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Synthesis of Sliding-Mode Controllers for Nonlinear Systems via Extended Linearization

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Abstract. In this article, a general synthesis method is proposed for the design of discontinuous feedback strategies leading to asymptotically stabilizing sliding regimes. The method is applicable to the class of nonlinear dynamical systems possessing constant equilibrium points. A family of nonlinear stabilizing sliding manifolds, parametrized by generic desired equilibrium point, is specified on the basis of the extended linearization approach. Some examples including simulations are presented for illustrative purposes.

1. Introduction

In this article, a new method is proposed for the synthesis of stabilizing sliding modes (Utkin [1]) in nonlinear controlled dynamical systems. The nonlinear sliding-surface design method is based, entirely, on the *extended linearization* approach for nonlinear systems, developed by Rugh and his co-workers [2–5]. We propose to specify a nonlinear sliding-mode controller by first resorting to parametrized linearization of the given nonlinear system about a general constant equilibrium point. Using well-known results (Utkin [1]; Itkis [6]), a standard stabilizing sliding-hyperplane design is then carried out on the basis of the parametrized family of linear systems, possibly transformed to *controllable canonical form*. The *ideal sliding dynamics*, corresponding to the linear design, is purposefully characterized by a set of stable eigenvalues that are independent of the constant operating point. A suitable *extension* of the sliding-hyperplane design yields a nonlinear switching manifold that is tangent to the prescribed hyperplane. The designed manifold contains the equilibrium point and is parametrizable in terms of the nominal operating conditions. Moreover, the corresponding ideal sliding dynamics can always be made locally linear (possibly, modulo a suitable local diffeomorphic state-coordinate transformation derivable from the linearized system). The nonlinear sliding manifold is nonuniquely obtained by standard "unlinearization" schemes carried out by direct integration of the synthesized sliding hyperplane. The nonlinear sliding-mode switching logic is synthesized on the basis of the obtained nonlinear *sliding-surface coordinate function* and the corresponding nonlinear *equivalent control.*

An important property of the proposed sliding-mode controller, aside from those already mentioned, lies in the fact that if a sudden change of the nominal operating conditions takes place, the control scheme exhibits *self-scheduling properties* by means of which a sliding regime is automatically formed that stabilizes the system trajectories to the new equilibrium point. This last property is clearly inherited from the well-known merits of the extended linearization technique, and it makes the "scheduling" process of the sliding manifold and of the switching "gains" totally unnecessary.

In this article, only single-input nonlinear systems are treated. The multi-input case will be presented elsewhere. Section 2 of this article presents a general procedure for synthesizing stable nonlinear sliding manifolds for single-input nonlinear systems via extended linearization. Section 3 presents several illustrative examples—some of them of a physical nature—accompanied by simulation experiments. The conclusions and suggestions for further research are given in section 4.

2. A synthesis procedure for sliding mode controllers via extended linearization

2.1. Problem formulation

Consider the n-dimensional nonlinear dynamical system:

$$\frac{dx(t)}{dt} = f(x(t), u(t)) \tag{1}$$

where $f(\cdot,\cdot) : R^n \times R \to R^n$ is a continuously differentiable function of its arguments. The controlled system (1) is assumed to have a continuous family of constant state-equilibrium points, $X(U)$, corresponding to nonzero constant inputs, $u = U$. In other words, $f(X(U), U) = 0$. The pair $[\partial f/\partial x(X(U), U), \partial f/\partial u(X(U), U)]$ is assumed to be *controllable*, where $\partial f/\partial x(X(U), U) \in R^{n\times n}$ and $\partial f/\partial u(X(U), U) \in R^{n\times 1}$ stand for the Jacobian matrices of $f(x, u)$ with respect to x and u, respectively.

It is desired to maintain locally, in a stable fashion, the trajectories of the nonlinear system (1) at the constant nominal equilibrium trajectory, $X(U)$, by means of a sliding motion suitably induced on a manifold S that contains such an equilibrium point. In other words, it is required to synthesize the following:

1. A nonlinear sliding surface S, parametrized by the nominal control input U, of the form

$$S = \{x \in R^n, s(x, U) = 0\}, \tag{2}$$

   where, for each fixed U, $s(.,U) : R^n \to R$, is a smooth function satisfying $s(X(U), U) = 0$;
2. An associated variable structure control law

$$u(x, U) = \begin{cases} u^+(x, U) & \text{for } s(x, U) > 0 \\ u^-(x, U) & \text{for } s(x, U) < 0 \end{cases} \tag{3}$$

that automatically forces every small state deviation, from the nominal operating conditions, to zero, via the local creation of a stable sliding regime taking place on S and leading the state trajectory to $X(U)$.

In order to specify such a parametrized sliding manifold, we propose to resort to the method of *extended linearization* (see [2–4]) as indicated in the following paragraphs. For the connections of this technique with the closely related method of pseudolinearization, the reader is referred to [5].

