\right] \cong -\frac{\alpha}{m_0}t - \frac{\alpha^2}{2m_0^2}t^2 \cong 2\frac{x_3 - m_0}{x_3 + m_0}. \tag{2.10}$$

The minimum-time switching surface is then, approximately, represented by the set of points $x^* = (x_1^*, x_2^*, x_3^*)$ satisfying (using also the notation $s(x)$ for the approximation)

$$s(x^*) = -\frac{\sigma\alpha^2}{m_0^2}\left[g - \frac{\sigma\alpha}{m_0}\right]^{-1} x_1^* + x_2^* - \sqrt{\left(g - \frac{\sigma\alpha}{m_0}\right)x_1^*} = 0. \tag{2.11}$$

Thus, the set of initial states (h_0, v_0, m_0) for which the fully thrusted trajectory can reach the sliding manifold $w = 0$, must, necessarily, comply (within the effects of 2.23[%] error in the approximation (2.10)) with the restrictions $w > 0$, $s < 0$, i.e.,

$$w > 0, \quad -\frac{\sigma\alpha^2}{m_0^2}\left[g - \frac{\sigma\alpha}{m_0}\right]^{-1} h_0 + v_0 - \sqrt{\left(g - \frac{\sigma\alpha}{m_0}\right)h_0} < 0. \tag{2.12}$$

It is obvious that any initial state starting *above* (2.11) leads, inevitably, to a crashing of the vehicle on the surface of the planet. A sliding mode approach therefore demands that the initial conditions, above the sliding surface, which possibly lead to a sliding regime, must be necessarily located below the minimum-time (i.e., minimum-fuel) switching surface.

Aside from this set of restrictions, one is also compelled to identify the subset of initial states satisfying (2.12) whose corresponding full thrusted trajectories hit the region of existence of a sliding regime, on $k^{-1}(0)$, determined by (2.4). To find this new subset of (2.12), one must find, by virtue of (2.4), an integral manifold of (2.1) with $u = \alpha$, whose intersection with the sliding plane is now represented by the hyperbolic trace of the cylinder: $(x_3/\sigma)[g + c_1x_2] = \alpha$ on $k^{-1}(0)$. This integral manifold, denoted by $s_1(x) = 0$, is easily seen to also satisfy the partial differential equation (2.11) with the boundary condition represented now by the line, defined on $w = 0$,

$$(x_3/\sigma)[g + c_1x_2] = \alpha; \; x_2 = -c_1x_1. \tag{2.13}$$

The *characteristics* parametrized by the initial data (h_0, v_0, m_0) are represented, as before, by the intersection of its projecting cylinders (2.9). In this case, however, it is very difficult to compute even an approximate expression

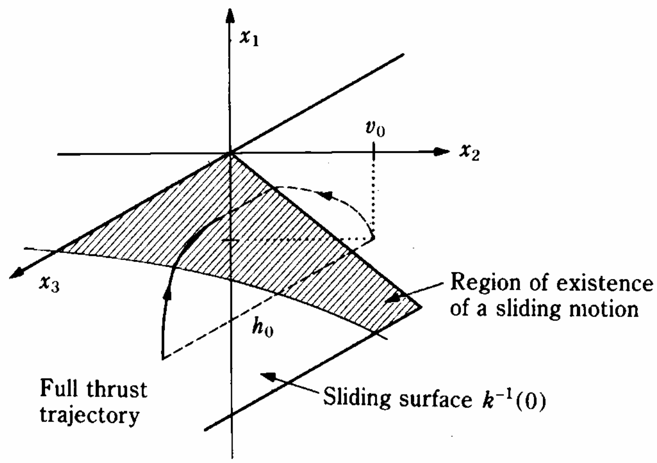


Fig. 3. Full thrust trajectory reaching the region of sliding mode existence on the sliding plane.

for the manifold $s_1(\boldsymbol{x})=0$.

Note that, due to the nearly parabolic form of an individual characteristic, each one of them intersects twice the sliding plane. By forcing one of the intersections to be on the boundary of the sliding mode existence region (i.e., to force it to satisfy the boundary condition (2.13)), the second intersection point is actually a boundary point of the set of initial states, on the sliding surface, that reach the boundary of the sliding mode existence region on $w=0$.

In summary, the initial states $\boldsymbol{x}_0$ that reach the region of existence of a sliding regime from above the sliding surface are restricted to satisfy

$$w = k(\boldsymbol{x}_0) > 0, \quad s(\boldsymbol{x}_0) < 0, \quad s_1(\boldsymbol{x}_0) > 0. \tag{2.14}$$

Figure 3 shows a typical state trajectory which starts above the sliding plane and reaches the region of existence of a sliding regime.

