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Sliding mode control of nonlinear first order distributed parameter systems [1]

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Abstract. A complete characterization of Distributed Sliding Modes is presented for dynamical systems described by controlled nonlinear first order partial differential equations. The proposed approach is geometric in nature and uses the notions of contract forms, canonical contact structures and characteristics defined in the space of 1-jets of functions. The results generalize the characterization of sliding modes in distributed controlled systems described by linear and quasi-linear first order partial differential equations.

AMS Subject Classifications. 35B37, 93C10.

1. Introduction

The theory of sliding regimes associated to variable structure controlled systems of the lumped type has been extensively developed in the literature during the last 40 years. Early fundamental contributions must be credited to researchers in the Soviet Union and Eastern Europe. For a comprehensive treatise on the subject, in the context of finite-dimensional nonlinear dynamical systems, the reader is referred to Emelyanov [1] and Utkin [2], and for the linear case to a book by Itkis [3]. The theoretical background, in its most general form, was provided by Filippov's fundamental contributions, collected now in a recently translated book [4]. Thorough surveys indicating the main results and the many applications of this field to technical problems have been written by Utkin, [5]-[7], during the years. A tutorial article of recommended reading is that by DeCarlo *et al* [8]. Generalizations for finite dimensional nonlinear systems can be found in the works of Slotine [9], Marino [10], and Sira-Ramirez [11], [12] and [21].

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In this article a geometric approach is presented for the characterization of sliding regimes, and their fundamental properties, in controlled dynamical systems described by nonlinear first order partial differential equations (NFOPDE). This article constitutes a generalization of the results presented in Sira-Ramirez [13] for the case of distributed sliding mode control in first order linear and quasi-linear partial differential equations. The proposed approach constitutes a fundamental departure from the methods, based on Banach spaces, used by Berger et al [14], Orlov and Utkin [15], and from the approximate finite-dimensional method presented by Orlov and Utkin in [16]. It also differs from the Lyapunov-based approach, presented by Utkin in [17], where a particular one-dimensional second order distributed heat process of the nonlinear parabolic type was treated.

A distributed sliding manifold may be portrayed as the smooth solution manifold of a desirable closed loop dynamical system represented by a NFOPDE. The prolongation to the space of 1-jets of functions of the sliding surface is shown to play a fundamental role in both the characterization of the invariance properties of the ideal sliding dynamics and the determination of the sliding mode existence conditions. A controlled NFOPDE is geometrically characterized as a smooth manifold, parametrized by the feedback control function, in the space of 1-jets of functions. The ideal sliding dynamics is obtained by impossing a local invariance condition on the Filippov average of the controlled characteristic direction fields defined in such a jet space, with respect to the prolongation of the sliding manifold. Local existence of a sliding regime on the given sliding surface is then characterized in terms of the appropriate transversality of the corresponding characteristic direction fields — associated to the extreme controlled systems — with respect to the tangent distribution to the sliding manifold representing the solution of the ideally desirable dynamics. The results not only characterize sliding regimes in general nonlinear first order distributed parameter systems but they can also rederive known results for the linear and quasilinear distributed cases [13], as well as those related to the explicit and implicit finite dimensional nonlinear cases [12]. The results can be extended to systems described by a finite set of NFOPDE controlled by multiple inputs.

Section 2 of this article contains some background material on the geometrical aspects of nonlinear partial differential equations. Section 3 contains the main results of the article. Section 4 is devoted to some conclusions and suggestions for further research. Further background material related to this article can be found in references [18], [19] and [20].

2. Mathematical background

In this section we present some basic results about integration of NFOPDE's by means of characteristics. The reader is referred to Olver [18] and Arnold [19], [20] for more thorough and enlightening details.

2.1. Elements of the geometric theory of nonlinear first order partial differential equiations

The developments in this section assume some familiarity with contact manifolds, differential forms, the space of jets of functions, symplectic structures and prolongations of functions, vector fields and 1-forms to jet spaces. All the background definitions are provided in the above cited references. We collect the fundamental results of the theory of NFOPDE slightly extending the exposition in Arnold [20].

Consider a NFOPDE:

$$\frac{\partial v}{\partial t} + \mathcal{F}(v, x, t, p) = 0 \tag{1}$$

where x represents the vector of local spatial coordinate functions x_i $(i = 1, \dots, n)$ defining points on an open set in R^n, t denotes time. The function v is the unknown scalar function and p is an n dimensional vector with components p_i, representing the partial derivatives, $\partial v/\partial x_i$ $(i = 1, \dots, n)$. We also denote by q the partial derivative $\partial v/\partial t$ and by π the vector of components (p, q). $\mathcal{F}$ is a smooth function of all its arguments.

All our considerations and results are of *local* character on a given open set (manifold) N of R^{n+2} described by the vector of local coordinate functions (v, x, t), denoted by η. The projection of such an open set N onto R^{n+1}, along the direction of v, is labeled as M, and it is equipped with local coordinates (x, t) which we simply denote by χ. Also, we denote by z the vector of local coordinates $(v, x, t, p, q) = (\eta, \pi)$ in R^{2n+3} which we identify as the manifold of 1-jets of functions defined on M, labeled here as $J^1(M, R)$. By TN and $TJ^1(M, R)$ we denote, respectively, the tangent bundles of N and of $J^1(M, R)$.

Equation (1) can be interpreted as the expression of a 2n+2 dimensional hypersurface in $J^1(M, R)$. We denote such a hypersurface by E and define it as:

$$E = \phi^{-1}(0) := \left\{ z \in J^{-1}(M, R) \mid \phi(z) = q + \mathcal{F}(v, x, t, p) = 0 \right\} . \tag{2}$$

The space $J^1(M, R)$ is equipped with the standard *contact structure* induced by the nowhere zero canonical 1-form $\alpha := dv - pdx - qdt = dv - \pi d\chi$.