2.2. *A nonlinear sliding-mode controller design based on extended linearization*

1. Linearize the dynamical system about each point in the family of constant operating trajectories, $[U, X(U)]$, obtaining the following parametrized family of linear systems:

$$\dot{x}_\delta = A(U)x_\delta + b(U)u_\delta, \tag{4}$$

where, for fixed U, the input and state perturbation variables are defined, respectively, as: $u_\delta = u(t) - U$, $x_\delta(t) = x(t) - x(U)$, while the $n \times n$ matrix $A(U)$ and the n-vector $b(U)$ are defined as

$$A(U) := \frac{\partial f}{\partial x}(X(U), U)\ ; \qquad b(U) := \frac{\partial f}{\partial u}(X(U), U). \tag{5}$$

Since the pair $[(A(U), b(U)]$ is assumed to be controllable, a similarity transformation exists of the form

$$z_\delta = P(U)x_\delta =: [p_1(U), p_2(U), \ldots, p_n(U)]x_\delta \tag{6}$$

such that system (4) may be represented as a *controllable canoncial realization*. The nonsingular matrix $P(U)$ is obtained from the well-known expression

$$P^{-1}(U) = [b(U), A(U)b(U), \ldots, A^{n-1}(U)b(U)] \begin{bmatrix} \alpha_1(U) & \alpha_2(U) & \ldots & 1 \\ \alpha_2(U) & \alpha_3(U) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1}(U) & 1 & \ldots & \vdots \\ 1 & 0 & \ldots & 0 \end{bmatrix} \tag{7}$$

where $\det[\lambda I\text{-}A(U)] = \lambda^n + \alpha_{n-1}(U)\lambda^{n-1} + \alpha_{n-2}(U)\lambda^{n-2} + \ldots + \alpha_0(U)$.

2. Obtain the transformed system in *controllable canonical form* as

$$\begin{aligned} \dot{z}_{1\delta} &= z_{2\delta} \\ \dot{z}_{2\delta} &= z_{3\delta} \\ &\vdots \\ \dot{z}_{(n-1)\delta} &= z_{n\delta} \\ \dot{z}_{n\delta} &= -\alpha_{n-1}(U)z_{n\delta} - \alpha_{n-2}(U)z_{(n-1)\delta} - \ldots -\alpha_0(U)z_{1\delta} + u_\delta. \end{aligned} \tag{8}$$

3. Use as a *sliding surface* the linear manifold

$$\Sigma_\delta = \{z_\delta \in R^n : \sigma_\delta(z_\delta) = \sum_{i=1}^{n} c_i z_{i\delta} = c^T z_\delta = 0;\ c_n = 1\}, \tag{9}$$

and choose the coefficients c_i, independently of the operating point $(X(U), U)]$, such that the roots of the *characteristic polynomial*

$$\sum_{i=1}^{n} c_i \lambda^{i-1} = 0 \tag{10}$$

for the (reduced) *linear ideal sliding dynamics* are specified at convenient locations in the open left half of the complex plane—i.e., so that the autonomous ideal sliding-mode dynamical system

$$\begin{aligned} \dot{z}_{1\delta} &= z_{2\delta} \\ \dot{z}_{2\delta} &= z_{3\delta} \\ &\vdots \\ \dot{z}_{(n-1)\delta} &= -c_{n-1}\, z_{(n-1)\delta} - c_{n-2} z_{(n-2)\delta} - \dots - c_1 z_{1\delta} \end{aligned} \tag{11}$$

is asymptotically stable toward the origin of transformed coordinates.

4. Obtain, on the basis of the previously described design steps, the parametrized sliding-hyperplane specification in terms of the original perturbed state coordinates x_δ, as follows:

$$S_\delta = \{x_\delta \in R^n : S_\delta(x_\delta, U) := \sigma_\delta(P(U)x_\delta) = c^T P(U)x_\delta = 0\}. \tag{12}$$

5. Obtain a nonlinear sliding manifold S such that its corresponding linearization about the operating point $[X(U), U]$ yields back the sliding hyperplane (12). In other words, find a nonlinear switching surface that is tangent to the sliding hyperplane (12) at the equilibrium point.

5a. *Sliding manifold.* We must, thus, find a nonlinear sliding-surface coordinate function $s(x, U)$, parametrized by the constant operating point U, such that the following relations are satisfied:

$$\left.\frac{\partial s(x, U)}{\partial x}\right|_{x = X(U)} = c^T P(U) = [c^T p_1(U), c^T p_2(U), \dots, c^T p_n(U)] \tag{13}$$

or, componentwise,

$$\left.\frac{\partial s(x, U)}{\partial x_i}\right|_{x = X(U)} = c^T p_i(U);\ i = 2, \dots, n, \tag{14}$$

with the additional (boundary) condition $s(X(U), U) = 0$.

Remark. In general, there are many parametrized sliding-surface coordinate functions, $s(x, U)$, that satisfy relations (14) and the boundary condition. Such a lack of uniqueness of solution may not be totally inconvenient, and it is a prevailing characteristic of the method of extended linearization. ■

We present below two alternative procedures for obtaining the required nonlinear sliding surface on the basis of the designed linear switching manifold.

1. Assume that $\partial X_i(U)/\partial U \neq 0$ for each i, i.e., according to the implicit function theorem, the ith component $X_i(U)$ of the vector $X(U)$ is invertible. In other words, there locally exists a unique solution, $X_i^{-1}(x_i)$, for U in each of the equations: $x_i = X_i(U)$. It then follows that the relations in equation (14) may be viewed as a particularization, on the equilibrium point, of the following relations:

$$\frac{\partial s(x, U)}{\partial x_i} = c^T p_i[X_i^{-1}(x_i)]; \; i = 1, \ldots, n. \tag{15}$$

These relations, indeed, define a trivially solvable set of first-order partial differential equations, to be satisfied by $s(x, U)$, with a given boundary condition. The solution of such a system may be obtained by *direct integration* (see Arnold [7], pp. 65–68) as follows.

$$s(x, U) = \sum_{j=1}^{n} \int_U^{X_j^{-1}(x_j)} c^T p_j[\vartheta] \frac{dX_j(\vartheta)}{d\vartheta} \, d\vartheta. \tag{16}$$

The validity of this integration procedure is immediately verified upon differentiation of equation (16) with respect to the ith component of the state vector. Indeed, such a differentiation yields

$$\frac{\partial s(x, U)}{\partial x_i} = c^T p_i[X_i^{-1}(x_i)] \frac{dX_i(X_i^{-1}(x_i))}{dX_i^{-1}(x_i)} \frac{dX_i^{-1}(x_i)}{dx_i} = c^T p_i[X_i^{-1}(x_i)].$$

This expression coincides, for each i, with relations (15) and yields back relations (14) when particularized at the equilibrium point $x_i(U) = X_i(U)$.

