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Non-linear discrete variable structure systems in quasi-sliding mode

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The problem of inducing convergent quasi-sliding regimes on smooth state-space surfaces of non-linear single-input single output discrete-time controlled systems is addressed in full generality. A suitable extension of the notion of relative degree is used in establishing the most salient features of quasi-sliding motions in non-linearly controlled systems. Several examples are given.

1. Introduction

Sliding regimes in continuous-time systems have been extensively studied in the literature during the last 30 years. For through surveys and the main theoretical developments in this field, the reader is referred to Utkin (1977, 1978, 1984), De Carlo *et al.* (1988) and Sira-Ramierz (1988, 1989 a, b, 1990). The discrete-time counterpart of sliding regimes, addressed as quasi-sliding regimes, has been less well studied. The available contributions in this field, for linear discrete-time systems, are represented by the work of Miloslavjevic (1985) in the context of sampled data systems, Opitz (1986), Magaña and Zak (1987), and Sarpturk *et al.* (1987) for various classes of discrete-time linear systems, and more recently by the contributions of Drakunov and Utkin (1989). Recently, interesting contributions have been provided by Yu and Potts (1989 a, b, 1990) and Furuta (1990), in this context of linear time-invariant systems.

In this article, we examine the properties of general discrete-time single-input single-output non-linear variable structure controlled systems operating in the quasi-sliding mode. The term 'quasi-sliding' regime was introduced by Miloslavjevic to express the fact that the extension to the discrete-time case of the usual continuous-time conditions for the existence of a sliding regime do not necessarily guarantee chattering motions close to the sliding manifold, in the same 'incipient overshoot and instantaneous correction' fashion that is exhibited in continuous-time systems. Moreover, in Sarpturk *et al.* (1987) it was demonstrated that, even for discrete linear dynamic systems, the straightforward extension of the continuous-time sliding mode conditions may lead to unstable chattering about the switching hyperplane (see example § 3.1).

In this article, it is found that a quasi-sliding regime exists on the zero level set of the output switching function, if and only if the non-linear system has relative degree (Byrnes and Isidori 1984, Isidori 1989) equal to 1. This condition, however, is not sufficient to guarantee a convergent quasi-sliding mode. It is easily seen that the corresponding zero dynamics precisely coincides with the ideal sliding dynamics. Using the ideas in Isidori (1989), the problem of inducing sliding regimes on systems with relative degree higher than 1 is also examined. The disturbance rejection properties of non-linear systems undergoing quasi-sliding motions are also analysed and a general matching condition is found.

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In § 2 we present some background material about an extension of the relative degree concept, normal forms and zero dynamics, to the discrete-time case closely following the work of Van der Schaft (1989). New and general results about quasi-sliding motions, in general non-linear systems, are presented in § 3. Section 4 contains the conclusions and suggestions for further work.

2. Background and main results

2.1. *Relative degree, normal forms and zero dynamics*

Consider a smooth single-input single-output discrete-time non-linear system of the form

$$\left.\begin{aligned} x(k+1) &= F(x(k), u(k)), \quad k = 0, 1, 2, \ldots \\ y(k) &= h(x(k)) \end{aligned}\right\} \tag{2.1}$$

defined for all x in $\mathbf{X}$, which is an open part of R^n. The scalar control function u and the scalar output y take values, respectively, in open neighborhoods $\mathbf{U}$ and $\mathbf{Y}$ of R. The mapping $F: \mathbf{X} \times \mathbf{U} \to \mathbf{X}$ and the function $h: \mathbf{X} \to \mathbf{Y}$ are assumed to be analytic. We often refer to (2.1) as the pair (F, h).

The level set $h^{-1}(0) := \{x \in \mathbf{X} : h(x) = 0\}$, defines the sliding manifold which is assumed to be sufficiently smooth.

Remark 2.1

Let $z = F(x, u)$ and suppose $h(z) = h(F(x, u))$ is independent of u, i.e. $h(F(x, u)) = (h \circ F)(x)$, then $\partial[h(F(x, u)]/\partial u = 0$ in all of $\mathbf{X} \times \mathbf{U}$. We recursively define $h \circ F^i(x, u) := (h \circ F^{i-1})[F(x, u)]$, $i = 1, 2, \ldots, r$. In general, one may assume that $h \circ F$, $h \circ F^2, \ldots$, etc are independent of u, up to r compositions. This idea introduces the concept of relative degree (also called relative order or characteristic number, see also Byrnes and Isidori 1984, Isidori 1989, and Van der Schaft 1989). □

Definition 2.1

Let x^0 be an arbitrary state in $\mathbf{X}$. The pair (F, h) has relative degree r if $h \circ F^k(x^0, u)$ does not explicitly depend on u, everywhere in $\mathbf{X} \times \mathbf{U}$, for all integers $k < r - 1$, and there exists an open dense submanifold of $\mathbf{X} \times \mathbf{U}$, with $\mathbf{X}$ containing x^0 such that $h \circ F^{r-1}(x, u)$ does depend upon u. In other words

$$\partial[h \circ F^k(x^0, u)]/\partial u = 0 \quad \text{for all } (x^0, u) \text{ in } \mathbf{X} \times \mathbf{U}, \text{ and all } k < r - 1.$$

$$\partial[h \circ F^{r-1}(x^0, u)]/\partial u \neq 0 \quad \text{for all } (x^0, u) \text{ in an open and dense submanifold of } \mathbf{X} \times \mathbf{U} \tag{2.2}$$

The relative degree r determines the time delay undergone by the input signals u before they influence the output y of the system. □

Proposition 2.1

Let (2.1) have relative degree r at x^0 and choose $\phi_i(x) = h \circ F^{i-1}(x)$ for $i = 1, 2, \ldots, r$, and let the functions $\phi_{r+j}(x)$, $j = 1, 2, \ldots, n - r$, be arbitrary functions chosen to be independent of the first r functions, with no additional requirements.