2.5 Ideal sliding dynamics The smooth control action $u^{EQ}(\boldsymbol{x})$ (see (2.4) and the Appendix), and the condition $w=k(\boldsymbol{x})=0$, on the basis of (2.1), define the following set of redundant differential equations representing the ideal sliding dynamics,

$$\frac{dx_1}{dt} = -c_1x_1, \quad \frac{dx_2}{dt} = -c_1x_2, \quad \frac{dx_3}{dt} = -(x_3/\sigma)[g+c_1x_2]. \tag{2.15}$$

Integrating the second and third equations above, one obtains an explicit expression for the line representing the ideal sliding mode state trajectory,

$$\left.\begin{aligned} x_3 &= x_3^*\left\{\left[\left(\frac{x_2}{x_2^*}\right)^{\frac{g}{c_1\sigma}}\right]\exp\left(-\frac{x_2-x_2^*}{\sigma}\right)\right\} \\ &= x_3^*\left\{\left[\left(\frac{x_1}{x_1^*}\right)^{\frac{g}{c_1\sigma}}\right]\exp\left(c_1\frac{x_1-x_1^*}{\sigma}\right)\right\} \\ x_2 &= -c_1x_1 \end{aligned}\right\}, \tag{2.16}$$

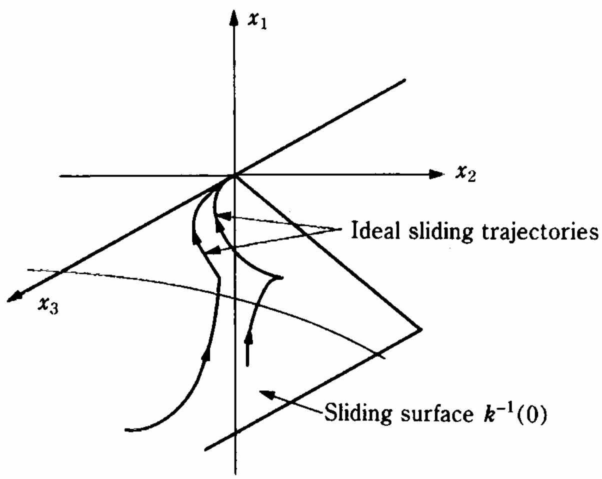


Fig. 4. Ideal sliding trajectories arising from free fall and full thrust trajectories.

where $x_1^\dagger$, $x_2^\dagger$, and $x_3^\dagger$ are the coordinates of the initial condition point lying on the sliding surface (typical ideal sliding trajectories are shown in Fig. 4).

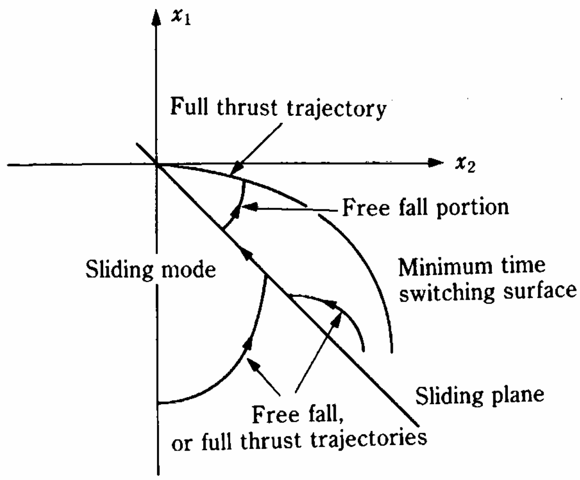
From (2.15), and the fact that $c_1>0$, it follows that $x_1 \to 0$ and $x_2 \to 0$ exponentially (i.e., in infinite time). It is easy to see from (2.16) that the ideal solution would lead to $x_3=0$ at touchdown (i.e., the entire residual fuel mass *and the satellite mass* would have disappeared in the infinite time effort for soft landing!). The proposed approach, however, is easy to modify in order to yield a finite time smooth landing. This is accomplished by modifying only the final stages of the sliding maneuver.

2.6 Some practical aspects of smooth landing via sliding regimes In order to avoid total exhaustion of residual fuel in the sliding mode controlled descent, one may resort to one of two possible control actions at the final stage.

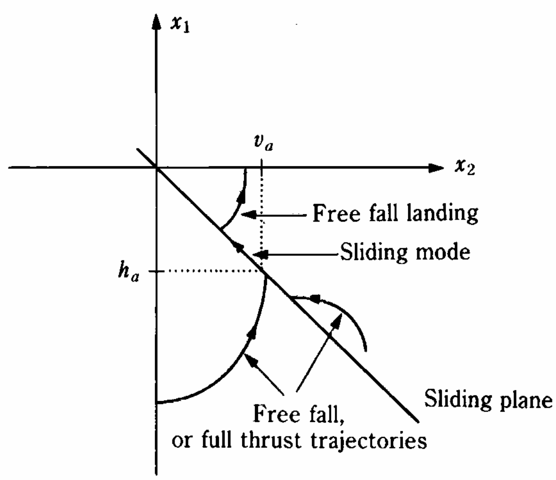
(1) Maintain a sliding mode strategy until certain prescribed height is reached. At this point, switch the engines off and let the vehicle follow a free fall descent trajectory, until reaching the minimum-time switching surface. From this time on, exercise full thrusting until touchdown.

(2) Keep the sliding mode strategy until a small height is achieved and switch the engines off to allow a free fall descent at 1g acceleration. Maximum height for switch off (typically 1 [mt] for lunar missions) is chosen to guarantee touchdown within an allowable downwards velocity (see Figs. 5(a) and 5(b)).