2. Alternatively, if not all the components of the equilbrium vector $X(U)$ are invertible, there will be, generally speaking, at least one (say the first one) that is invertible indeed. One can then resort to a similar integration procedure, which will be justified in the same manner as in the previous case, and obtain an integration formula for the sliding surface. Such an explicit *integration formula* is inspired by the results in Rugh [2] and allows one to obtain a nonlinear sliding manifold in a rather systematic manner:

$$S = \{x \in R^n : s(x, U) = \int_U^{X_1^{-1}(x_1)} c^T P(\vartheta) \frac{dX(\vartheta)}{d\vartheta} d\vartheta + \sum_{j=2}^{n} c^T p_j(X_1^{-1}(x_1))[x_j - X_j(X_1^{-1}x_1))] = 0\}. \tag{17}$$

It can be verified, after partial differentiation with respect to the components of the vector x, and substitution of the equilibrium point $(U, X(U))$, that the manifold defined in equation (17) represents one possible solution for the required parametrized nonlinear sliding manifold that satisfies relations (14).

Indeed, it should be evident from equation (17) that $s(X(U), U) = 0$. (For this, notice that $X_1^{-1}(x_1(U)) = X_1^{-1}(X_1(U)) = U$, and also $x_j(U) = X_j(U)$. Hence, the integral term and every term in the sum in equation (17) vanish when evaluated at the equilibrium point.) Differentiating $s(x, U)$ with respect to x_i, $(i = 2, \ldots, n)$, one obtains

$$\frac{\partial s(x, U)}{\partial x_i} = c^T p_j(X_1^{-1}(x_1)); \; i = 2, \ldots, n$$

which, particularized on the equilibrium point, satisfies the last of the n-1 relations (14).

Differentiating now equation (17) with respect to x_1 yields

$$\begin{aligned}
\frac{\partial s(x, U)}{\partial x_1} &= \sum_{j=1}^{n} c^T p_j(X_1^{-1}(x_1)) \frac{dX_j(X_1^{-1}(x_1))}{dX_1^{-1}(x_1)} \frac{dX_1^{-1}(x_1)}{dx_1} \\
&\quad + \sum_{j=2}^{n} c^T \frac{dp_j(X_1^{-1}(x_1))}{dx_1} [x_j - X_j(X_1^{-1}(x_1))] \\
&\quad - \sum_{j=2}^{n} c^T p_j(X_1^{-1}(x_1) \frac{dX_j(X_1^{-1}(x_1))}{dx_1} \\
&= c^T p_1(X_1^{-1}(x_1)) \frac{dX_1(X_1^{-1}(x_1))}{dx_1} + \sum_{j=2}^{n} c^T \frac{dp_j(X_1^{-1}(x_1))}{dx_1} [x_j - X_j(X_1^{-1}(x_1))] \\
&= c^T p_1(X_1^{-1}(x_1)) + \sum_{j=2}^{n} c^T \frac{dp_j(X_1^{-1}(x_1))}{dx_1} [x_j - X_j(X_1^{-1}(x_1))],
\end{aligned}$$

which satisfies the relation in equation (14) for $i = 1$, in view of the fact that every term in the sum vanishes when particularized at the equilibrium point.

It should be remarked, however, that in obtaining a suitable nonlinear sliding surface by any integration procedure carried out on the linearized surface design, one should always be led by the idea of obtaining a manifold where the constrained dynamics exhibits the property of linearity and asymptotic stability as the main requirements. The second property is always guaranteed by the imposed nature of the linear design; however, the first property is not always trivially achievable except in special cases (see the examples in section 3).

5b. *Equivalent control and ideal sliding dynamics.* The smooth feedback control obtained as the solution of the equation

$$\frac{ds(x, U)}{dt} = \frac{\partial s(x, U)}{\partial x} f(x, u^{EQ}(x, U)) = 0 \tag{18}$$

plays a fundamental role in the theoretical developments of the sliding-mode control technique. Such a feedback control, usually known as the *equivalent control*, is here denoted by $u^{EQ}(x, U)$.

By virtue of the implicit function theorem, local existence and uniqueness of the equivalent control is guaranteed under the assumption that the following relation is locally verified:

$$\frac{\partial}{\partial u}\left[\frac{\partial s(x, U)}{\partial x} f(x, u)\right] = \frac{\partial s(x, U)}{\partial x}\frac{\partial f(x, u)}{\partial u} \neq 0. \tag{19}$$

The equivalent control $u^{EQ}(x, U)$ generates state trajectories that locally satisfy $s(x, U) = constant$. When the equivalent control is used on the system with initial conditions precisely located on the sliding manifold $s(x, U) = 0$, the resulting state trajectories leave the switching manifold locally invariant, i.e., motions stay locally constrained to the switching manifold. The dynamics obeyed on the manifold, under such idealized control actions, is known as the *ideal sliding dyanmics.* The ideal sliding motions are thus obtained from the well-known *invariance conditions,* which make the switching manifold a *local integral manifold* of the smoothly controlled system:

$$s(x, U) = 0, \qquad \frac{d}{dt} s(x, U) = 0. \tag{20}$$

Once the nonlinear sliding-surface coordinate function $s(x, U)$ is known, computation of the switching strategy is carried out on the basis of the equivalent control.