Define new z coordinates as $z = \Phi(x)$ with $\Phi(x) = \text{col}\,[\phi_1(x), \ldots, \phi_n(x)]$. Φ is then a diffeomorphism on $\mathbf{X}$. The transformed system, said to be in normal coordinates, is expressed, around $z^0 = \Phi(x^0)$ as

$$\begin{aligned} z_i(k+1) &= z_{i+1}(k) \quad i = 1, 2, \ldots, r-1 \\ z_r(k+1) &= [h \circ F^r](\Phi^{-1}(z(k)), u(k)) \\ z_{r+j}(k+1) &= q_j(z(k), u(k)), \quad j = 1, 2, \ldots, n-r \\ y(k) &= z_1(k) \end{aligned} \tag{2.3}$$

Proof

The proof is obvious from the choice of coordinates. □

Remark 2.2

Any initial state z^0 of (2.3) may be driven, in finite time (as a matter of fact, in precisely r steps), to the manifold $h^{-1}(0)$ by an appropriate feedback control function $u(k)$. Indeed, let $u(k) = \alpha(z(k))$ be such that $[h \circ F^r](\Phi^{-1}(z(k)), \alpha(z(k))) = 0$ for all k. That such a control exists and is unique follows by virtue of the Implicit Function Theorem and the definition of relative degree. Once the state trajectory $z(k)$ belongs to this manifold, it can be kept there indefinitely by use of the feedback control action, $u(k) = \alpha(z(k))$ for all remaining ks. The components $z_1, \ldots, z_r$ of the normal coordinate vector z are all zero from time $k = r$ onwards. Any point x^0 on $h^{-1}(0)$ is therefore expressed, in normal coordinates, as $(0, \eta)$, where $\eta = \text{col}\,(z_{r+1}, \ldots, z_n)$. These considerations show that with no magnitude restrictions on the feedback control actions $u(k)$, the evolution of the controlled system may be forced to remain on $h^{-1}(0)$ for ever. In our considerations, the available feedback control laws are, unless otherwise explicitly stated, *a priori* restricted to be fixed analytic functions of the form: $u(k) = u^+(x(k))$ or $u(k) = u^-(x(k))$. Hence, it is not necessarily true that any arbitrary initial state x^0 of (2.1) can always be driven to the manifold $h^{-1}(0)$, in finite time, by the use of the available feedback control laws. □

Definition 2.2

The dynamic behaviour of system (2.1), with initial conditions set on $h^{-1}(0)$, and feedback control input $\alpha(z(k))$ such that the quantity $z_r(k+1) = [h \circ F^r](\Phi^{-1}(z(k)), \alpha(z(k))) = 0$, for all k, described in normal coordinates by:

$$\eta(k+1) = q(0, \eta(k)) = q_0(\eta(k)) \tag{2.4}$$

is addressed as 'zero dynamics' (Byrnes and Isidori 1984). □

Assumption

The zero dynamics crucially depend upon the nature of the manifold $h^{-1}(0)$. It is assumed, throughout, that $y = h(x)$ is such that system (2.4) is asymptotically stable (in such a case, the system is said to be 'globally minimum phase'—Byrnes and Isidori 1984). □

Lemma 2.1

The relative degree of a system is a feedback invariant, i.e. it cannot be modified by feedback. □

Proof

Let **U** be an open and dense submanifold of R contained in **U**. For a system with feedback $u = u(x, v)$, the quantities $h \circ F^k\ (x, u)$ are independent of u everywhere in $\mathbf{X} \times \mathbf{U}$ if and only if they are independent of v everywhere in $\mathbf{X} \times \mathbf{U}$. □

3. Quasi-sliding regimes in general non-linear discrete-time systems

3.1. *Generalities about local quasi-sliding regimes*

A variable structure feedback control law for (F, h) is obtained by letting

$$u = \begin{cases} u^+(x) & \text{for } h(x) > 0 \\ u^-(x) & \text{for } h(x) < 0 \end{cases} \tag{3.1}$$

with $u^+(x) > u^-(x)$, everywhere in **X**.

Definition 3.1 (Sarpturk *et al.* 1987)

The controlled system (2.1) is said to exhibit a quasi-sliding motion locally about $h^{-1}(0)$ whenever,

$$y_k(y_{k+1} - y_k) < 0 \tag{3.2}$$

□

Remark 3.1

Notice that the quasi-sliding motion is not defined as one necessarily created by a discontinuous control law of the form (3.1). Indeed, it is a substantial departure from continuous-time sliding modes that, in discrete time systems, quasi-sliding regimes can be achieved with continuous feedback laws (see the examples below). □

Definition 3.1 represents the discrete-time counterpart of the continuous time sliding mode condition, $y\, dy/dt < 0$ and it is also trivially equivalent to have $y_k y_{k+1} < y_k^2$. Condition (3.2), however, allows for unstable quasi-sliding regimes as the following simple example demonstrates.

Example 3.1

Consider the non-linear discrete-time system: $x_1(k+1) = 1 + x_1(k)$; $x_2(k+1) = x_1(k)u(k)$; $y(k) =: y_k = x_2(k)$ with $x_1(0) = 1$ and $x_2(0) \neq 0$. As a variable structure feedback control we choose $u(k) = -\text{sign}\, y_k$. In this case:

$$y_k(y_{k+1} - y_k) = y_k[-x_1(k)\ \text{sign}\, y_k - y_k] = -x_1(k)|y_k| - y_k^2 < 0$$

for all $x_1(k) \geqslant 0$. The controlled motions are clearly unstable about $x_2 = 0$ (see Fig. 1). □

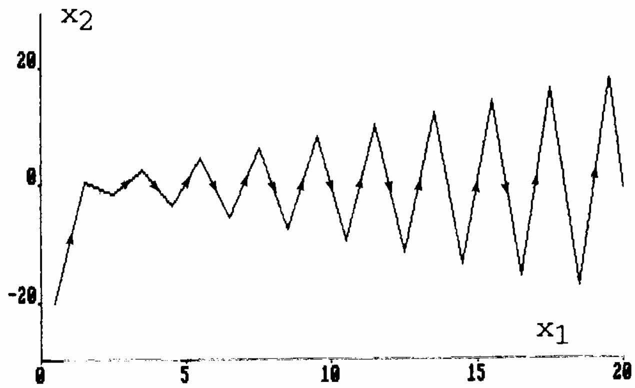


Figure 1. An unstable quasi-sliding regime.