The *contact structure* in the space of 1-jets, $J^1(M,R)$, is represented by the field of 2n+2 dimensional planes which are called *the contact planes* (or *contact distribution*), annihilating the 1-form α. A contact plane at the point z in $J^{-1}(M,R)$ is denoted by Π_z and it is a subspace of the tangent space of $J^1(M,R)$ at z, $T_zJ^1(M,R)$. Since the exterior derivative of α, denoted by $\omega := d\alpha = dx \wedge dp + dt \wedge dq$, is a nondegenerate *skew symmetric bilinear form* on the field of even dimensional contact planes Π, the contact planes are, indeed, *symplectic vector spaces*. The distribution in $TJ^{-1}(M,R)$ annihilating the 1-form $d\phi$ is constituted by the field of planes Θ tangent to the hypersurface E (*tangent distribution*). The plane tangent to E at z is denoted by Θ_z.

DEFINITION 1. A surface E in $J^{-1}(M,R)$ is *noncharacteristic* if its tangent planes Θ and the contact planes Π are transversal, i.e., if their direct sum at each z spans the tangent space $T_zJ^1(M,R)$.

It is generally assumed that the manifold E is noncharacteristic at all points z under consideration of the space $J^1(M,R)$. The intersections of the contact planes and tangent planes at points z in E are, again, a field of planes, of dimension 2n+1, which are tangent to E. These are the *characteristic planes*. It is not difficult to see how the *symplectic structure* associated to the contact structure, naturally and uniquely, determines a *jet-field of directions* belonging to the characteristic planes. Such a field of directions, in turn, determines the *characteristics* of the nonlinear equation (1) as described below.

Consider the 2-form ω, obtained by exterior differentiation of the contact 1-form α. Since the 1-form $d\phi$ is not identically zero at z, $d\phi$ acts on vectors belonging to Π_z as a nonzero linear form. Notice that Π_z is an even dimensional vector space, while ω is a nondegenerate skew-symmetric bilinear form on Π_z. Hence, Π_z is actually a *symplectic vector space*. Every symplectic vector space is known to be isomorphic to its dual space. Hence, one can identify 1-forms (covectors) with vectors of Π_z. We proceed to identify, by the above discussion, a nonzero vector ξ of Π_z with the nonzero 1-form $d\phi$, by imposing the equality $d\phi(\cdot) = \omega(\xi,\cdot)$. The vector ξ, will be known as the *jet-characteristic vector* of E at the point z. The jet-characteristic vector ξ belongs to the characteristic distribution $\Pi \cap \Theta$ and it is skew-orthogonal to itself $d\phi(\xi) = \omega(\xi,\xi) = 0$. This vector, however, is nonuniquely defined. The nonuniqueness of ξ arises from the fact that one may take any vector $\mu = \beta(z)\xi$ in the span of ξ, and still obtain the same previously described identification with the 1-form $d\phi$. Indeed, let $d\phi(\cdot) = \omega(\mu,\cdot)$ and then $d\phi(\mu) = \beta^2(z)\omega(\xi,\xi) = 0$, and μ is also characteristic. The above procedure does, however, uniquely determine, at each point z, the *jet-characteristic line* or, *jet-characteristic direction*, containing all vectors that are in the span of ξ, The $n+1$-dimensional integral curves of the jet-characteristic directions, contained in E, are *the jet-characteristics* of the partial differential equation.

One can explicitly compute the set of ordinary first order differential equations generating the jet-characteristics for the equation (1) as follows:

Consider a nonzero vector

$$\varsigma = V\frac{\partial}{\partial v} + X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + P\frac{\partial}{\partial p} + Q\frac{\partial}{\partial q}$$

belonging to Θ_z, the tangent plane to E in $T_z J^1(M,R)$. For such a vector we have

$$d\phi(\varsigma) = \phi_v V + \phi_x X + \phi_t T + \phi_p P + Q = 0\,. \tag{3}$$

The vector ς lies in the contact distribution Π_z if it belongs to the null space of the 1-form α, i.e., $\alpha(\varsigma) = V - pX - qT = 0$. The vector ς is then, necessarily, of the form

$$\varsigma = (pX + qT)\frac{\partial}{\partial v} + X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + P\frac{\partial}{\partial p} + Q\frac{\partial}{\partial q}\,.$$

The vector ς belongs to the characteristic plane $\Pi_z \cap \Theta_z$ if and only if

$$[\phi_v p + \phi_x]\,X + [\phi_v q + \phi_t]\,T + \phi_p P + Q = 0\,. \tag{4}$$

The jet-characteristic vector ξ has components $(\dot{v} = \pi\,\dot{\chi}, \dot{x}, 1, \dot{p}, \dot{q})$. These components are determined from the condition that the skew scalar product of ξ with all vectors of the form

$$(pX + qT)\frac{\partial}{\partial v} + X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + P\frac{\partial}{\partial p} + Q\frac{\partial}{\partial q}\,,$$

is identically zero. Evaluating the 2-form $d\alpha = d\chi \wedge d\pi$ on the pair of vectors

$$(p\,\dot{x} + q)\frac{\partial}{\partial v} + \dot{x}\,\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \dot{p}\,\frac{\partial}{\partial p} + \dot{q}\,\frac{\partial}{\partial q} \quad \text{and} \quad \varsigma\,,$$

one readily obtains, that

$$-\,\dot{p}\,X - \dot{q}\,T + \dot{x}\,P + Q = 0\,. \tag{5}$$

Consequently, letting the coefficients of X, T, P in (4) and (5) be identical and using $\dot{v} = \pi\,\dot{\chi} = p\,\dot{x} + q$, one obtains an expression for the components of the jet-characteristic vector yield ξ as follows