5c. *Sliding-Mode Switching Logic.* A nonlinear sliding-mode switching strategy is usually synthesized such the sliding-mode existence (Utkin [1]) are satsified, at least in a local fashion. Such well-known conditions are given by

$$\lim_{s\to 0^+} \frac{ds(x, U)}{dt} < 0; \qquad \lim_{s\to 0^-} \frac{ds(x, U)}{dt} > 0. \tag{21}$$

It has been shown that, for nonlinear systems that are linear in the scalar control input, a necessary and sufficient condition for the local existence of a sliding mode is that the equivalent control locally exhibits values that are *intermediate* between the extreme values of feedback laws among which the switching take place (i.e., $u^+(x, U) < u^{EQ}(x, U) < u^-(x, U)$). The region of existence, on the sliding surface, of such a sliding regime coincides precisely with the region where such an *intermediacy* condition is satisfied by the equivalent control. One may therefore synthesize the nonlinear sliding-mode switching logic for such a large class of nonlinear systems from knowledge of the equivalent control function, $u^{EQ}(x, U)$, and of the sliding-manifold coordinate function, $s(x, U)$ (see Sira-Ramirez [8]), as follows:

$$u(x, U) = -k \mid u^{EQ}(x, U) \mid \text{sgn}\, s(x, U); \mid k \mid > 1, \tag{22}$$

where the sign of the constant k is locally chosen so as to satisfy the ideal sliding-mode conditions (21) (see examples 1 and 3 in section 3).

In more general cases, where there is no special input structure to the system such as linearity, the above switching logic (22)—or any other satisfying the equivalent control intermediacy condition—may still locally create a sliding regime, provided that the system exhibits a *control foliation property* through its input channel (see Sira-Ramirez [9, 10]).

An alternative discontinuous feedback-control strategy, possibly yielding weaker discontinuities in the controls, is represented (see Slotine and Li [11], Chapter 7, and also Dwyer and Sira-Ramirez [12]) by

$$u(x, U) = u^{EQ}(x, U) - \kappa\, \text{sgn}\, s(x, U), \tag{23}$$

where the sign of the constant κ is appropriately chosen to guarantee locally the validity of the sliding-mode conditions (20) (see sections 3.1 and 3.2).

For the class of application examples presented in the next section, switching strategies of the form (22) or (23) suffice for the local creation of a sliding regime.

3. Some application examples

In this section, we present some illustrative examples of sliding-mode control synthesis for nonlinear plants, using the method of extended linearization. We begin with a somewhat general second-order example, in which the synthesized sliding surface is seen to entirely coincide with the intuitive solution that one would propose, in general, for achieving a linear ideal sliding dynamics. The proposed sliding-mode design process therefore appears as a natural synthesis procedure. The rest of the examples in this section represent simple applications of a physical nature.

3.1. A general second-order example

Consider the nonlinear controlled system, defined in R^2, expressed in *regular canonical form* (see Luk'yanov and Utkin [13]):

$$\dot{x}_1 = \varphi(x_1, x_2)$$
$$\dot{x}_2 = \gamma(x_1, x_2) + g(x_1, x_2)u. \tag{24}$$

We assume the existence of a contimuum of constant equilibrium points, parametrized by the corresponding constant value U of the control input u (parametrization with respect to equilibrium values of the state variables is also possible in cases where $U = 0$; see [2] and section 3.2 below):

$$u = U;\ x_1(U) = X_1(U); \qquad x_2(U) = X_2(U) \tag{25}$$

such that $\partial\varphi/\partial x_2(X_1(U), X_2(U)) \neq 0$ and $g(X_1(U), X_2(U)) \neq 0$.

3.1.1. Design of the stabilizing switching line for the family of linearized systems.

Linearization of system (24) about an equilibrium point of the form (25) yields

$$\dot{x}_{1\delta} = \varphi_1(X_1(U, X_2(U))x_{1\delta} + \varphi_2(X_1(U), X_2(U))x_{2\delta}$$
$$\dot{x}_{2\delta} = [\gamma_1(X_1(U), X_2(U)) + g_1(X_1(U), X_2(U))U]x_{1\delta}$$
$$+ (\gamma_2(X_1(U), X_2(U)) + g_2(X_1(U), X_2(U))U]x_{2\delta} + g(X_1(U), X_2(U))u_\delta, \tag{26}$$

where $\varphi_i := \partial\varphi/\partial x_i$; $\gamma_i := \partial\gamma/\partial x_i$; $g_i := \partial g/\partial x_i$; $i = 1, 2$.