Definition 3.2 (Sarpturk *et al.* 1987)

A convergent quasi-sliding regime is said to exist locally on $h^{-1}(0)$, if there is a feedback control policy for which there exists an open and dense subset of **X**, containing $h^{-1}(0)$, such that the state trajectories of (2.1) starting on such an open subset satisfy, for all k

$$|y_{k+1}| < |y_k| \tag{3.3}$$

□

Remark 3.2

Intuitively, one expects that discrete-time quasi-sliding motions exhibit the same 'chattering' or 'zigzag' behaviour, characteristic of non-ideal sliding regimes occurring in continuous time systems. In this respect, quasi-sliding motions, even if they are convergent, are quite counter-intuitive. In fact, under condition (3.2) or (3.3), zigzag behaviour may or may not exist about the proposed quasi-sliding surface. It may even exist without, necessarily, being achieved by means of discontinuous controllers of the form (3.1). The following example depicts these possibilities. □

Example 3.2

Consider the system: $x_1(k+1) = x_2(k)$, $x_2(k+1) = u(k)$, $y(k) = x_2(k) + ax_1(k)$ with $|a| < 1$. The quasi-sliding mode condition (3.3) may be enforced on the difference equation derived for $y(k)$ as: $y(k+1) = u(k) - a^2x_1(k) + ay(k)$. A non-discontinuous feedback control law of the form $u(k) = a^2x_1(k)$ yields a convergent quasi-sliding regime characterized by the autonomous system: $y(k+1) = ay(k)$. If $0 < a < 1$, the sliding mode asymptotically approaches the line $y = 0$, without overshooting it. If, on the contrary, $-1 < a < 0$, the motions zigzag about the line $y = 0$. Consider now the auxiliary output $w(k) = a^{-1}x_2(k) - x_1(k)$. The controlled motions described above yield now $w(k+1) = -aw(k)$. Hence, those quasi-sliding motions that do not zigzag about $y = 0$, $(a > 0)$ in fact appear as if they were creating a quasi-sliding regime about the line $w = 0$ and vice versa. Thus, zigzag motions do not necessarily occur on the chosen sliding surface when the quasi-sliding mode conditions (3.3) are valid and they are not, necessarily, due to variable structure control actions (see Figs 2 (*a*)–(*d*)). □

Lemma 3.1

A necessary condition for the existence of convergent quasi-sliding regime about $h^{-1}(0)$ is that a quasi-sliding regime exists about such a manifold. □

Proof

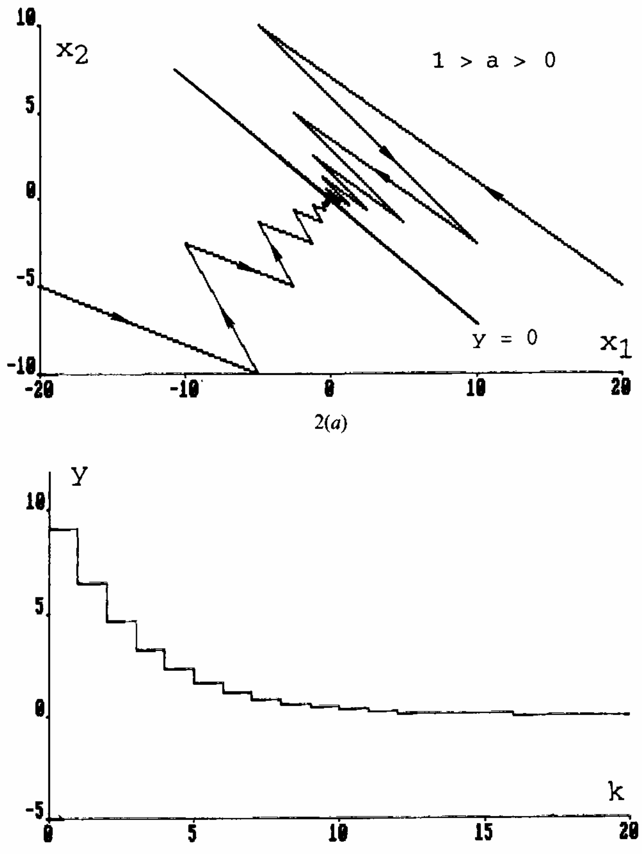
Suppose a convergent quasi-sliding regime exists on $h^{-1}(0)$. Then, from (3.3) it follows that $|y_{k+1}|\,|y_k| < |y_k|^2$, i.e., $|y_{k+1}y_k| < y_k^2$, but then, $y_{k+1}y_k < y_k^2$. Equivalently $(y_{k+1} - y_k)y_k < 0$. Condition (3.2) holds true. □

Theorem 3.1

A convergent quasi-sliding regime exists on $h^{-1}(0)$ if and only if

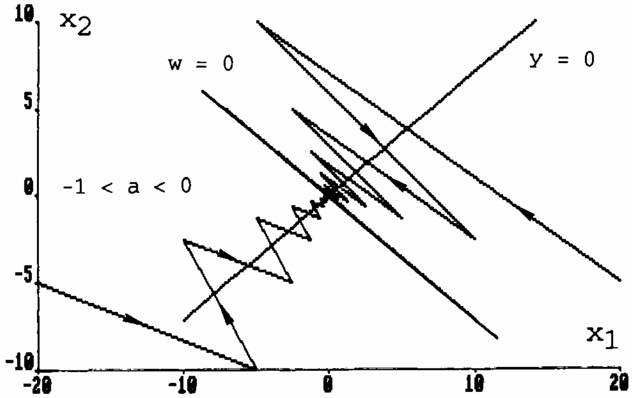
$$|y_{k+1}y_k| < y_k^2 \tag{3.4}$$

□



2(*a*)

2(*b*)



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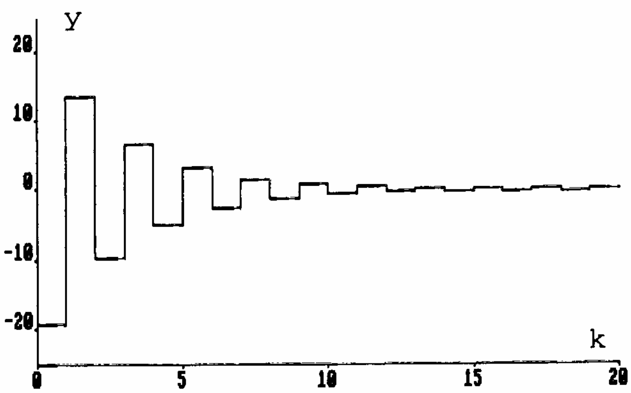

(*d*)

Figure 2. (*a*) A non-chattering convergent quasi-sliding regime; (*b*) quasi-sliding surface coordinate evolution; (*c*) a chattering convergent quasi-sliding regime; (*d*) quasi-sliding surface coordinate evolution.