$$
\begin{aligned}
\dot{v} &= p\phi_p + q = p\mathcal{F}_p + q\,, \\
\dot{x} &= \phi_p = \mathcal{F}_p\,, \quad \dot{t} = \phi_q = 1\,, \\
\dot{p} &= -\phi_v p - \phi_x = -\mathcal{F}_v p - \mathcal{F}_x\,, \\
\dot{q} &= -\phi_v q - \phi_t = -\mathcal{F}_v q - \mathcal{F}_t\,,
\end{aligned} \tag{6}
$$

or briefly, in terms of the local coordinates z of $J^1(M,R)$

$$
\dot{z} = \xi(z)\,. \tag{7}
$$

DEFINITION 2. Let γ be an n-dimensional submanifold of M and let $\varphi : \gamma \mapsto R$ be a smooth function. The *initial manifold* $\mathcal{G}$ defined in N, is constituted by the set $\{(v,x,t)|v = \varphi(x,t)\,,\text{ for } (x,t) \in \gamma\}$. The pair (φ,γ) is referred to as *the Cauchy data*. The *initial jet-manifold* Γ constructed in $J^1(M,R)$ on the basis of the Cauchy data is the set consisting of all 1-jets of functions on M satisfying the following requirements:

1) the base point χ of the jet (η,π) lies on γ,

2) the value of the function v at the point χ is equal to φ,

3) the value of the total differential of the function v at $\chi \in \gamma$ is such that its restriction to the tangent planes (tangent distribution) to γ is equal to the total differential of the initial condition φ evaluated on such planes,

4) the jet $z = (\eta,\pi)$ is a point of $J^1(M,R)$ belonging to E.

In other words, an initial condition for the equation $\phi = 0$ is an assignment of a particular value φ to the unknown function v on the points of an n-dimensional hypersurface γ defined in the $n+1$-dimensional space of coordinates $\chi = (x,t)$.

DEFINITION 3. A point of the initial jeet-manifold Γ is said to be *noncharacteristic* for the system (1) if the projection of the *jet-characteristic direction* at this point onto M is transversal to γ. It can be shown [20, pp. 82] that an explicit condition, over the initial data (γ,φ) and the function ϕ, for which an initial point $z^0 = (\eta^0,\pi^0) = (v^0,x^0,t^0,p^0,q^0)$ is noncharacteristic, with respect to the equation $\phi = 0$, is that the $n+1$-dimensional vector $\phi_\pi(z^0) = [\mathcal{F}_p\,(\eta^0,p^0)\,,1]$ is not tangent to γ.

For a given noncharacteristic point z^0 of the initial jet-manifold Γ in $J^1(M,R)$ there exists, defined in some open neighbourhood U of $\chi^0 = (x^0,t^0)$, a locally unique solution v to the equation $\phi = 0$. This means that any two solutions of (1), which are made to satisfy the same noncharacteristic initial condition $v|_{U\cap\gamma} = \varphi|_{U\cap\gamma}$, $v(x^0,t^0) = v^0$, $dv(x^0,t^0) = (p^0,q^0) = \pi^0$, necessarily coincide over some open subset of U.

DEFINITION 4. The 1–graph of a function $f : M \mapsto R$ of $n+1$ variables is the submanifold constituted by 1–jets of f at all points of M, i.e., it is an $n+1$–dimensional surface in a $2n+3$–dimensional space.

The solution of equation (7) generates a one–parameter group of diffeomorphic transformations, on the subset E of the jet space $J^1(M,R)$, for which the field of contact planes Π remains invariant. From each noncharacteristic point of the initial manifold Γ in $J^1(M,R)$ one locally obtains a *jet–characteristic* defined as the integral curve of the vector field ξ. The solutions of equation (1) are constituted by those functions on M whose 1–graphs coincide with jet–characteristics on E. To find such a solution, one solves the set of $2n+3$ first order ordinary differential equations (7) for the jet–characteristics of (1) on E, and performs a number of algebraic operations (see Arnold [19, pp. 370]) to obtain the graph of the mapping $v = v(x,t)$, $p = \partial v(x,t)/\partial x$, $q = \partial v(x,t)/\partial t = -\mathcal{F}(v(x,t), x, t, p(x,t))$ in $J^1(M,R)$. The function $v(x,t)$ is a solution of (1) with initial condition $v|_\gamma = \varphi$. We call a *characteristic* for (1) the functions defined on M which correspond to the jet–characteristic of (1) in E. In other words, a function f defined on M with values in N is a characteristic of (1) if its prolongation to $J^1(M,R)$ is a jet–characteristic of (1). The field of directions in TN whose associated flow coincides locally with the characteristics of (1) will be termed *the characteristic direction* of (1). We specify a *characteristic vector field* $\kappa(\eta)$ defined in TN, as a vector field whose span is the characteristic direction of (1).

Notice that since $\xi(z)$ is a prolongation of $\kappa(\eta)$ to $TJ^1(M,R)$, the first $n+2$ components of $\xi(z)$ necessarily coincide with those of κ. According to the prolongation formula for vector fields (Olver [18, pp. 108–111]), the last $n+1$ components of $\xi(z)$ are therefore discarded when an explicit expression for $\kappa(\eta)$ is sought once ξ is explicitly known as $\xi(\eta)$, i.e., with its components written as explicit functions of the local coordinates $\eta = (v, x, t)$ of N.

DEFINITION 5. A given $n+1$ dimensional manifold S in N characterized by the smooth graph of the function $v = s(x,t)$ in N is said to be *locally invariant with respect to the dynamical system* (1) if for some noncharacteristic set of initial data (φ, γ), whose graph lies on S, the solution of (1) locally coincides with S.

An equivalent definition of invariance can be given in terms of the field of jet–characteristic directions or, alternatively, in terms of the characteristic vector field associated to (1).

DEFINITION 6. An $n+1$–dimensional manifold S in N is locally *invariant with respect to the system* (1) if the jet–characteristics of (1), i.e., the integral curves of $\xi(z)$ in E, locally belong to the prolongation $S^{(1)}$ of

S to $J^1(M,R)$. Alternatively, S is locally invariant with respect to (1) if the characteristic vector field $\kappa(\eta)$ in TN locally belongs to the distribution tangent to S.