We briefly express such a linearized system by

$$\dot{x}_{1\delta} = \varphi_1 x_{1\delta} + \varphi_2 x_{2\delta}$$
$$\dot{x}_{2\delta} = [\gamma_1 + g_1 U]\, x_{1\delta} + [\gamma_2 + g_2 U]\, x_{2\delta} + g u_\delta. \tag{27}$$

As can be easily seen, the linearized system (27) may be placed in *controllable canonical form* by means of the following similarity transformation, parametrized by the equilibrium point:

$$z_{1\delta} = \frac{1}{g\varphi_2} \cdot x_{1\delta}$$
$$z_{2\delta} = \frac{1}{g\varphi_2} \cdot [\varphi_1 x_{1\delta} + \varphi_2 x_{2\delta}]. \tag{28}$$

The previous assumption about the nonvanishing of g and φ_2 at the equilibrium point $[X_1(U), X_2(U)]$ locally guarantees the nonsingularity of such a linear coordinate transformation. Evidently, a sliding line rendering an asymptotically stable *ideal sliding dynamics* for the transformed system is given by $\sigma_\delta(z_\delta) = c_1 z_{1\delta} + z_{2\delta} = [c_1 \ 1]^T z_\delta =: c^T z_\delta$. Using form (28), the sliding-line equation in original coordinates is obtained after multiplication by the nonzero factor $g\varphi_2$ as

$$S_\delta = \{x_\delta \in R^2 : s_\delta(x_\delta, U) = (\varphi_1 + c_1)x_{1\delta} + \varphi_2 x_{2\delta} = 0;\ c_1 > 0\}. \tag{29}$$

Indeed, if the system is ideally maintained on such a sliding line, the resulting dynamics (known as the *ideal sliding dynamics*) is simply governed, according to system (27), by

$$\dot{x}_{1\delta} = -c_1 x_{1\delta};\ c_1 > 0 \tag{30}$$

which is asymptotically stable to zero and independent of the operating point.

3.1.2. Synthesis of the sliding-mode controller for the nonlinear system. In general, the key idea behind the method of extended linearization for obtaining a nonlinear controller design, once a linear feedback stabilizing controller has been properly synthesized for any member of the parametrized family of linear systems, consists in finding a *nonlinear regulator* that when linearized about the nominal operating trajectory yields back the designed linear regulator specified for the family of linearized systems.

In nonlinear sliding-mode design, the method of extended linearization consists in specifying a (nonlinear) sliding manifold, and its associated equivalent control, on the basis of the linearized surface design. This manifold must be such that when it is linearized about the constant equilibrium point, it yields back the designed stabilizing sliding hyperplane corresponding to the linearized system. The linearization of the corresponding nonlinear equivalent control about the operating point yields the linear equivalent control previously obtained.

Expressions (14) yield, in this case, the following conditions:

$$\left.\frac{\partial s(x, U)}{\partial x_1}\right|_{\substack{x_1 = X_1(U)\\ x_2 = X_2(U)}} c_1 + \varphi_1(X_1(U), X_2(U));\ \left.\frac{\partial s(x, U)}{\partial x_2}\right|_{\substack{x_1 = X_1(U)\\ x_2 = X_2(U)}} = \varphi_2(X_1(U), X_2(U)).$$

In is easy to verify, from the definition of φ_1 and φ_2, that the following nonlinear sliding manifold S is such that its surface coordinate function $s(x_1, x_2, U)$ satisfies the above conditions:

$$S = \{x \in R_2 : s(x_1, x_2, U) = \varphi(x_1, x_2) + c_1(x_1 - X_1(U)) = 0;\ c_1 > 0\}. \tag{31}$$

In view of $\varphi(x_1(U), x_2(U)) = \varphi(X_1(U)), X_2(U)) = 0$, it can also be immediately verified that $s(X(U), U) = 0$ as required, ie., S contains the equilibrium point.

The ideal sliding dynamics corresponding to the manifold (31), according to the first equation in system (24), is clearly given by the linear system

$$\dot{x}_1 = -c_1(x_1 - X_1(U)); \; c_1 > 0, \tag{32}$$

which represents an asymptotically stable linear dynamics whose solution converges toward the first component of the equilibrium point. Since φ_2 is nonzero at the equilibrium point, the implicit function theorem guarantees that the local isolated solution for x_2 of $s(X_1(U), x_2, U) = \varphi(X_1(U), x_2(U) = 0$ exists and is unique. Thus, the solution for x_2 coincides precisely with $X_2(U)$.

The equivalent control, obtained from the condition $ds/dt = 0$, is given by

$$u^{EQ}(x, U) = -\frac{1}{g(x_1, x_2)\varphi_2(x_1, x_2)} \{[\varphi_1(x_1, x_2) + c_1]\, \varphi(x_1, x_2) + \varphi_2(x_1, x_2)\gamma(x_1, x_2)\}. \tag{33}$$

For the control function u, a switching strategy that locally accomplishes sliding-mode existence for the discontinuously controlled system is given by

$$u = -k \mid u^{EQ}(x, U) \mid \operatorname{sgn} s(x, U); \; \mid k \mid > 1 \tag{34}$$

where the sign of k locally coincides with that of $\varphi_2 g$, i.e.,

$$\operatorname{sgn}(k) = \operatorname{sgn}(g\varphi_2). \tag{35}$$

It is easy to verify, in this case, that equation (34) leads to a sliding regime on $s(x, U) = 0$. Computing the time derivative of $s(x, U)$, substituting into the resulting expression the control law given by equation (34), and letting $k = \mid k \mid \operatorname{sign}(\varphi_2 g)$, one obtains:

$$\frac{ds(x, U)}{dt} = |(\varphi_1 + c_1)\varphi + \varphi_2\gamma \mid \{\operatorname{sgn}[(\varphi_1 + c_1)\varphi + \varphi_2\gamma] - \mid k \mid \operatorname{sgn}[s(x, U)]\}.$$

from which it should be evident that $ds(x, U)/dt < 0$ for $s(x, U) > 0$ and $ds(x, U)/dt > 0$ for $s(x, U) < 0$. Hence, in the vicinity of the switching surface, conditions (21) hold valid.