Proof

Let there be a convergent quasi-sliding regime on $h^{-1}(0)$. Then, from Lemma 3.1, $(y_{k+1} - y_k)y_k < 0$ holds true; i.e., $y_{k+1}y_k < y_k^2$. Also, from (3.3), we have:y_{k+1} sign $y_{k+1} - y_k$ sign $y_k < 0$. But then, either y_{k+1} differs in sign with respect to y_k, or it does not. If it does not, then the last inequality is equivalent to $(y_{k+1} - y_k)$ sign $y_k < 0$, i.e., $(y_{k+1} - y_k)y_k < 0$ and we are back at the quasi-sliding condition, $y_{k+1}y_k < y_k^2$. If, on the other hand, the signs differ then $-(y_{k+1} + y_k)$ sign $y_k < 0$, i.e., $(y_{k+1} + y_k)$ sign $y_k > 0$. But this is equivalent to $(y_{k+1} + y_k)y_k > 0$, i.e., $y_{k+1}y_k > -y_k^2$. Hence, the existence of a covergent quasi-sliding regime is equivalent to: $-y_k^2 < y_{k+1}y_k < y_k^2$, which is the desired result. Sufficiency is easily established by the fact that $|y_{k+1}y_k| = |y_{k+1}|\,|y_k| < y_k^2 = |y_k|^2$, i.e., $|y_{k+1}| < |y_k|$. □

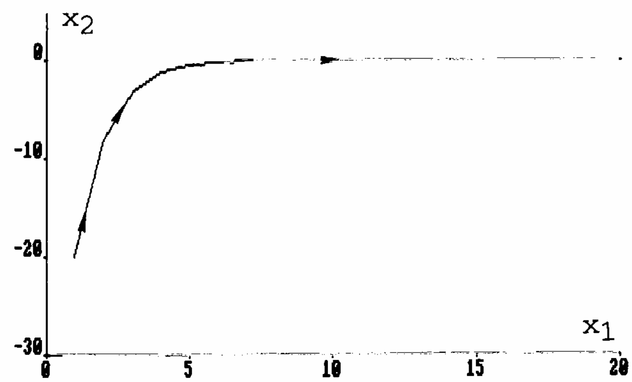

Figure 3. A convergent variable structure quasi-sliding regime.

Example 3.3

In the system of example 3.1, consider the control law: $u(x(k)) = -\lambda x_2(k) x_1^{-1}(k)$ sign y_k with $|\lambda| < 1$. In this case one has $y_k(y_{k+1} - y_k) = -y_k^2(\lambda \text{ sign } y_k + 1) < 0$, i.e., a quasi-sliding motion exists about $x_2 = 0$. However, $|y_{k+1}| = |x_2(k+1)| = |x_1(k)u(k)| = |\lambda x_2(k) \text{ sign } y_k| = |\lambda|\,|x_2(k)| = |\lambda|\,|y_k| < |y_k|$. The controlled motions now result in asymptotically stable motions converging toward $x_2 = 0$ (see Fig. 3). □

The following theorem is useful when the variable structure feedback control actions $u^+(x)$ and $u^-(x)$ of (3.1) have not been specified *a priori.*

Theorem 3.2

Let $u(k)$ be of the form (3.1) but otherwise unspecified. Then, if the system (F, h) has relative degree equals to 1, there exist a variable structure feedback control law of the form (3.1) which creates a quasi-sliding regime on $h^{-1}(0)$. □

Proof

Suppose $\partial[h(F(x, u))]/\partial u \neq 0$ in an open and dense submanifold of $\mathbf{X} \times \mathbf{U}$ and let x be an arbitrary point in $\mathbf{X}$, located sufficiently close to the manifold $h^{-1}(0)$. Then, $\partial[h(F(x, u)) - h(x)]/\partial u \neq 0$. Let $\varepsilon^+(x)$ be an analytic, strictly positive function of x. Let, at the same time, $y = h(x)$ be strictly negative. Then, by virtue of the Implicit Function Theorem, the equation $h(F(x, u)) - h(x) = \varepsilon^+(x)$ has a unique solution in $\mathbf{U}$ given by $u = u^{-\varepsilon}(x)$, i.e., $h(F(x, u^{-\varepsilon}(x)) - h(x) = \varepsilon^+(x) > 0$. By the same arguments, choosing x such that, now, $h(x)$ is strictly positive, and given an analytic strictly negative function $\varepsilon^-(x)$, a control law, $u(x) = u^{+\varepsilon}(x)$, exists with values in $\mathbf{U}$ such that $h(F(x, u^{+\varepsilon}(x))) - h(x) = \varepsilon^-(x) < 0$. Hence, a quasi-sliding regime exists oin $h^{-1}(0)$ since (3.2) is satisfied by the variable structure feedback control law: $u^+(x) = u^{-\varepsilon}(x)$ for $h(x) < 0$ and $u^-(x) = u^{+\varepsilon}(x)$ for $h(x) > 0$. □

Corollary 3.1

Suppose the system (F, h) has relative degree 1. Then, there always exists a continuous feedback control law $\upsilon(x(k))$ such that a covergent quasi-sliding regime exists about $h^{-1}(0)$. □

Proof

By virtue of the Implicit Function Theorem, for any given state vector $x(k)$, the scalar equation $y(k+1) = h(F(x(k), u(k))) = \alpha y(k) = \alpha h(x(k))$ has a unique solution in $\mathbf{U}$ given by $u = \upsilon(x(k))$. It only suffices then to take $|\alpha| < 1$. □

The following simple example clearly shows that the condition given in Theorem 3.2 is not necessary for the existence of a quasi-sliding regime.