An important restriction to be incorporated on any landing maneuver strategy, especially in manned missions, is the availability of residual fuel at touchdown (so as to guarantee the ascent of the vehicle, later on). Typically a 25[%]–30[%] of the total mass of the landing spacecraft is assumed to be constituted by fuel (Meditch, 1964). Since the ascending maneuver is only accomplished with full thrusting, a significant percentage of the available mission fuel must be used for orbital re-injection of the spacecraft. This is so, even taking into account the fact that, usually, a "platform" and considerable payload is left behind, at the landing site.



(a) Combination of sliding mode and time-optimal (fully thrusted) landing strategy.



(b) Combination of sliding mode and free fall (shut-off) landing strategy.

Fig. 5.

The prespecified total mass of fuel needed for the ascent plus the satellite payload and its dead mass are to be viewed as a restriction on the state space, bounded by a plane of the form, say $x_3 = m_p$. This plane evidently intersects the sliding surface $w = 0$, on a straight line parallel to the plane x_1, x_2. If a final free-fall landing is permissible, the engine must be completely shut-off at the moment the ideal sliding dynamics trajectory reaches this mass restriction line. Hence, maximum allowable switch-off height must be guaranteed by that moment. Alternatively, if a minimum-time path strategy is adopted for the final stage, a different plane, parallel to $x_3 = m_p$, given by $x_3 = m_t + m_p$, must be considered as the switch-off line. Here, m_t would represent the needed fuel for the fully thrusted minimum-time touchdown maneuver. The height at which the switch off takes place is conditioned by the expendable fuel mass m_t.

2.7 The sliding-mode-free-fall strategy In a sliding-mode-free-fall strategy, the height at which the engine is shut off is extremely important. Suppose the allowable downwards velocity at touchdown is given by v_a, then this value determines, in turn, the maximum height h_a at which the engines can be shut off. This height is easily obtained by intersecting the parabola (representing the free fall trajectory) which crosses the x_2 axis at $x_2=v_a$, with the sliding plane, and then taking the only physically meaningful solution. Such maximum height is given by

$$h_a = \frac{g}{c_1^2}\left[1-\sqrt{1+\frac{2c_1v_a^2}{g}}\right]. \tag{2.17}$$

The planes $x_1=0$, $x_1=h_a$ determine a line segment, on the sliding plane, on which $x_3=m_p$. It is clear that the ideal sliding dynamics must reach this target segment in order to guarantee: 1) that final downward velocity at landing is smaller or equal than v_a and 2) landing occurs within the total required payload mass, m_p. The set of initial states (x_1^*, x_2^*, x_3^*), defined on the sliding plane $w=0$, from which this reaching condition can be ideally achieved, may be obtained from the expressions (2.16) as the set of states (x_1^*, x_2^*, x_3^*) satisfying the restriction,

$$\left.\begin{aligned} x_3^* &> m_p\left\{\left[\left(\frac{h_a}{x_1^*}\right)^{-\frac{g}{c_1\sigma}}\right]\exp\left(-c_1\frac{h_a-x_1^*}{\sigma}\right)\right\} \\ x_2^* &= -c_1x_1^* \end{aligned}\right\}. \tag{2.18}$$

When these requirements are enforced, the restriction (2.18), together with (2.4) (expressed now as $(x_3^*/\sigma)[g+c_1x_2^*]<\alpha$) must be satisfied by the initial point on the sliding surface reached by the controlled, or uncontrolled, state trajectory (see Fig. 6).

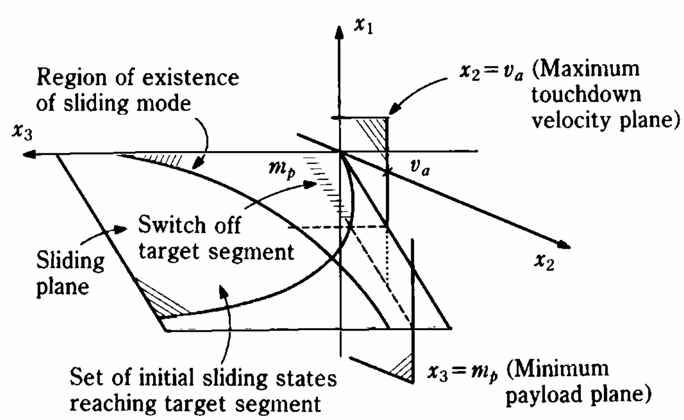


Fig. 6. Set of initial states, on the sliding surface, transferable to minimum payload switch off line with prescribed final free fall velocity at touchdown.

2.8 A sliding-mode-minimum-time strategy In a sliding-mode-time-optimal combination of controlled descent, the moment at which the engine should be shut off is determined by the intersection of the sliding trajectory with the switching line specified by the plane $x_3 = m_p + m_t$. This instant, however, determines in turn, the height at which the full thrust (minimum-time) descent will occur later on, following a portion of the free fall descent trajectory.

Suppose m_t is given. Then, according to (2.7), the height at which the fully thrusted descent must start on the minimum-time surface is given by

$$h_0 = -(v_0-\sigma)\frac{m_t}{\alpha} - \frac{g}{2}\frac{m_t^2}{\alpha^2} + \frac{\sigma m_p}{\alpha}\log\left[1-\frac{m_t}{m_t+m_p}\right] \quad (2.19)$$

with

$$v_0 = -g\frac{m_t}{\alpha} - \sigma\log\left[1-\frac{m_t}{m_t+m_p}\right]. \quad (2.20)$$

Using expressions (2.19), (2.20) in the equations describing the free-fall parabolic descent from the sliding surface to the minimum time switch surface, one finds that the height x_1^*, on the sliding surface at which the switch off must take place is given by

$$x_1^* = \frac{g}{c_1^2}\left\{1-\sqrt{1-\frac{2c_1^2}{g}\left(h_0-\frac{v_0^2}{2g}\right)}\right\} \quad (2.21)$$

with h_0 and v_0 given by (2.19) and (2.20).