3. Distributed sliding mode characterization in variable structure controlled NFOPDE

In this section we characterize sliding regimes in feedback controlled distributed systems described by NFOPDE. Restrictions to the case of distributed sliding regimes in controlled systems described by first order quasi-linear partial differential equations [13], and to the case of ordinary differential equations (ODE) [2], [11], [12], [21], can easily be accomplished by direct particularization of the adopted framework.

3.1. Distributed variable structure controlled systems discribed by NFOPDE and their associated sliding regimes

Consider a dynamical distributed system described by a controlled NFOPDE

$$\frac{\partial v}{\partial t} + \Phi(v,x,t,u,p) = 0\,, \tag{8}$$

$$y = h(v,x,t)\,, \tag{9}$$

where y is the scalar-valued smooth output function defined on the open set N of R^{n+2}, x represents the vector of local spatial coordinate functions x_i $(i = 1,\ldots,n)$, t denotes time, while $u = u(v,x,t)$ is a distributed smooth time-varying feedback control law taking values in R. The function v is regarded as the distributed "state" of the controlled system. p is an n-dimensional vector with components p_i representing the partial derivatives, $\partial v/\partial x_i$ $(i = 1,\ldots,n)$. We will be using q to denote $\partial v/\partial t$. Φ and h are smooth functions of all their arguments.

Available to the controller is a distributed variable structure feedback switching law

$$u = \begin{cases} u^+(v,x,t) & \text{for } y > 0 \\ u^-(v,x,t) & \text{for } y < 0 \end{cases} \tag{10}$$

with $u^+(v,x,t) > u^-(v,x,t)$ locally on N.

The condition $y = 0$ is assumed to locally define an isolated smooth manifold solution $v = s(x,t)$ in N, i.e., $h(s(x,t),x,t) \equiv 0$. The graph of v is assumed to be a smooth time-varying surface with locally nonzero gradient except possibly on a set of measure zero. The zero level set of h is addressed as the *sliding manifold*, or the *sliding surface*, and is locally defined as

$$S = \{\eta = (v,x,t) \in N \subset R^{n+2} \mid v = s(x,t)\} \tag{11}$$

The manifold S can be *prolonged* (See Olver [18, p. 97–119]) to the space of 1–jets of functions $J^1(M,R)$ defined on M. For this, we simply complete the set of coordinate values at each point of S with those of the partial derivatives $s_x(x,t)$, $s_t(x,t)$ at the point of S. In this manner we obtain the 1–graph of the sliding surface in $J^1(M,R)$ as

$$(12)\quad S^{(1)}(z)=\{z\in J^1(M,R)\mid z=(s(x,t),x,t,\ s_x(x,t),\ s_t(x,t))\}.$$

Alternatively to the above definition, one may characterize a sliding surface S as a solution of an unforced NFOPDE regarded as a *desirable distributed dynamics*. Associated to such a desirable dynamics there is defined an arbitrary set of known initial data $(\gamma^\sigma,\varphi^\sigma)$ such that the sliding manifold $v=s(x,t)$ *is locally a solution* for the corresponding Cauchy problem, i.e., we may assume that the NFOPDE

$$(13)\qquad \begin{aligned}\partial v/\partial t+\mathcal{F}^d(v,x,t,p)&=0\\ v|_{\gamma^\sigma}&=\varphi^\sigma\end{aligned}$$

is such that $\varphi^\sigma(x,t)|_{\gamma^\sigma}=s(x,t)|_{\gamma^\sigma}$ and

$$s_t(x,t)+\mathcal{F}^d(s(x,t),x,t,\ s_x(x,t))\equiv 0.$$

Hence, within such a characterization of the sliding surface S, the prolongation $S^{(1)}(z)$ in (12) is also expressed as

$$(14)\qquad S^{(1)}(z)=\{z\in J^1(M,R)\mid q+\mathcal{F}^d(v,x,t,p)=0\}.$$

All developments below can be reproduced with ease for this alternative, though implicit, characterization of the sliding manifold. However, we only deal with the more natural characterization of a sliding surface given explicitly by an expression of the form (11), from which the corresponding prolongation (12) can be computed.

For an unspecified control u, system (8) $\partial v/\partial t+\Phi(v,x,t,u,p)=q+\phi(v,x,t,u,p)=0$ can be interpreted as a hypersurface E^u, defined on the manifold $J^1(M,R)$, parametrized by the control function u. If $u=u(v,x,t)$ is a fixed smooth distributed feedback control function, then the closed loop system (1) $\partial v/\partial t+\Phi(v,x,t,u(v,x,t),p)=q+\mathcal{F}(v,x,t,p)=0$ corresponds to a hypersurface denoted by E in the manifold $J^1(M,R)$. In order to introduce a suitable parametrization of the hypersurfaces representing the variable structure controlled system (8), (9) we rewrite the controlled system (8), (9) in an equivalent form using a *distributed switch position function* ν, taking values in the discrete set $\{0,1\}$ and defined on each point of coordinates $\eta=(v,x,t)$, according to the value of the scalar output function

$y = h(v, x, t)$. The switch function ν acts then as a distributed control parameter

$$\frac{\partial v}{\partial t} + \nu\Phi(v, x, t, u^+(v, x, t), p) + (1 - \nu)\Phi(v, x, t, u^-(v, x, t), p) = 0$$

with

$$\nu = \begin{cases} 1 & \text{for } y > 0 \\ 0 & \text{for } y < 0 \end{cases}$$

or,

$$\frac{\partial v}{\partial t} + \Phi(v, x, t, u^-(v, x, t), p) +$$
$$+\nu[\Phi(v, x, t, u^+(v, x, t), p) - \Phi(v, x, t, u^-(v, x, t), p)] = 0\,.$$