Example 3.4

Consider the system $x_1(k+1) = x_2(k)$; $x_2(k+1) = u(k)$, $y(k) = -x_1(k)$. The system is clearly relative degree 2. The variable structure control law, $u = -\text{sign}\,(x_2(k))$, evidently creates a quasi-sliding regime about $y = 0$. □

Definition 3.3

For all initial states located on $h^{-1}(0)$, the control function constraining the system trajectories to manifold $y = h(x) = 0$, in the region of the existence of a quasi-sliding regime, is known as the *equivalent control* and it is denoted by, $u^{\mathrm{EQ}}(x)$. The dynamics resulting from the use of such a control law constitute the trajectories of the *ideal sliding system* (Utkin 1978). This notion evidently coincides with the notion of *zero dynamics* of the pair (F, h). A description of such a system is

$$\left.\begin{aligned} x(k+1) &= F(x(k), u^{\mathrm{EQ}}(x(k))) \\ y(k) &= 0 \quad \text{for all } k \end{aligned}\right\} \tag{3.5}$$

Under ideal sliding conditions on $h^{-1}(0)$

$$y_{k+1} = y_k = 0, \quad \text{for all } k \tag{3.6}$$

or

$$h(x(k+1) = h(F(x(k), u^{\mathrm{EQ}}(k))) = h(x(k)) = 0$$

i.e., the manifold $h^{-1}(0)$ is invariant with respect to the ideal sliding system. □

Proposition 3.1

Suppose the system (F, h) is relative degree 1, then the equivalent control exists and it is uniquely defined. □

Proof

If (F, h) has local relative degree 1, then $\partial[h(F(x, u)]/\partial u \neq 0$ for all (x, u) in an open and dense submanifold of $\mathbf{X} \times \mathbf{U}$. By the Implicit Function Theorem, for any x in $h^{-1}(0)$, the equation $h(F(x, u)) - h(x) = 0$, has a unique solution $u = u^{\mathrm{EQ}}(x)$ with values in $\mathbf{U}$ for which (3.6) holds valid. □

Theorem 3.3

Suppose the system (F, h) has relative degree 1. Then, there exists a local quasi-sliding regime on $h^{-1}(0)$, for a given variable structure control law of the form (3.1), only if, locally on $h^{-1}(0)$, the equivalent control function, $u^{\mathrm{EQ}}(x)$, satisfies

$$u^{-}(x) < u^{\mathrm{EQ}}(x) < u^{+}(x) \tag{3.7}$$

□

Proof

By Proposition 3.1, if the system $((F, h)$ is relative degree 1, then the equivalent control $u^{\mathrm{EQ}}(x)$ exists and it is uniquely defined. Suppose now, that a quasi-sliding regime exists on $h^{-1}(0)$ for the switching feedback control law (3.1). Then, locally, for x in $h^{-1}(0)$, the following three relations hold valid:

$$h(F(x, u^{+}(x))) < h(x) = 0 \tag{3.8}$$

$$h(F(x), u^{\mathrm{EQ}}(x))) = h(x) = 0 \tag{3.9}$$

$$h(F(x, u^{-}(x))) > h(x) = 0 \tag{3.10}$$

Consider the control law $u^{\alpha}(x) = \alpha u^{+}(x) + (1-\alpha)u^{-}(x)$ and define, $\gamma(x, \alpha) := h(F(x, u^{\alpha}(x)))$. The function $\gamma(x, \alpha)$ is clearly analytic. From the relative degree 1 assumption, together with the feedback invariance property of the relative degree, it follows that $\partial\gamma(x, \alpha)/\partial\alpha \neq 0$. Moreover, from (3.8) and (3.10), locally for x in $h^{-1}(0)$, $\gamma(x, 1) < 0$ and $\gamma(x, 0) > 0$. It then follows that there is, at least, one value of $\alpha = \alpha_{\mathrm{EQ}}$, possibly depending on x, bounded by the interval $(0, 1)$ where $\gamma(x, \alpha_{\mathrm{EQ}}) = h(F(x, u^{\alpha}\mathrm{EQ}(x))) = 0$. From the uniqueness of the equivalent control, one concludes that $u^{\mathrm{EQ}}(x) = \alpha_{\mathrm{EQ}}(x)u^{+}(x) + (1-\alpha_{\mathrm{EQ}}(x))u^{-}(x)$. It follows that $u^{-}(x) < u^{\mathrm{EQ}}(x) < u^{+}(x)$. □

Definition 3.4

Let (F, h) have relative degree 1 locally on $x \in h^{-1}(0)$. The system (X, h) is said to exhibit a local *control foliation property* about the manifold $h^{-1}(0)$ for some triple of analytic feedback functions, $\{u_1(x), u_2(x), u_3(x)\}$, which are locally defined on M and such that $u_1(x) < u_2(x) < u_3(x)$, if, $h \circ F\ (x, u_1) > h \circ F\ (x, u_2) > h \circ F\ (x, u_3)$. □

Theorem 3.4

Let (F, h) be relative degree 1 on $h^{-1}(0)$. The control law (3.1) locally induces a quasi-sliding regime, for system (F, h), about $h^{-1}(0)$, if and only if (F, h) exhibits a control foliation property about $h^{-1}(0)$ for the triple $\{u^{-}(x), u^{\mathrm{EQ}}(x), u^{+}(x)\}$, with u^{EQ} defined as in (3.5), or (3.9). □

Proof

Suppose, that, thanks to a control action of the form (3.1), the system exhibits a local quasi-sliding regime about $h^{-1}(0)$. Then, from the hypothesis and according to Theorem 3.3, there necessarily exists a unique smooth $u^{\mathrm{EQ}}(x)$ satisfying $u^{-}(x) < u^{\mathrm{EQ}}(x) < u^{+}(x)$ on $h^{-1}(0)$. Since a quasi-sliding motion exists locally on

$h^{-1}(0)$, it follows that for $x \in h^{-1}(0)$: $h \circ F(x, u^-(x)) > 0$, $h \circ F(x, u^{EQ}(x)) = 0$ and $h \circ F(x, u^+(x)) < 0$, i.e., $h \circ F(x, u^-(x)) > h \circ F(x, u^{EQ}(x)) > h \circ F(x, u^+(x))$ on $h^{-1}(0)$. Hence, (F, h) exhibits a control foliation property for the given triple $\{u^-(x), u^{EQ}(x), u^+(x)\}$.