2.9 A smoothly controlled alternative If fast switchings are not technically possible one may think of using, upon hitting of the sliding surface, a nonlinear controller, which instruments the smooth equivalent control law defined previously as

$$u^{\mathrm{EQ}}(x) = (x_3/\sigma)[g+c_1x_2]. \quad (2.22)$$

This control action represents a smooth feedback policy of the amplitude modulation type, producing the ideal sliding behavior previously discussed. However, in comparison with the discontinuous feedback policy, its lack of robustness results in no safety margin guarantee. We propose in this section a different smooth control alternative which is also robust.

A well known and efficient way of avoiding high frequency switchings inherent in the sliding mode policy (2.3) is represented by the use of a piecewise smooth actuator of the high-gain saturation type (see Slotine, 1984). This technique substitutes the ideal discontinuous policy (2.3) by the piece-wise smooth policy,

$$u = \begin{cases} \alpha & \text{for} \quad w > \frac{1}{\beta}, \\ \alpha + \frac{\alpha\beta}{2}\left(w-\frac{1}{\beta}\right) & \text{for} \quad |w| \le \frac{1}{\beta}, \\ 0 & \text{for} \quad w < -\frac{1}{\beta}, \end{cases} \quad (2.23)$$

where β is a constant defining the linear gain of the saturation-type actuator. For a sufficiently large β, the linear portion of (2.23) exhibits high-gain properties and thus performs an adequate averaging of the controller actions between the full thrust and the shut off conditions. Thrust amplitude modulation is hence assumed. The asymptotic stability properties of the controlled system (2.1)–(2.23) toward the $w=0$ manifold are examined in the Appendix.

This possibility, aside from retaining the robust features of the sliding mode control policy, conveniently smoothes the actuator action and the resulting state trajectories in the vicinity of the surface $w=0$. (see Figs. 8(a)–8(e) in Example).

2.10 The effect of atmospheric resistance If the density of the planet atmospheric gases is not entirely negligible, then the model (2.1) must be slightly modified (see Arnol'd, 1988, p. 4),

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = g - \left(\frac{\gamma}{x_3}\right)x_2^2 - \left(\frac{\sigma}{x_3}\right), \quad \frac{dx_3}{dt} = -u, \qquad (2.24)$$

where γ is a positive quantity representing the "air" resistance. It is easy to show in this case that for a free fall trajectory, the downwards velocity never exceeds the quantity $(m_0 g/c)^{1/2}$, independently of the initial state of the falling spacecraft.

The equivalent control corresponding to the sliding surface $x_2 = -c_1 x_1$ is now represented by $u^{EQ}(\boldsymbol{x}) = (x_3/\sigma)[g + c_1 x_2 - (\gamma/x_3)x_2^2]$. According to Theorem A.2, the region of existence of a sliding regime on $k^{-1}(0)$ is thus represented by

$$x_2 = -c_1 x_1, \quad 0 < (x_3/\sigma)[g + c_1 x_2 - (\gamma/x_3)x_2^2] < \alpha, \qquad (2.25)$$

i.e., the set of states that must be reached on the sliding surface is quite more restrictive than in the case where the atmospheric resistance was negligible. Further details of this case are left to the interested reader.

2.11 A simulation example Simulations were performed for the model (2.1) with the following parameters:

$$\sigma = 200\,[\text{mt/sec}], \qquad \alpha = 50\,[\text{kg/sec}],$$
$$g = 1.63\,[\text{m/sec}^2], \qquad c_1 = 1.2\,[\text{sec}^{-1}],$$