Alternatively, the switching logic specified in equation (23),

$$u(x, U) = u^{EQ}(x, U) - \kappa \operatorname{sgn} (s(x, U), \tag{36}$$

also locally generates a sliding regime on $s(x, U) = 0$, provided sign (κ) = sign $(\varphi_2 g)$— i.e., here also $\kappa = \mid \kappa \mid$ sign $(\varphi_2 g)$. Proceeding in the same manner as described above, one obtains

$$\frac{ds(x, U)}{dt} = -\mid \kappa \mid \mid \varphi_{2g} \mid \operatorname{sgn} s(x, U),$$

which evidently satisfies the sliding mode conditions (21).

3.2. State-scheduled sliding-mode controlled reorientation maneuvers for a single-axis spacecraft

Consider the kinematic and dynamic model of a single-axis externally controlled spacecraft whose orientation is given in terms of the Cayley–Rodrigues representation of the attitude parameter, denoted by x_1 (see Dwyer and Sira-Ramírez [12]). The angular velocity is represented by x_2, while J stands for the moment of inertia and u is the applied external torque:

$$\frac{dx_1}{dt} = 0.5\,(1 + x_1^2)x_2; \qquad \frac{dx_2}{dt} = \frac{1}{J}\,u. \tag{37}$$

Given arbitrary initial conditions, a slewing maneuver is required that brings the attitude parameter to a final desired value X_1 and the angular velocity to a rest equilibrium. We summarize below the design steps leading to a nonlinar sliding surface where the ideal sliding dynamics is linear and asymptotically stable toward the desired equilibrium point: $x_1 = X_1$, $x_2 = 0$, $u = 0$.

3.2.1. Family of linearizations parametrized by constant equilibrium point

$$\begin{bmatrix} \dfrac{dx_{1\delta}}{dt} \\ \dfrac{dx_{2\delta}}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 0.5\,(1 + X_1^2) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1\delta} \\ x_{2\delta} \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{J} \end{bmatrix} u_\delta \tag{38}$$

with $x_{1\delta} = x_1 - X_1$, $x_{2\delta} = x_2 - 0$, $u_\delta = u - 0$.

3.2.2. State-coordinate transformation to controllable canonical form

$$\begin{bmatrix} z_{1\delta} \\ z_{2\delta} \end{bmatrix} = \begin{bmatrix} \dfrac{2J}{(1 + X_1^2)} & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} x_{1\delta} \\ x_{2\delta} \end{bmatrix} \tag{39}$$

$$\begin{bmatrix} \dfrac{dz_{1\delta}}{dt} \\ \dfrac{dz_{2\delta}}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_{1\delta} \\ z_{2\delta} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_\delta \tag{40}$$

3.2.3. Linear sliding surface and ideal sliding dynamics in transformed and original coordinates

In transformed coordinates:

$$\sigma_\delta(z_\delta) = z_{2\delta} + c_1 z_{1\delta} = 0; \; c_1 > 0 \tag{41}$$

$$\dot{z}_{1\delta} = - c_1 z_{1\delta} \tag{42}$$

In original coordinates:

$$s_\delta(x_\delta) = c_1 x_{1\delta} + 0.5\,(1 + X_1^2)\, x_{2\delta} = 0 \tag{43}$$

$$\dot{x}_{1\delta} = -c_1 x_{1\delta} \tag{44}$$

3.2.4. Nonlinear sliding surface, ideal sliding dynamics, and nonlinear sliding-mode controller. The nonlinear parametrized sliding-surface coordinate function $s(x, X_1)$ must satisfy the following relations:

$$\left.\frac{\partial s(x, X_1)}{\partial x_1}\right|_{\substack{x_1 = X_1 \\ x_2 = 0}} = c_1; \qquad \left.\frac{\partial s(x, X_1)}{\partial x_2}\right|_{\substack{x_1 = X_1 \\ x_2 = 0}} = 0.5\,(1 + X_1^2) \tag{45}$$

with the condition $s([X_1, 0]^T, X_1) = 0$.

In this example, we illustrate in detail how to obtain the nonlinear sliding surface $s(x, X_1) = 0$, using the *direct integration method* described in section 2.

The relations (45) may be viewed as representing a particularization on the equilibrium point $[X_1, 0]$ of the more general relations

$$\frac{\partial s(x, X_1)}{\partial x_1} = c_1; \; \frac{\partial s(x, X_1)}{\partial x_2} = 0.5\,(1 + x_1^2), \tag{46}$$

with the boundary condition $s([X_1, 0], X_1) = 0$.

Integration of the second equation in relations (46) yields

$$s(x, X_1) = 0.5\,(1 + x_1^2)\, x_2 + \zeta(x_1, X_1), \tag{47}$$

where $\zeta(x_1, X_1)$ is an arbitrary function of x_1 that must satisfy $\zeta(X_1, X_1) = 0$, in accordance with the boundary condition imposed on $s(x, X_1)$.

Substituting the solution (47) on thefirst equation in relations (46), one obtains now an ordinary differential equation for the unknown function $\zeta(x_1, X_1)$:

$$\frac{\partial s(x, X_1)}{\partial x_1} = \frac{d\zeta(x_1, X_1)}{dx_1} = c_1,$$

which evidently has as a solution $\zeta(x_1, X_1) = c_1(x_1 - X_1)$. This solution clearly satisfies the condition $\zeta(X_1, X_1) = 0$.