We will simply write the obtained switching system, with the obvious indentifications, as

$$\frac{\partial v}{\partial t} + F(v, x, t, p) + \nu G(v, x, t, p) = 0\,; \tag{15}$$

$$y = h(v, x, t)$$

$$\nu = \begin{cases} 1 & \text{for } y > 0\,, \\ 0 & \text{for } y < 0\,. \end{cases} \tag{16}$$

Thus, corresponding to the controlled system (15)–(16) one has two hypersurfaces E^+ and E^- respectively, defined in $J^1(M, R)$ as

$$E^+ = \{z \in J^1(M, R) | q + F(v, x, t, p) + G(v, x, t, p) = 0\} \tag{17}$$

$$E^- = \{z \in J^1(M, R) | q + F(v, x, t, p) = 0\}\,. \tag{18}$$

The set of corresponding controlled jet–characteristics in E^+ and E^- are, according to the notation of Section (2.1), generated by the vector field $\xi^+(z)$ and $\xi^-(z)$, which we notationally unify under the parametrized vector field $\xi(z, \nu)$, with $\nu \in \{0, 1\}$. We thus denote by $\xi(z, 1)$ the vector field $\xi^+(z)$ and by $\xi(z, 0)$ the vector field $\xi^-(z)$, i.e., $\xi(z, \nu) = \nu\xi^+(z) + (1 - \nu)\xi^-(z)$. The components of the vector fields $\xi^+(z)$, $\xi^-(z)$ are described, respectively, by the right hand sides of the following set of ordinary differential equations

$$\begin{array}{ll} \dot{v} = p(F_p + G_p) + q & \dot{v} = pF_p + q \\ \dot{x} = F_p + G_p & \dot{x} = F_p \\ \dot{t} = 1 & \dot{t} = 1 \\ \dot{p} = -(F_v + G_v)p - (F_x + G_x) & \dot{p} = -F_v p - F_x \\ \dot{q} = -(F_v + G_v)q - (F_t + G_t) & \dot{q} = -F_v q - F_t \end{array} \tag{19}$$

or, briefly, in the local coordinates z of $J^1(M,R)$ by

$$\dot{z} = \xi^+(z)\,, \qquad \dot{z} = \xi^-(z)\,. \tag{20}$$

It follows, from (19), that the controlled vector field $\xi(z,\nu)$, $\nu \in \{0,1\}$, is described by

$$\begin{aligned} \dot{v} &= p(F_p + \nu G_p) + q \\ \dot{x} &= F_p + \nu G_p \\ \dot{t} &= 1 \\ \dot{p} &= -(F_v + \nu G_v)p - (F_x + \nu G_x) \\ \dot{q} &= -(F_v + \nu G_v)q - (F_t + \nu G_t)\,, \end{aligned} \tag{21}$$

i.e.,

$$\begin{aligned} \dot{z} &= \xi(z,\nu) = \nu\xi^+(z) + (1-\nu)\xi^-(z) = \\ &= \xi^{-1}(z) + \nu\left[\xi^+(z) - \xi^-(z)\right]\,. \end{aligned} \tag{22}$$

The contact distribution Π intersects the field of planes tangent to E^+ and E^- and creates on such tangent distributions Θ^+ and Θ^- fields of $2n+1$–dimensional planes called the *extreme characteristic planes*. In the manner indicated in Section 2, such fields of characteristic planes uniquely determine the corresponding fields of jet–characteristic directions specified in $TJ^1(M,R)$ by the vector fields $\xi^+(z)$ and $\xi^-(z)$. There exist integral submanifolds (curves) of these fields of directions in E^+ and E^- which correspond to 1–graphs of functions on M. Such submanifolds will be termed in E^+ and E^- the *extreme controlled jet-characteristics*.

The extreme jet–characteristics in $J^1(M,R)$ uniquely define *extreme characteristics* in the open set N by simple projection. These are obtained by identification of integral submanifolds of the jet–characteristics with 1–graphs of functions defined on M which constitute solutions of the corresponding NFOPDE. Associated to such characteristics one defines *extreme characteristic vector fields* $\kappa^+(\eta)$ and $\kappa^-(\eta)$ in TN whose prolongations to $TJ^1(M,R)$ (See Olver [18, p. 104]) coincide with the jet–characteristic vector fields $\xi^+(z)$ and $\xi^-(z)$ respectively. It is evident that the vector fields $\kappa^+(\eta)$ and $\kappa^-(\eta)$ defined in TN are uniquely determined. It is easy to see that the parametrization previously adopted for the extreme jet–characteristic fields $\xi^+(z)$ and $\xi^-(z)$ in terms of the switching control function ν, denoted as $\xi(z,\,\nu)$, is trivially inherited by the corresponding extreme characteristic vector fields $\kappa^+(\eta)$ and $\kappa^-(\eta)$ (the argument being that the projection from $TJ^1(m, R)$ to TN does not destroy the parametrization since such an operation simply amounts to deleting the prolongation components in $\xi^+(z)$ and $\xi^-(z)$). In correspondence with such an inherited parametrization, we also denote by $\kappa(\eta, 1)$ the vector field $\kappa^+(\eta)$ and by $\kappa(\eta, 0)$ the vector field $\kappa^-(\eta)$. One trivially has $\kappa(\eta, \nu) := \nu\kappa^+(\eta) + (1-\nu)\kappa^-(\eta)$. It is easy to see from the prolongation formula for vector fields (Olver [18, p. 104]) that, conversely, such a parametrization is preserved, via prolongation, in the controlled jet–characteristic vector fields, i.e., denoting by "Pr κ" the prolongation of the vector field κ we have

$$\begin{aligned} \text{Pr } \kappa(\eta, \nu) &= \text{Pr } [\nu\kappa^+(\eta) + (1-\nu)\kappa^-(\eta)] = \\ &= \nu \text{Pr } \kappa^+(\eta) + (1-\nu)\text{Pr } \kappa^-(\eta) = \\ &= \nu\xi^+(z) + (1-\nu)\xi^-(z) = \xi(z, \nu)\,. \end{aligned}$$