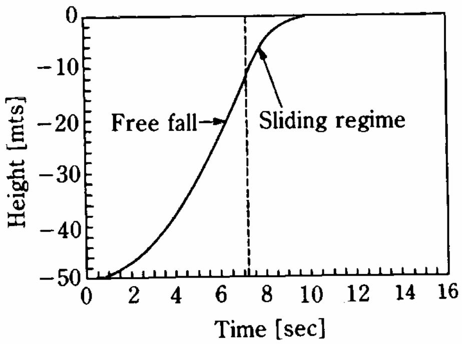
two cases of initial data were used,

$$h_0 = -50\,[\text{mt}], \quad v_0 = 0\,[\text{mt/sec}] \qquad (2.26)$$

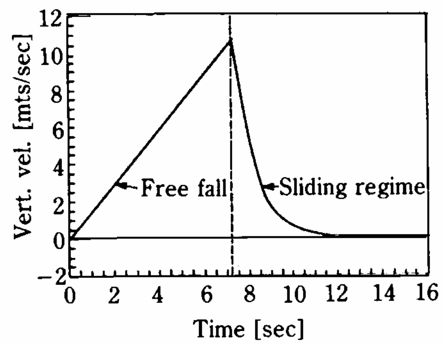
and

$$h_0 = -5\,[\text{mt}], \quad v_0 = 8\,[\text{mt/sec}], \qquad (2.27)$$

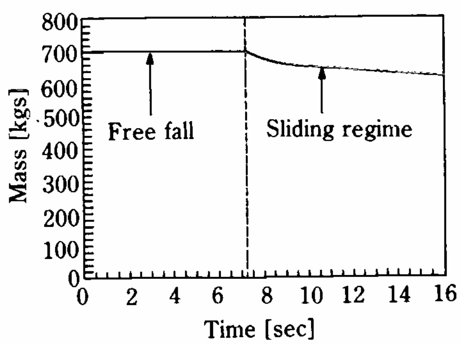
corresponding to initial conditions below and above, the sliding surface, respectively. Figures 7(a)–7(d) show the evolution of the controlled state variables during the free fall and the sliding mode stages (i.e., from initial conditions



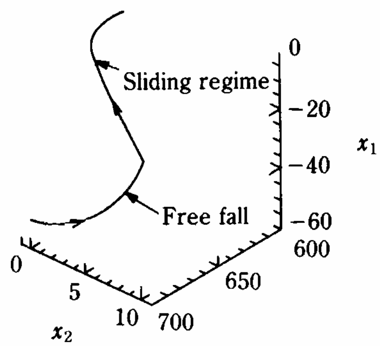
(a) Sliding mode behavior of height variable with initial free fall trajectory for the simulation example.



(b) Sliding mode behavior of downwards velocity variable with initial free fall trajectories for the simulation example.



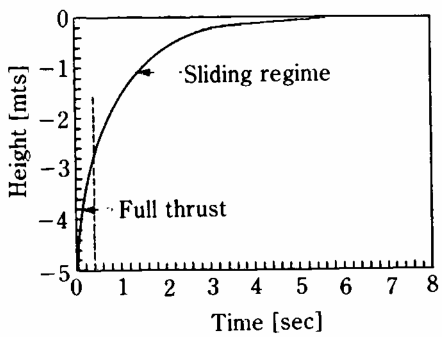
(c) Sliding mode behavior of spacecraft total mass with initial free fall trajectory for the simulation example.



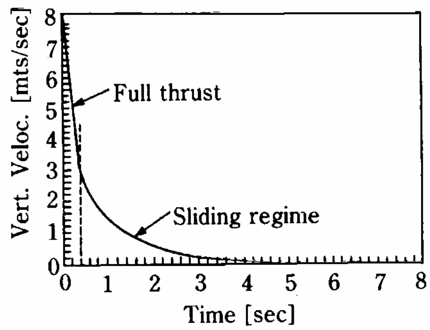
(d) Sliding mode trajectory in R^3.

Fig. 7.

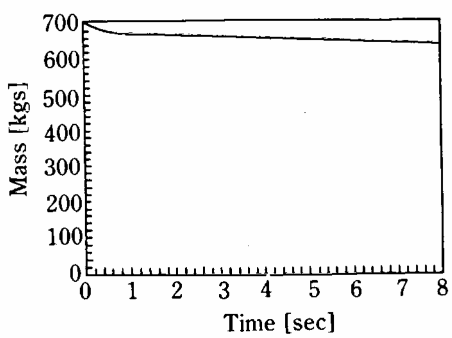
(2.26)). A variable structure controller of the form (2.3) was used in obtaining these simulations. Figures 8(a)–8(d) correspond to the state variables evolution during the full thrust and sliding mode stages (i.e., from initial conditions (2.27)). Figures 8(a) to 8(d) were obtained by using a high-gain actuator of the form (2.23) with the value of β set equals to 50. The state-space performance of the controlled vehicle, in this case, approximates that obtained with an ideal switch of the form (2.3) but with the switching frequency, and inherent state chattering, significantly reduced as depicted in the control trajectory of Fig. 8(e).



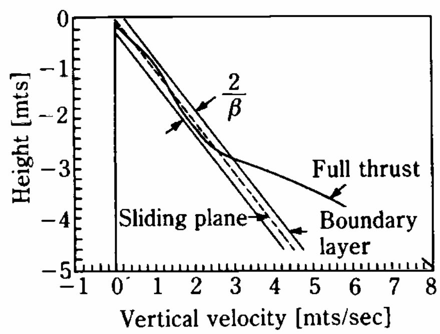
(a) Sliding mode behavior of height variable with initial full thrust trajectory for the simulation example.



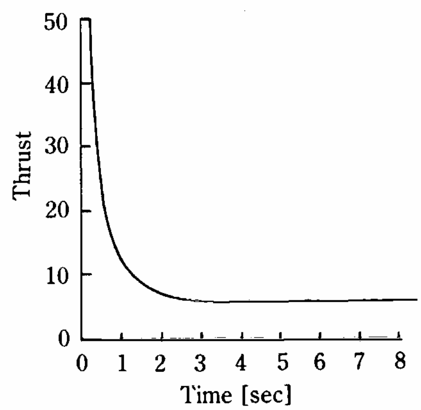
(b) Sliding mode behavior of downwards velocity variable with initial full thrust trajectory for the simulation example.



(c) Sliding mode behavior of spacecraft total mass with initial full thrust trajectory for the simulation example.