The sliding-surface coordinate function is therefore given by

$$s(x, X_1) = c_1(x_1 - X_1) + 0.5\,(1 + x_1^2)\,x_2. \tag{48}$$

From equation (36) one obtains, on $s(x, X_1) = 0$, the ideal sliding dynamics as

$$\dot{x}_1 = -\,c_1(x_1 - X_1). \tag{49}$$

The equivalent control $u^{EQ}(x, X_1)$, associated with equations (36) and (48), is given by

$$u^{EQ}(x, X_1) = -\,J\,x_2(c_1 + x_1x_2). \tag{50}$$

Finally, according to equation (23), the adopted switching logic for this example is given by

$$u(x, U) = u^{EQ}(x, U) - \kappa \operatorname{sgn} s(x, U) = J\,x_2(c_1 + x_1x_2) - \kappa \operatorname{sgn} s(x, U), \tag{51}$$

where, in view of the fact that the quantity $\varphi_2 g$ of equation (35) is in this case represented by $0.5(1 + x_1^2)/J$, the sign of κ is globally positive.

Remark. Notice that from equations (39) and (41), one could have also obtained, instead of equation (43), the following linear sliding manifold:

$$s_\delta(x_\delta, X_1) = \frac{2\,c_1}{(1 + X_1^2)}\,x_{1\delta} + x_{2\delta} = 0. \tag{52}$$

The linearized ideal sliding dynamics corresponding to this manifold is still the same as in equation (44). However, the nonlinear sliding-surface coordinate function $s(x, X_1)$ must now satisfy the following relations:

$$\left.\frac{\partial s(x, X_1)}{\partial x_1}\right|_{\substack{x_1 = X_1 \\ x_2 = 0}} = \frac{2\,c_1}{(1 + X_1^2)}; \qquad \left.\frac{\partial s(x, X_1)}{\partial x_2}\right|_{\substack{x_1 = X_1 \\ x_2 = 0}} = 1 \tag{53}$$

with the condition $s([X_1, 0]^T, X_1) = 0$.

Relations (53) are again viewed as particularizations at the equilibrium point of

$$\frac{\partial s(x, X_1)}{\partial x_1} = \frac{2\, c_1}{(1 + x_1^2)}; \qquad \frac{\partial s(x, X_1)}{\partial x_2} = 1.$$

Using the direct integration procedure, as described above, one obtains

$$s(x, X_1) = 2\, c_1[\tan^{-1}(x_1) - \tan^{-1}(X_1)] + x_2 = 0. \tag{54}$$

The nonlinear equivalent control associated with equation (54) is given by

$$u^{EQ}(x, X_1) = Jc_1x_2 \tag{55}$$

The corresponding ideal sliding dynamics taking place on the sliding surface (54) is no longer a linear system, in the original coordinates. However, a suitable nonlinear state-coordinate transformation reveals the underlying linear nature of such an ideal sliding dynamics. Indeed, one obtains by virtue of equations (37) and (54) that

$$\dot{x}_1 = -\, c_1\, (1 + x_1^2)[\tan^{-1}(x_1) - \tan^{-1}(X_1)]. \tag{56}$$

Letting $\xi = \tan^{-1}(x_1)$, and denoting the constant equilibrium point by $\Xi = \tan^{-1}(X_1)$, one readily obtains

$$\frac{d\xi}{dt} = -\, c_1\, (\xi - \Xi). \tag{57}$$

3.2.5. Simulations. Computer simulations were carred out for the synthesized sliding-mode controller (48), (51) on a spacecraft with moment of inertia $J = 90$ N-mt-sec^2, with c_1 chosen as 0.11 sec^{-1}, and $k = 1.2$. Figure 1 shows a family of state trajectories of the sliding-mode controlled system when the reference operating point for the Cayley–Rodrigues attitude orientation parameter x_1 is set to $X_1 = -0.4$ rad. Figure 2 shows a typical controlled state-variables time response. Figure 3 also shows a family of state trajectories of the sliding-mode controlled system when the reference operating point abruptly changes from the stabilized value of $X_1 = -0.4$ rad to a new operating point set at $X_1 = 0.6$ rad. The parametrized sliding surfaces corresponding to both operating points are also depicted in this figure and labeled as S_1, S_2. Figure 4 depicts the time response of the state variables under such an abrupt change of operating conditions taking place precisely at $t = 70$ sec.

3.3. Vertical motions of a glider

Consider the motion of a glider on a vertical plane of symmetry (Rouche et al. [14], pp. 17–19). Let x_1 represent the angle between the normalized velocity vector of the glider's center of inertia and a horiztonal line. Let x_2 represent the magnitude of such a velocity vector. In appropriate time-scaled coordinates, the controlled system equations can be written as