DEFINITION 7. A distributed sliding regime is said to locally exist on an open set $\mathcal{N}$ $(= N \cap S)$ of the sliding manifold S if the total time derivative of the output function of the controlled system (15)–(16) satisfies

$$\lim_{y \to +0} \frac{dy}{dt} < 0 \quad \text{and} \quad \lim_{y \to -0} \frac{dy}{dt} > 0\,. \tag{23}$$

THEOREM 1. *Given Cauchy data (φ, γ) defining an initial submanifold $\mathcal{G}$ of noncharacteristic points in N. There locally exists a distributed sliding regime for the solutions of system (15), (16) on an open set $\mathcal{N}$ of S if and only if the extreme characteristics (or phase flows corresponding to the extreme controlled characteristic direction fields $\kappa^+(\eta)$, $\kappa^-(\eta)$) which arise from the initial submanifold $\mathcal{G}$ exhibit such a local sliding regime on $\mathcal{N}$, in correspondence with the distributed control policy (16).*

Proof. Suppose a distributed sliding mode locally exists for (15), (16) on an open set $\mathcal{N}$ of S. Then the total time derivatives of y computed on any point η of an open $n+2$ dimensional negihbourhood of S with nonempty intersection with $\mathcal{N}$ satisfy conditions (23) on the regions $y < 0$ and $y > 0$ respectively. The total time derivative at any point η in N can be computed in terms of the directional derivative of the scalar function h along the controlled characteristic direction fields $\kappa^+(\eta)$, $\kappa^-(\eta)$. The directional derivative depends on the location of the point η with respect to the sliding surface S and, hence, it is given by for $y > 0$

$$\frac{dy}{dt} = \frac{\partial h}{\partial v}\frac{dv}{dt} + \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial t} =$$

$$= \frac{\partial h}{\partial \eta}\kappa^+(\eta) = L_{\kappa^+(\eta)}h = L_{\kappa(\eta,1)}h < 0\,.$$

for $y < 0$

$$\frac{dy}{dt} = \frac{\partial h}{\partial v}\frac{dv}{dt} + \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial t} = \frac{\partial h}{\partial \eta}\kappa^-(\eta) =$$

$$= L_{\kappa^-(\eta)}h = L_{\kappa(\eta,0)}h > 0\,.$$

Hence the flows corresponding to the characteristic vector fields $\kappa^+(\eta)$ and $\kappa^-(\eta)$ satisfy the conditions for the existence of a sliding regime on S (See [12]).

Sufficiency is easily obtained by assuming that a sliding regime locally exists for the flows corresponding to the extreme controlled characteristic direction fields $\kappa^+(\eta)$ and $\kappa^-(\eta)$, about an open set $\mathcal{N}$ of the sliding manifold $y = 0$, while hypothesizing that a distributed sliding mode *does not exist* on such a set. By reversing the above arguments, a contradiction is readily established.

Remark. Existence of local distributed sliding regimes for (15)–(16) on open subsets $\mathcal{N}$ of the sliding manifold S are completely characterized in terms of the existence of local sliding regimes — on the same sliding manifold S — for the $(n+2)$–dimensional time–varying dynamical system generating the control parametrized characteristics of (15) in N

$$\begin{aligned}(24)\qquad \frac{d\eta}{dt} = \kappa(\eta,\nu) &= \nu\kappa^+(\eta) + (1-\nu)\kappa^-(\eta) = \\ &= \kappa^-(\eta) + \nu[\kappa^+(\eta) - \kappa^-(\eta)]\end{aligned}$$

$y = h(\eta)$
with ν given by (16), according to the sign of y, i.e.,

$$(25)\qquad \nu = \begin{cases} 1 & \text{for } y > 0\ , \\ 0 & \text{for } y < 0\ . \end{cases}$$

We refer to (24)–(25) as the *controlled characteristic system.*

The problem of characterizing distributed sliding regimes in controlled dynamical systems described by NFOPDE is thus reduced to the problem of characterizing sliding regimes for an associated nonlinear dynamical system described by a finite set of discontinuously controlled ordinary differential equations representing the controlled characteristic system. All known results for the description of sliding motions in finite dimensional nonlinear controlled dynamical systems thus become immediately available for the above formulated distributed control problem. The following theorems characterize the existence of distributed sliding motions for (15)–(16) in terms of the associated controlled characteristic system (24)–(25). Some additional results can also be found in Sira–Ramirez [11] and [21].

THEOREM 2. *Let $L_\kappa h$ denote the directional (Lie) derivative of the scalar function h with respect to the vector field κ. A distributed sliding regime locally exists on an open set $\mathcal{N}$ of S for the distributed system (15), (16) only if the extreme characteristic vector fields $\kappa^+(\eta)$, $\kappa^-(\eta)$, associated to the system (24)–(25), satisfy, on the open set $\mathcal{N}$, the following distributed transversality condition*

$$L_{[\kappa^+(\eta)-\kappa^-(\eta)]}h < 0\,. \tag{26}$$

Proof. Let a distributed sliding regime exist on an open set $\mathcal{N}$ of S, then, according to Theorem (1) one has that $L_{\kappa^+(\eta)}h < 0$ and $L_{\kappa^-(\eta)}h > 0$ on any point η of S. Then $L_{\kappa^+(\eta)}h - L_{\kappa^-(\eta)}h = L_{[\kappa^+(\eta)-\kappa^-(\eta)]}h < 0$. □

3.2. Characterization of the distributed ideal sliding dynamics and the distributed equivalent control

In this section a characterization is presented for the *Distributed Ideal Sliding Dynamics* and the *Distributed Equivalent Control* (Utkin [2]). Intuitively, under ideal distributed sliding mode conditions, the closed loop system solution must locally adopt the zero output level set S as a local solution manifold of the resulting controlled partial differential equation, provided the (noncharacteristic) initial Cauchy data is specified precisely on the sliding manifold S. The characteristics of the ideal sliding dynamics must then be invariant with respect to the sliding manifold. The smooth distributed feedback control law responsible for the idealized controlled response is known as the *Distributed Equivalent Control* and it formally replaces the distributed switch position function ν in the system equation (24). Such a smooth distributed feedback control law is here denoted by $\nu^{EQ}(v,x,t)$ or $\nu^{EQ}(\eta)$.