(d) Projection of high gain sliding mode trajectory on the $x_2 - x_1$ plane.



(e) High-gain control trajectory for the simulation example.

Fig. 8.

3. Conclusions and Suggestions for Further Research

A non-optimal feedback control scheme of discontinuous nature has been presented for the soft landing of a vertically controlled vehicle on the surface of a planet. A sliding mode control approach was shown to allow ideal exponentially controlled trajectories towards a soft landing condition. In order to avoid total residual fuel exhaustion, the discontinuous control approach must be necessarily combined with either a free fall policy or a free-fall-time-optimal strategy at the final stages of the landing maneuver.

It was also shown that a sliding motion does not globally exist on the sliding surface guaranteeing exponential descent. A detailed study was conducted on the initial state restrictions from which reachability of the region of existence of the sliding mode can be accomplished. Finally, some consideration was devoted to the computations of the set of initial states on the sliding surface from which a prespecified terminal velocity can be guaranteed in a free fall landing policy executed from a small terminal height. This restriction was also combined with a fixed terminal payload requirement. The set of initial states on the sliding manifold, from which the ideal transfer can be accomplished, was also explicitly computed.

Chattering avoidance can be accomplished by using a high-gain actuator in place of the ideal switch in the feedback control strategy. This results in a feedback scheme sharing the robustness of sliding mode control with reduced bang-bang effort on the part of the actuator. A different but intimately related approach to the problem here analyzed is constituted by a Pulse-Width-Modulation feedback control strategy (See Sira-Ramirez, 1989 b).

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Appendix

A.1 The concept of relative degree and sliding regimes in nonlinear systems A smooth (C^∞) single input single output system of the form,

$$\left.\begin{aligned} \frac{dx}{dt} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}\right\} \qquad (A.1)$$

is said to have local relative degree r (Byrnes and Isidori, 1984) at x^0, if

$$\left.\begin{array}{ll} L_g L_f^{k-1} h(\boldsymbol{x}) = 0 & \text{for all } x \text{ in a neighborhood of } x^0 \\ & \text{and for all } k = 1, 2, \cdots, r-1 \\ L_g L_f^{r-1} h(\boldsymbol{x}^0) = 0 & \end{array}\right\}, \tag{A.2}$$

where $L_X\phi$ stands for the directional or Lie derivative of the smooth scalar function ϕ in the direction of the vector field $\boldsymbol{X}$. System (A.1) is usually addressed as the triple $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$.

The state coordinate transformation $z_i = \phi_i(\boldsymbol{x}) \triangleq L_f^{i-1}(\boldsymbol{x})$, $i = 1, 2, \cdots, r$ and $z_{r+j} = \phi_{r+j}(\boldsymbol{x})$, $j = 1, 2, \cdots, n-r$ places the system in normal form coordinates whenever $\boldsymbol{z} = \boldsymbol{\Phi}(\boldsymbol{x}) = \text{col}[\phi_1, \phi_2, \cdots, \phi_n]$ defines a local (or global) diffeomorphism around x^0. For this, the coordinate functions $\phi_{r+j}(\boldsymbol{x})$, $j = 1, 2, \cdots, n-r$, are chosen to be functionally independent among themselves and from the first r coordinate functions. Let $\xi \triangleq \text{col}[z_1, z_2, \cdots, z_r]$ and $\eta \triangleq \text{col}[z_{r+1}, z_{r+2}, \cdots, z_n]$. The transformed system, defined in new coordinates $\boldsymbol{z}$, and said to be in *normal form*, is expressed as

$$\left.\begin{array}{l} \dfrac{dz_i}{dt} = z_{i+1}, \quad i = 1, 2, \cdots, r-1 \\ \dfrac{dz_r}{dt} = L_f^r h(\phi^{-1}(z)) + L_g L_f^{r-1} h(\phi^{-1}(z)) u \\ \dfrac{d\eta}{dt} = q(\xi, \eta, u) \end{array}\right\}. \tag{A.3}$$

The dynamics associated with the evolution of the system, starting with initial conditions such that $y = 0$, and controlled by the smooth feedback law,

$$u^0(z) \triangleq -\frac{L_f^r h(\phi^{-1}(z))}{L_g L_f^{r-1} h(\phi^{-1}(z))} = -\frac{L_f^r h(\boldsymbol{x})}{L_g L_f^{r-1} h(\boldsymbol{x})}$$

is known as the *zero dynamics*. The zero dynamics is described from (A.3), by

$$\frac{d\eta}{dt} = q(0, \eta, u^0(0, \eta)) \triangleq q_0(\eta). \tag{A.4}$$

Let $h^{-1}(0)$ denote the set, $\{x \in R^n: y = h(\boldsymbol{x}) = 0\}$. Then, *a sliding regime* is said to exist locally on $h^{-1}(0)$, if and only if there exists an open set of R^n containing $h^{-1}(0)$ where (See Utkin, 1978; Sira-Ramirez, 1988 b)

$$\lim_{s \to 0^+} L_{f+u^+(x)g} h < 0 \quad \text{and} \quad \lim_{s \to 0^-} L_{f+u^-(x)g} h > 0 \tag{A.5}$$

with $u^+(\boldsymbol{x})$ and $u^-(\boldsymbol{x})$ which are fixed smooth feedback control laws defining the switching policy triggered by the sign of y on each side of the manifold $h^{-1}(0)$, i.e.,

$$u = \begin{cases} u^+(\boldsymbol{x}) & \text{for } y > 0 \\ u^-(\boldsymbol{x}) & \text{for } y < 0 \end{cases} \tag{A.6}$$

with $u^+(\boldsymbol{x}) > u^-(\boldsymbol{x})$ locally.