Suppose, on the other hand, that the system (F, h) exhibits a control foliation property for the triple $\{u^-(x), u^{EQ}(x), u^+(x)\}$ such that $u^-(x) < u^{EQ}(x) < u^+(x)$. From the hypothesis, and Theorem 3.3, it follows that (1) $u^{EQ}(x)$ uniquely exists such that on $h^{-1}(0)$: $h \circ F(x, u^{EQ}(x)) = 0$ and (2) $h \circ F(x, u^-(x)) > h \circ F(x, u^{EQ}(x)) = 0 > h \circ F(x, u^+(x))$. Hence, necessarily, $h \circ F(x, u^+(x)) < 0$ and $h \circ F(x, u^-(x)) > 0$ holds true on $h^{-1}(0)$. It follows that conditions (3.2) are locally satisfied. Thus, a quasi-sliding regime exists about $h^{-1}(0)$. □

Example 3.5: *Bisection method*

Given a continuous scalar function $h(x)$ with an isolated root in the interval $(x_1(0), x_2(0))$, the bisection method (see Dahlquist and Björk 1974) generates, under certain conditions, a convergent sequence of intervals $I_k := (x_1(k), x_2(k)) \supset I_{k+1} = (x_1(k+1), x_2(k+1))$ which all contain the root of the equation $h(x) = 0$. The midpoint of the interval I_{k-1} is defined as $x_3(k)$ and it is taken as an estimate of the (isolated) root of $h(x)$ after a certain number of steps in the algorithm. The algorithm is started under the basic assumption that $h(x_1(0))h(x_2(0)) < 0$. By construction, the algorithm guarantees that from this initial time on the product $h(x_1(k))h_2(k)) < 0$ for all k, unless the process has already converged.

The bisection algorithm can be expressed as a variable structure, bilinear, discrete-time controlled dynamical system with non-linear output function $h(x)$ as

$$
\begin{aligned}
x_1(k+1) &= x_1(k) + u[x_3(k) - x_1(k)] \\
x_2(k+1) &= x_3(k) + u[x_2(k) - x_3(k)] \\
x_3(k+1) &= 0{\cdot}5[x_1(k) + x_3(k)] + u0{\cdot}5[x_2(k) - x_1(k)] \\
y(k) &= h(x_3(k))
\end{aligned}
$$

The control function u takes values in the discrete set $\{0, 1\}$ according to the following variable structure control law:

$$
u = \begin{cases} 1 & \text{if } y(k) < 0 \\ 0 & \text{if } y(k) > 0 \end{cases}
$$

The local convergence properties of the bisection method can be, evidently, assessed from the convergence condition of a quasi-sliding mode taking place about the non-linear surface represented by $y = 0$.

The system is evidently relative degree 1. Otherwise $h(x_3(k+1))$ would be independent of u. But this would mean that $h(0{\cdot}5[x_1(k) + x_3(k)]) = h(0{\cdot}5[x_2(k) + x_3(k)])$ and then, either $x_1(k) = x_2(k)$, in which case the algorithm has already converged, or else the basic property $h(x_1(k))h(x_2(k)) < 0$ has been violated, which is a contradiction.

Under the ideal sliding conditions one would have convergence of the algorithm to the root being sought

$$
y(k+1) = h(x_3(k+1)) = h(x_3(k)) = y(k) = 0
$$

which evidently occurs when $x_3(k+1) = x_3(k)$, i.e., $x_1(k) = x_2(k) = x_3(k)$. From the invariance condition, the equivalent control is obtained as the following, apparently indeterminate, expression

$$u^{EQ}(x) = \frac{x_1(k) - x_3(k)}{x_2(k) - x_1(k)}$$

However, by virtue of the definition of $x_3(k)$ as the midpoint of the interval $(x_1(k), x_2(k))$, it follows that $u^{EQ}(x) = 0{\cdot}5$, for all k, i.e., $0 < u^{EQ}(x) = 0{\cdot}5 < 1$. The control foliation property is evidently satisfied around a given root of $h(x)$ for the triple $\{0, 0{\cdot}5, 1\}$ and hence a quasi-sliding regime is induced by the switching policy imposed on u. Using the convergence condition for the underlying quasi-sliding regime, one concludes that the algorithm locally converges toward $y = 0$ provided one has for all k

$$|h[0{\cdot}5(x_1(k) + x_3(k))]| < |h(x_3(k))| \quad \text{if } h(x_3(k)) > 0$$

or

$$|h[0{\cdot}5(x_2(k) + x_3(k))]| < |h(x_3(k))| \quad \text{if } h(x_3(k)) < 0$$

Such convergence conditions are evidently satisfied whenever the function $h(x)$ is locally strictly increasing or locally strictly decreasing around the isolated root. A simulation for the case in which $h(x) = (x/2)^2 - \sin(x)$ is shown in Fig. 4. □

Example 3.6

Consider the system (Grizzle 1985 a)

$$x_1(k+1) = [x_1^2(k) + x_2^2(k) + u(k)] \cos(x_2(x))$$

$$x_2(k+1) = [x_1^2(k) + x_2^2(k) + u(k)] \sin(x_2(k))$$

$$y(k) = h(x_1(k), x_2(k)) = x_1^2(k) + x_2^2(k) - R^2$$

The set $h^{-1}(0)$ is constituted by a circle of radius R with its centre at the origin. The system is relative degree 1 with respect to the output since

$$y(k+1) = (y(k) + R^2 + u(k))^2 - R^2$$

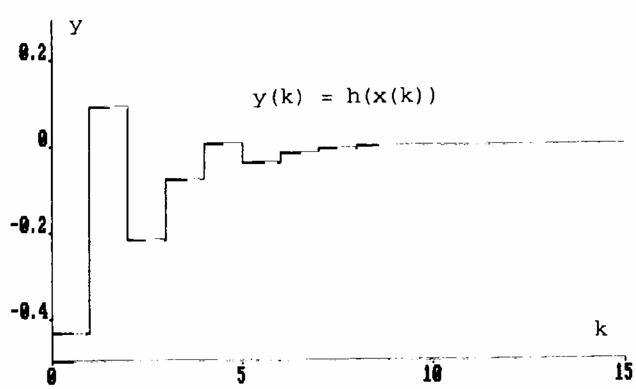

Figure 4. Quasi-sliding regime for the bisection algorithm.