DEFINITION 8. Consider the $n+1$–dimensional distribution in TN, tangent to the sliding manifold S. Any smooth vector field $\kappa(\eta)$, locally belonging to such a distribution, annihilates the 1–form dh, i.e., $\langle dh, \kappa(\eta)\rangle = 0$. We denote such a distribution by Ker dh and define it as follows:

$$\text{Ker } dh = \{\kappa(\eta) \in TN \,|\, \langle dh,\ \kappa(\eta)\rangle = 0\}\,.$$

The following proposition is just a restatement, in simple terms of the annihilating distribution of a 1–forms, of an invariance criterion for partial differential equations appearing, in terms of symmetry groups and infinitesimal generators, in Olver [18, pp. 103, Theorem 2.27 and pp. 165, Theorems 2.71 and 2.72].

PROPOSITION 1. *The prolongation $\xi(z)$ of a vector field $\kappa(\eta)$ to $TJ^1(M,R)$ (denoted by $\xi = \text{Pr } \kappa$), belongs to the distribution tangent to $S^{(1)}$ in $TJ^1(M,R)$ if and only if $\kappa(\eta) \in \text{Ker } dh$.*

Hence, there exists a 1–form ω in the cotangent space $T \star J^1(M,R)$ such that $\langle \omega, \ \xi(z) \rangle = 0$. We define such a 1–form ω as the prolongation of the 1–form dh and denote it by "Pr dh". In other words, Proposition (1) states that $\langle dh, \kappa \rangle = 0$ if and only if $\langle \text{Pr } dh, \text{Pr } \kappa \rangle = 0$.

The next proposition states that the smoothly controlled jet–characteristics, generated by the controlled vector field $\xi(z,\nu)$ in $TJ^1(M,R)$, locally adopt the prolongation $S^{(1)}$ of the sliding manifold S as their integral manifold if and only if the corresponding characteristics, generated by the controlled vector field $\kappa(\eta,\nu)$ in TN, locally adopt S as their integral manifold. Hence the invariance of the controlled jet–characteristics with respect to $S^{(1)}$ is equivalent to the invariance of the controlled characteristics with respect to S.

PROPOSITION 2. *Consider the prolongation $S^{(1)}$ of S to $J^1(M,R)$. The controlled jet–characteristic direction field $\xi(z,\nu) = \nu\xi^+(z) + (1-\nu)\xi^-(z)$ belongs, for some smooth control function $\nu(\eta)$, to the distribution in $TJ^1(M,R)$ tangent to $S^{(1)}$ if and only if the smoothly controlled characteristic direction field $\kappa(\eta,\nu) = \nu\kappa^+(z) + (1-\nu)\kappa^-(z)$, whose prolongation to $TJ^1(M,R)$ coincides with $\xi(z,\nu)$, belongs to the distribution tangent to S, i.e., to the annihilating distribution of the 1–form dh, here denoted by* Ker dh.

Proof. Let $\kappa(\eta,\nu) \in \text{Ker } dh$, locally in $\mathcal{N}$ for some smooth function $\nu(\eta)$. Then $\langle dh, \ \kappa(\eta,\nu)\rangle = 0$ locally in $\mathcal{N}$. Hence, by the result of Proposition (1) above, $\kappa(\eta,\nu) \in \text{Ker } dh$ if and only if $\langle \text{Pr } dh, \ \text{Pr } \kappa(\eta,\nu)\rangle = \langle \text{Pr } dh, \ \xi(z,\nu)\rangle = 0$, i.e., if and only if $\xi(z,\nu) \in \text{Ker Pr } dh$. □

The *Distributed Ideal Sliding Dynamics* is thus obtained by imposing an invariance condition, with respect to the manifold S, on the ideally smooth controlled characteristic flows arising from the controlled characteristic system (24)-(25). This amounts to constraining the corresponding smoothly controlled characteristic vector field $\kappa(\eta,\nu) = \kappa^-(\eta) + \nu(\eta) \ [\kappa^+(\eta) - \kappa^-(\eta)]$

to the distribution Ker dh, tangent to the given sliding surface S. Thus,

$$\langle dh,\ \kappa^-(\eta) + \nu^{EQ}(\eta)\ [\kappa^+(\eta) - \kappa^-(\eta)]\rangle\,,$$

$$L_{\kappa^-(\eta)+\nu^{EQ}(\eta)\ [\kappa^+(\eta)-\kappa^-(\eta)]}h = 0$$

it follows that the equivalent control is *uniquely* given by

$$\nu^{EQ}(\eta) = -\frac{\langle dh,\ \kappa^-(\eta)\rangle}{\langle dh,\ [\kappa^+(\eta) - \kappa^-(\eta)]\rangle} = -L_{\kappa^-(\eta)}h / L_{[\kappa^+(\eta)-\kappa^-(\eta)]}h\,. \tag{27}$$

Hence, by virtue of (26), the distributed equivalent control $\nu^{EQ}(\eta)$ is locally well defined on the open set $\mathcal{N}$ of S where a distributed sliding regime exists. The following theorem gives s sufficient condition for the existence of a local distributed sliding regime in terms of the computable distributed equivalent control function $\nu^{EQ}(\eta)$.