If a sliding mode exists, the state trajectories evolution remain, at least locally, constrained to $h^{-1}(0)$ and the state variables are algebraically related by a single constraint. Ideally, if the motions were smoothly controlled to remain locally restricted to $h^{-1}(0)$, they would be described by the *ideal sliding dynamics* (Utkin, 1978) represented by the redundant set of differential equations,

$$\left.\begin{aligned} \frac{dx}{dt} &= f(x) + g(x)u^{EQ}(x) \\ u^{EQ}(x) &= -\frac{L_f h(x)}{L_g h(x)} \end{aligned}\right\} \tag{A.7}$$

i.e., $u^{EQ}(x)$ is the smooth feedback control law that locally guarantees satisfaction of the invariance conditions (Itkis, 1976): $y=0$ and $dy/dt=L_f h(x) + u^{EQ}(x)L_g h(x)=0$. $u^{EQ}(x)$ is known as the *equivalent control* (Utkin, 1978). Hence, the ideal sliding dynamics coincides with the zero dynamics of the triple (f, g, h).

The following theorem assumes that the feedback control policies $u^+(x)$, $u^-(x)$ can be chosen at will, i.e., they are part of the sliding regime design problem.

Theorem A.1. A sliding regime locally exists on $h^{-1}(0)$, if and only if the system (f, g, h) has local relative degree equal to one.

Proof. Necessity is obvious, for suppose the system is not locally relative degree one, then $dy/dt=L_f h(x)+uL_g h(x)$ is locally independent of u, i.e., locally $L_g h=0$. Hence, switching from $u^+(x)$ to $u^-(x)$ in the vicinity of $h^{-1}(0)$ does not change the sign of dy/dt and conditions (2.5) cannot be satisfied. Suppose now that $L_g h \neq 0$ and let $\varepsilon(x)$ be an arbitrary local strictly positive function. Then, defining

$$u^-(x) = \frac{\varepsilon(x) - L_f h}{L_g h} \quad \text{and} \quad u^+(x) = \frac{-\varepsilon(x) - L_f h}{L_g h},$$

one obtains a switching logic that locally defines a sliding regime on $h^{-1}(0)$.

By subtracting, on $y=0$, the expressions in (A.5) one obtains that locally $L_g h<0$. This condition is known as the *transversality condition* (Sira-Ramirez, 1988 b).

From the definition of $u^{EQ}(x)$ given in (A.7), one has that locally on $h^{-1}(0)$, $u^+(x) = -\varepsilon(x)/L_g h + u^{EQ}(x)$ and $u^-(x) = \varepsilon(x)/L_g h + u^{EQ}(x)$ is a switching law locally creating a sliding regime on $h^{-1}(0)$. It then follows that the inequality, $u^-(x) < u^{EQ}(x) < u^+(x)$, is locally verified. On the other hand, assume that this last inequality is known to be locally valid around $h^{-1}(0)$. Then, it follows that $u^-(x) < -L_f h/L_g h < u^+(x)$, i.e., locally

$$L_f h + u^-(x)L_g h = L_{f+u^-(x)g} h > 0$$

and

$$L_f h + u^+(x)L_g h = L_{f+u^+(x)g} h > 0.$$

It is clear then that there exists an open set containing a subset of $h^{-1}(0)$ where conditions (A.5) are valid and a sliding regime locally exists on the zero output manifold. We have thus proved the following theorem:

Theorem A.2. A sliding regime locally exists on $h^{-1}(0)$, if and only if, locally on $h^{-1}(0)$

$$u^{-}(x) < u^{\mathrm{EQ}}(x) < u^{+}(x). \qquad \text{(A.8)}$$

Relation (2.8) actually allows the computation of the region of existence of a sliding regime on $h^{-1}(0)$.

If a given system does not locally exhibit a relative degree equal to one, it is still possible to create a sliding regime that asymptotically approaches the zero output manifold $y=0$. Indeed, suppose $(f,\ g,\ h)$ has relative degree $r>1$ and consider the auxiliary output, defined by original coordinates as (see Isidori (1987) for the original procedure related to local feedback stabilization of nonlinear systems),

$$w = k(x) = c_1 h(x) + c_2 L_f h(x) + \cdots + c_{r-1} L_f^{r-2} h(x) + L_f^{r-1} h(x). \qquad \text{(A.9)}$$

By definition of the relative degree of $(f,\ g,\ h)$, it follows that $L_g h(x) = L_g L_f^{r-1} h(x) \neq 0$. i.e., $(f,\ g,\ k)$ has relative degree one. Therefore, a sliding regime may be created on $k^{-1}(0) \triangleq \{x \in R^n:\ w=k(x)=0\}$ by use of an appropriate variable structure control law.