From the invariance condition $y(k+1) = y(k) = 0$ one obtains the equivalent control function as the constant

$$u^{EQ}(x) = R(1 - R)$$

(the second solution: $u^{EQ}(x) = -R(1-R)$ can be discarded since it actually corresponds to an imaginary trajactory solution when replaced on the system described in polar coordinates). The corresponding ideal quasi-sliding regime is characterized by:

$$x_1(k+1) = R\cos(x_2(k)), \quad x_2(k+1) = R\sin(x_2(k))$$

$$y(k) = 0 \quad \text{for all } k.$$

It is easy to see that the system satisfies a control foliation property on $h^{-1}(0)$ for the triple of constant feedback control functions: $\{-R^2, R(1-R), R(\sqrt{2}-R)\}$. It follows from the previous theorems that the variable structure control policy, $u = R(\alpha(y) - R)$, with $\alpha(y) = \alpha^- = 0$ for $y > 0$ and $\alpha(y) = \alpha^+ = \sqrt{2}$ for $y < 0$, locally creates a quasi-sliding mode around the circle of radius R in the $x_1 - x_2$ plane.

A convergent quasi-sliding motion zigzagging about $y = 0$, and characterized by $y(k+1) = \beta y(k)$ can also be obtained with a continuous output feedback control function u of the form $u(k) = -y(k)R^2 + [R^2 - \beta y(k)]^{1/2}$ with $0 \in \beta < 1$. Figure 5 depicts a convergent quasi-sliding regime taking place on a circle of radius $R = 1$ and obtained by using $\beta = 0{\cdot}75$. □

3.2. *Sliding regimes in variable structure systems with relative degree higher than* 1

If, on an open and dense subset of **X**, the sytem exhibits relative degree r, higher than 1, for the proposed output function $y(k) = h(x(k))$, then an alternative to creating a convergent quasi-sliding motion, which eventually reaches the manifold

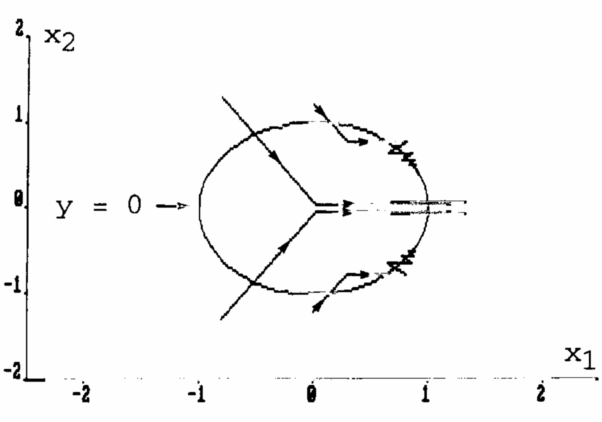

Figure 5. Convergent quasi-sliding regime on the circle.

$h^{-1}(0)$, is to use the auxiliary output function (see Isidori 1989 for the basic idea in local feedback stabilization) of the form

$$w(x(k)) = c_0 h(x(k)) + c_1 h \circ F(x(k)) + \dots + c_{r-2} h \circ F^{r-2}(x(k)) + h \circ F^{r-1} \quad (x(k)) \tag{3.11}$$

or, in normal form coordinates

$$w = z_r(k) + c_{r-2} z_{r-1}(k) + \dots + c_1 z_2(k) + c_0 z_1(k) \tag{3.12}$$

Evidently, $\partial[w \circ F(x(k), u(k))]/\partial u \neq 0$. That is, the system (F, w) has local relative degree 1, and a quasi-sliding motion can now be locally created on an open set of $w^{-1}(0/)$. Then, ideally, $w = 0$ and $z_r(k) = -c_{r-2} z_{r-1}(k) - \dots - c_1 z_2(k) - c_0 z_1(k)$. The ideal sliding system or zero dynamics, associated with the new quasi-sliding surface $w^{-1}(0)$, is expressed as

$$\begin{aligned} z_i(k+1) &= z_{i+1}(k), \quad i = 1, 2, \dots, r-2 \\ z_{r-1}(k+1) &= z_r(k) = -c_{r-2} z_{r-1}(k) - \dots - c_1 z_2(k) - c_0 z_1(k) \\ z_{r+j}(k+1) &= q(z_1(k), z_2(k), \dots, z_n(k)), \quad j = 1, \dots, n-r \\ y(k) &= z_1(k), \quad w(k) = 0 \end{aligned} \tag{3.13}$$

It is easy to see that by a suitable choice of the parameters $c_0, \dots, c_{r-2}$, an asymptotically stable motion can be obtained for the linear equation describing the first $r-1$ normal coordinates. Thus, if a convergent quasi-sliding motion locally takes place on the manifold $w^{-1}(0)$, the original output function y and its first $r-1$ differences asymptotically approach zero (i.e., the state vector of the system approaches the manifold $h^{-1}(0)$, as initially desired). The corresponding equivalent control is now given, in original coordinates, as the unique solution of $w \circ F(x, u^{EQ}(x)) = 0$.

The use of the auxiliary output $w(x(k))$ implies the possibility of either being able to measure completely the original state variables $x(k)$, and then proceed to use (3.11) as a quasi-sliding manifold, or else being able to generate $r-1$ delayed valued of the original output function $y(k)$. The last possibility is usually accomplished by means of a discrete-time high gain, phase lead, 'post-processor', fed by the output signal $y(k)$, similar in nature to the one proposed in Isidori (1989) for continuous systems.