THEOREM 3. *A local distributed sliding regime exists on an open set $\mathcal{N}$ of S if and only if the distributed equivalent control, $\nu^{EQ}(\eta)$, satisfies*

$$0 < \nu^{EQ}(\eta) < 1\,. \tag{28}$$

Proof. From (27), it readily follows that if (28) holds valid on an open set $\mathcal{N}$ of the sliding manifold S, then

$$0 < -\frac{\langle dh,\ \kappa^-(\eta)\rangle}{\langle dh,\ [\kappa^+(\eta) - \kappa^-(\eta)]\rangle} < 1\,. \tag{29}$$

By virtue of the transversality condition (26) and the local smoothness assumption on the involved vector fields $\kappa^+(\eta)$, $\kappa^-(\eta)$, and the function $h(\eta)$ the left hand side inequality in (29) yields $\langle dh,\ \kappa^-(\eta)\rangle > 0$, for points η located in $\mathcal{N}$ and located on any arbitrarily small neighbourhood of $\mathcal{N}$ in R^{n+2}. Similarly, the right hand side inequality in (29) yields $\langle dh,\ \kappa^+(\eta)\rangle < 0$. The result in Theorem 1 completes the sufficiency part of the proof.

To prove necessity, suppose that a distributed sliding regime locally exists on the open subset $\mathcal{N}$ of the manifold S. Then, from Theorem 2 the transversality condition $L_{[\kappa^+(\eta)-\kappa^-(\eta)]}h < 0$ holds locally valid. Also, for any η in $\mathcal{N}$, the relations $L_{\kappa^+(\eta)}h < 0$, and $L_{\kappa^-(\eta)}h > 0$, are locally true. It follows that there exist strictly positive smooth function $a(\eta)$ and $b(\eta)$ such that locally in $\mathcal{N}$,

$$\begin{aligned} & a(\eta)L_{\kappa^+(\eta)}h + b(\eta)\ L_{\kappa^-(\eta)}h = \\ = \ & L_{a(\eta)\kappa^+(\eta)+b(\eta)\kappa^-(\eta)}h = L_{[a(\eta)+b(\eta)]\kappa^-(\eta)+a(\eta)[\kappa^+(\eta)-\kappa^-(\eta)]}h = \\ = \ & [a(\eta) + b(\eta)]L_{\kappa^-(\eta)+(a(\eta)/[a(\eta)+b(\eta)])[\kappa^+(\eta)-\kappa^-(\eta)]}h = 0\,. \end{aligned}$$

From the uniqueness of the distributed equivalent control it follows that $\nu^{EQ}(\eta) = a(\eta)/[a(\eta) + b(\eta)]$. Hence $0 < \nu^{EQ}(h) < 1$, as claimed. □

A more explicit characterization of ideal distributed sliding motions and the corresponding distributed equivalent control is still possible in terms of the functions F, G and h, defining the controlled NFOPDE in (15). Indeed, the components of the controlled characteristic field $\kappa(\eta, \nu)$, can be obtained by projection of those of the controlled jet-characteristic direction vector $\xi(z, \nu)$ given in (21). However, recall that such projected components are supposed to be ultimately explicit functions of v, x, t, i.e., of η. The components of $\kappa(\eta, \nu)$ have the form

$$
\begin{aligned}
\dot{v} &= p(F_p + \nu G_p) + q \\
\dot{x} &= F_p + \nu G_p \\
\dot{t} &= 1
\end{aligned}
\tag{30}
$$

with $\nu \in \{0, 1\}$ determining the vector fields $\kappa^-(\eta)$ and $\kappa^+(\eta)$. The distributed transversality condition (26) is readily interpreted as

$$
\left[\frac{\partial h}{\partial v} p + \frac{\partial h}{\partial x}\right] G_p = 2\frac{\partial h}{\partial x} G_p < 0
\tag{31}
$$

and the equivalent control is just given by

$$
\nu^{EQ}(\eta) = -\frac{L_{\kappa^-(\eta)}h}{L_{[\kappa^+(\eta)-\kappa^-(\eta)]}h} =
$$

$$
= -\frac{\left[\frac{\partial h}{\partial v}(pF_p + q) + \frac{\partial h}{\partial x}F_p + \frac{\partial h}{\partial t}\right]}{2\frac{\partial h}{\partial x}G_p} = -\frac{\left[\frac{\partial h}{\partial x}F_p + \frac{\partial h}{\partial t}\right]}{\frac{\partial h}{\partial x}G_p}.
\tag{32}
$$

The ideal sliding dynamics is obtained by formally replacing the smooth distributed control $\nu^{EQ}(\eta)$, from (32), into the system equation (15). The initial Cauchy data for such an ideal dynamics is to be specified as a non-characteristic (but otherwise arbitrary) n-dimensional submanifold $\mathcal{G}$ of the sliding surface S.

4. Conclusions and suggestions for further research

The theory of Variable Structure Systems undergoing local sliding motions has been extended to distributed controlled systems described by first order nonlinear PDE's. The key property of such a class of dynamical systems is the possibility of relating properties of their controlled solutions to those of a controlled system described by a finite set of ordinary differential equations known as the characteristic equations. This property was used in this article to establish conditions for the local existence of a distributed sliding regime on a given sliding manifold for nonlinear first order distributed dynamical systems solved with respect to the time derivative. A distributed sliding mode locally exists for the infinite dimensional dynamical system whenever the corresponding finite dimensional controlled characteristic system exhibits such kind of motions on the prescribed sliding manifold. The given sliding manifold also qualifies as a local integral manifold for the flows of a smoothly controlled (equivalent) characteristic direction field. The equivalent characteristic direction field is the Filippov average direction field [4] also prescribed by the Equivalent Control Method [2] on the finite dimensional discontinuously controlled characteristic system.

The case of distributed sliding regimes in dynamical systems described by higher order nonlinear partial differential equations, of the implicit or explicit type, can be adequately treated from a Lie group theoretic viewpoint by using notions of symmetry groups, invariance, and prolongations to appropriate jet spaces, of associated functions and infinitesimal generators of such symmetry groups. This avenue is left as a topic for further research.

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