In the normal form, coordinates (A.9) is expressed as

$$w(z) = k(x) = c_1 z_1 + c_2 z_2 + \cdots + c_{r-1} z_{r-1} + z_r. \qquad \text{(A.10)}$$

Under ideal sliding conditions, $w=0$, $z_r = -\sum_{i=1}^{r-1} c_i z_i$, and the corresponding equivalent control is given by $u^{\mathrm{EQ}}(z) = -L_f k(\Phi^{-1}(z))/L_g k(\Phi^{-1}(z))$. The corresponding zero dynamics or ideal sliding dynamics is obtained as

$$\left.\begin{aligned} \frac{dz_i}{dt} &= z_{i+1}, \qquad i = 1,\ 2,\ \cdots,\ r-2 \\ \frac{dz_{r-1}}{dt} &= z_r = -\sum_{i=1}^{r-1} c_i z_i \\ \frac{dz_r}{dt} &= L_f k(\Phi^{-1}(z)) + L_g k(\Phi^{-1}(z)) u^{\mathrm{EQ}}(z) = 0 \\ \frac{d\eta}{dt} &= q_0(z), \qquad y = z_1, \qquad w = 0 \end{aligned}\right\}. \qquad \text{(A.11)}$$

It is easy to see that by appropriately choosing the constants c_i $(i=1,2,\cdots,r-1)$, the ideal sliding dynamics can be made locally asymptotically stable towards the manifold: $z_i=0$ $(i=1,2,\cdots,r)$, with the poles of the linear part of the approaching dynamics placed, entirely at will, in the left half of the complex plane. Hence, while sliding takes place on $k^{-1}(0)$, all components of the vector col$[z_1,\cdots,z_r]$ can be made asymptotically go to zero (notice that z_r is a linear combination of $z_1,\cdots,z_{r-1}$) in the open region where the relative degrees of $(f,\ g,\ h)$ and $(f,\ g,\ k)$ remain constant. In particular, z_1 goes asymptotically to zero and $h^{-1}(0)$ is locally reached.

A.2 A high gain approach to sliding mode creation Here, we will show that the state trajectories of the nonlinear system (A.1)–(2.23) locally asymptotically converge toward the $w=0$ manifold provided β is chosen sufficiently large in (2.23) and a sliding regime is known to exist locally on $w=0$ for the discontinuous controlled system (A.1)–(2.3). The results here constitute a particularization of those found in Marino (1985). High gain controllers usually represent a robust and practical substitute for a given variable structure control strategy. The idea has been reported extensively in the Russian literature (see Utkin, 1978 a; b, and references therein) and has also been brought to practical use by Slotine (see Slotine, 1984) in recent times.

The following theorem establishes that the existence of a local sliding mode on a given smooth manifold is a sufficient condition for the stabilization of the system trajectories around such a surface by means of a high-gain controller.

Theorem A.3. Suppose system (A.1), (2.3) exhibits a local sliding regime on the surface $k^{-1}(0)=\{x\in R^n:\ w=k(x)=0\}$. Then, there exists a high gain controller of the form (2.23) with a sufficiently large but finite β, such that the controlled trajectories locally asymptotically converge toward $w=0$.

Proof. Suppose that the hypothesis of the theorem holds true. Then, from the smoothness assumptions, one can always find an open neighborhood around $k^{-1}(0)$, where the sliding mode conditions (A.5) (with $u^+(x)=\alpha$ and $u^-(x)=0$) are valid. Suppose that points of this neighborhood can still be located at a distance greater than a small number ε from $w=0$. Choose now $\beta=1/\varepsilon$ in (2.23). It should be evident from the construction just made from (A.5) and (2.23) that outside the boundary layer defined by β, the controlled trajectories are directed towards the manifold. Note that from the sliding mode existence assumption, the transversality condition $L_g h<0$ necessarily holds locally true. On the other hand, within the boundary layer, the surface coordinate function w is governed by

$$\frac{dw}{dt} = L_f k(x) + \left[\alpha + \frac{\alpha\beta}{2}\left(w - \frac{1}{\beta}\right)\right] L_g k(x)$$
$$= \left[L_f k(x) + \frac{\alpha}{2} L_g k\right] + \frac{\alpha\beta}{2} w L_g k. \tag{A.12}$$

It is evident that for a sufficiently high β, the controlled surface coordinate dynamics (A.12) exhibits a two-time scale separation property. Indeed, dividing the above expression by β and letting $\beta\to\infty$ on sees that since $L_g k<0$, then $(\alpha\beta/2)wL_g k=0$ implies that $w=0$ is a *slow manifold* of the controlled system. The corresponding fast subsystem described in the fast time scale $\tau=\beta t$ is given by

$$\frac{dw}{d\tau} = \frac{\alpha}{2} w L_g k. \tag{A.13}$$

Consider the Lyapunov function $V(w)=w^2>0$ around $k^{-1}(0)$, then it is easy to see from (A.13) that $dV(w)/dt=\alpha w^2 L_g k<0$ locally around $k^{-1}(0)$, i.e., the fast system trajectories are locally asymptotically stable towards $k^{-1}(0)$ and the controlled trajectories eventually adopt $w=0$ as an integral manifold. It then follows from Tikhonov's theorem (see Marino, 1985) that $k^{-1}(0)$ is locally an asymptotically stable manifold for the high gain controlled system (A.1)–(2.23).

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