Example 3.7

Consider the system of Example 3.6 with an output function now given by $y(k) = h(x(k)) = \tan^{-1}[x_2(k)/x_1(k)] - \theta$, where θ is a known constant. The manifold $h^{-1}(0)$ is represented now by a straight line, passing through the origin, forming an angle θ with respect to the x_1 axis. In polar coordinates, defined by:

$$\rho(k) = (x_1^2(k) + x_2^2(k))^{1/2}, \quad \varphi(k) = \tan^{-1}(x_2(k)/x_1(k))$$

the system is clearly seen to be relative degree 2 with respect to y

$$\begin{aligned} \rho(k+1) &= \rho^2(k) + u(k) \\ \varphi(k+1) &= \rho(k) \sin \varphi(k) \\ y(k) &= \varphi(k) - \theta \end{aligned}$$

In normal form coordinates $z_1(k) = \varphi(k) - \theta$; $z_2(k) = z_1(k+1) = \rho(k) \sin \varphi(k) - \theta$, the system is written as

$$z_1(k+1) = z_2(k)$$

$$z_2(k+1) = \left\{ \left[\frac{z_2(k) + \theta}{\sin (z_1(k) + \theta)} \right]^2 + u(k) \right\} \sin (z_2(k) + \theta) - \theta$$

$$y(k) = z_1(k)$$

We choose the auxiliary output function w as $w(z(k)) = z_2(k) - az_1(k)$, with $|a| < 1$. Since a convergent quasi-sliding regime can always be created on $w = 0$, ideally an asymptotically stable zero dynamics or ideal quasi-sliding regime, described by $z_1(k+1) = az_1(k)$, is obtained for the output coordinate. The trajectories corresponding to these ideal quasi-sliding dynamics, in turn, converge toward $z_1 = 0$, i.e., toward the line $y = \varphi - \theta = 0$ as originally desired.

With respect to the output function w, the system is now relative degree 1. Hence, a convergent quasi-sliding motion can be created on $w = 0$, with a continuous feedback control law. From the linearity in $u(k)$, the system trivially exhibits a control foliation property and, hence, a variable structure control law exists creating a quasi-sliding motion about $w = 0$. □

3.3. *Disturbance rejection properties of systems undergoing quasi-sliding regimes*

Consider a smooth non-linear perturbed system of the form:

$$x(k+1) = F(x(k), u(k), w(k)) \quad y(k) = h(x(k)) \tag{3.14}$$

where w is a scalar perturbation signal affecting the system behaviour. Let us assume that the system has relative degree 1 on $\mathbf{X}$, and suppose the input w is assumed to have relative degree higher than 1, with respect to the output function h. In normal form coordinates, the perturbed system is written as:

$$\left.\begin{aligned} z_1(k+1) &= h \circ F(\Phi^{-1}(z(k)), u(k), w(k)) = h \circ F(\Phi^{-1}(z(k)), u(k)) \\ \eta(k+1) &= q(z_1(k), \eta(k), u(k), w(k)), \quad y(k) = z_1(k) \end{aligned}\right\} \tag{3.15}$$

where $\eta(k) = (z_2(k), \ldots, z_n(k))$ and $z(k) = (z_1(k), \eta(k))$. If a convergent quasi-sliding motion can be created on the manifold $z_1 = 0$, the ideal motions on $h^{-1}(0)$, and the equivalent control $u^{\mathrm{EQ}}(z(k))$ itself, are clearly unaffected by the perturbation signal w, although the ideal sliding motion, or zero dynamics, taking place on $h^{-1}(0)$, may still be affected by this perturbation quantity as (3.15) clearly indicates. The following lemma follows immediately.

Lemma 3.2

Let (F, h) have relative degree 1. The dynamics of a convergent quasi-sliding motion on $h^{-1}(0)$, is independent of the perturbation signal $w(k)$ if and only if, with respect to the perturbation input w, the system with output function $y = h(x)$ has local relative degree at least, equal to 2. □

Theorem 3.5

The ideal quasi-sliding dynamics are totally unaffected by perturbation signals w, of any kind, if and only if the following matching condition is satisfied.

$$\partial F/\partial w \in \text{range}\{\partial F/\partial u\} \tag{3.16}$$

□

Proof

Let the system have relative degree 1. Choose the first coordinate $z_1 = \phi_1(x) = h(x)$. Notice that one may always choose the remaining coordinates $\eta = (\phi_2, \ldots, \phi_n) = (z_2, \ldots, z_n)$ in such a way that $\eta(k+1) = q(z_1(k), \eta(k), u(k), w(k))$ is independent of $u(k)$. For this it suffices that $\partial(\phi_j \circ F)(x, u, w)/\partial u = [\partial(\phi_j \circ F)(x, u, w)/\partial F]\,\partial F/\partial u = 0$, $j = 2, \ldots, n$. That is, one chooses ϕ_j such that, locally for $u \in \mathbf{U}$, $\partial F/\partial u \in \cap_j \ker[\partial(\phi_j \circ F)(x, u, w)/\partial F]$. Since the ϕ_js are chosen as independent, the above intersection of kernels is clearly one-dimensional and it is, hence, necessarily spanned by the vector $\partial F/\partial u$. Suppose now, in order to prove necessity, that $q(\xi(k), \eta(k), u(k), w(k))$ is independent of $w(k)$. It follows, similarly, that $\partial(\phi_j \circ F)(x, u, w)/\partial w = [\partial(\phi_j \circ F)(x, u, w)/\partial F]\,\partial F/\partial w = 0$. That is, $\partial F/\partial w \in \cap_j \ker[\partial(\phi_j \circ F)(x, u, w)/\partial F] = \text{span}\{\partial F/\partial u\}$. Sufficiency is obvious. □

4. Conclusions and suggestions for further research

In this article, the relevance of the relative degree concept has been examined in the analysis, along with the design issues related to the creation of quasi-sliding regimes and convergent quasi-sliding regimes, for general non-linear discrete-time systems. The results indicate that the simplest possible structure at infinity must be exhibited by non-linear systems undergoind sliding motions on the zero level set of the output function. General necessary, as well as necessary and sufficient, conditions for the existence of quasi-sliding regimes have been presented. The disturbance rejection properties of sliding mode control were also examined and a generalization of the well-known 'matching condition' was found. The results need to be generalized to non-linear discrete multivariable systems. The implications of formulating the above problems in the fibre space approach of Grizzle (1985 b), and the recently introduced difference algebraic approach of Fliess (1990), remain to be explored.

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