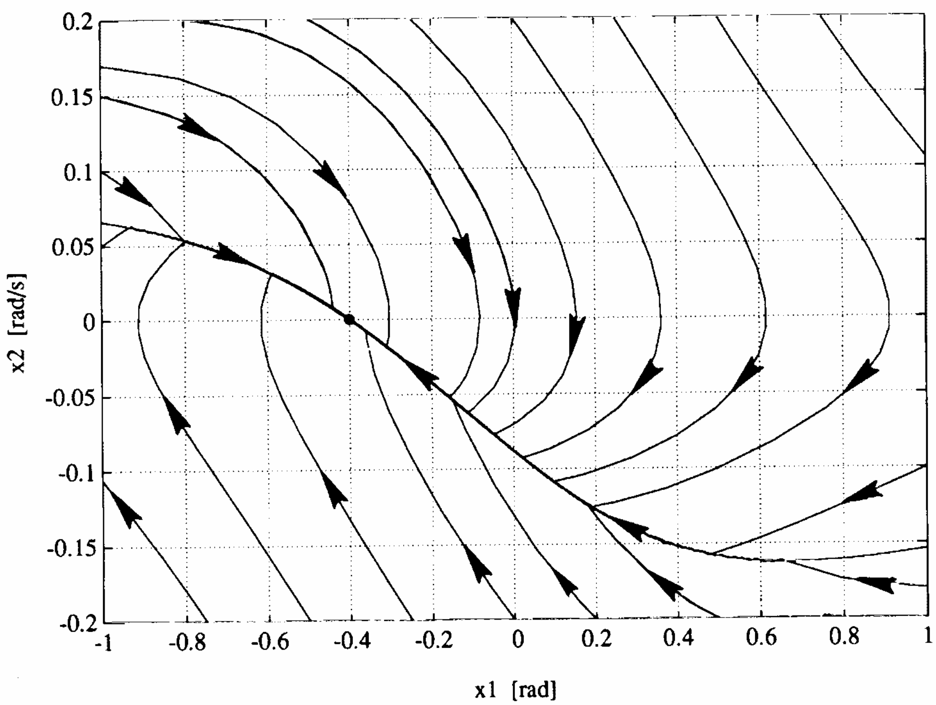

Figure 1. State trajectories of sliding-mode controlled spacecraft.

$$\dot{x}_1 = \frac{-\cos x_1 + x_2^2}{x_2}$$
$$\dot{x}_2 = -\sin x_1 - u\,x_2^2, \tag{58}$$

where u is the ratio of the drag and lift coefficients, acting here as a control parameter via suitable modifications of the glider's angle of attack. It is desired to devise a sliding-mode controller to maintain the glider state variables x_1 and x_2 at their nominal equilibrium points representing a rectilinear down motion at constant velocity (see [14], p. 33).

3.3.1. Constant equilibrium points

$$u = U;\ x_1 = X_1(U) = -\tan^{-1}(U);\ x_2 = X_2(U) = \frac{1}{\sqrt[4]{1 + U^2}} \tag{59}$$

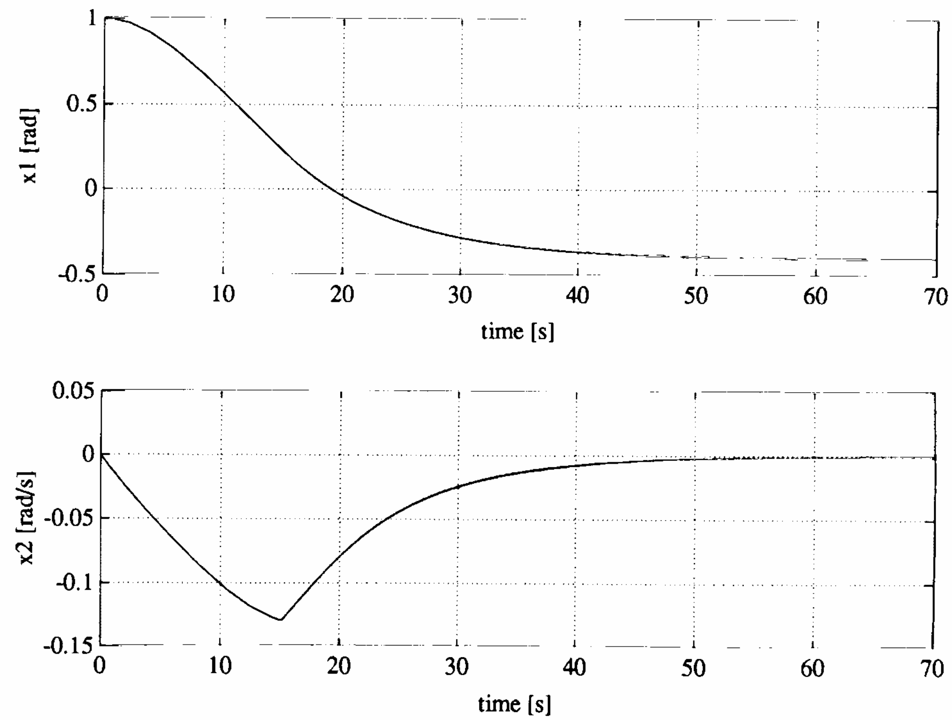

Figure 2. Typical time response of sliding-mode controlled spacecraft state variables.

3.3.2. Family of linearization about equilibrium points

$$\begin{bmatrix} \dot{x}_{\delta 1} \\ \dot{x}_{2\delta} \end{bmatrix} = \begin{bmatrix} -\dfrac{U}{\sqrt[4]{1+U^2}} & 2 \\ -\dfrac{1}{\sqrt{1+U^2}} & -\dfrac{2U}{\sqrt[4]{1+U^2}} \end{bmatrix} \begin{bmatrix} x_{1\delta} \\ x_{2\delta} \end{bmatrix} + \begin{bmatrix} 0 \\ -\dfrac{1}{\sqrt{1+U^2}} \end{bmatrix} u_\delta \quad (60)$$

with $x_{1\delta} = x_1 - X_1(U)$, $x_{2\delta} = x_2 - X_2(U)$, $u_\delta = u - U$.

3.3.3. Transformation to controllable canonical form

$$\begin{bmatrix} x_{1\delta} \\ z_{2\delta} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{2}\sqrt{1+U^2} & 0 \\ \dfrac{1}{2}U\sqrt[4]{1+U^2} & -\sqrt{1+U^2} \end{bmatrix} \begin{bmatrix} x_{1\delta} \\ x_{2\delta} \end{bmatrix} \quad (61)$$

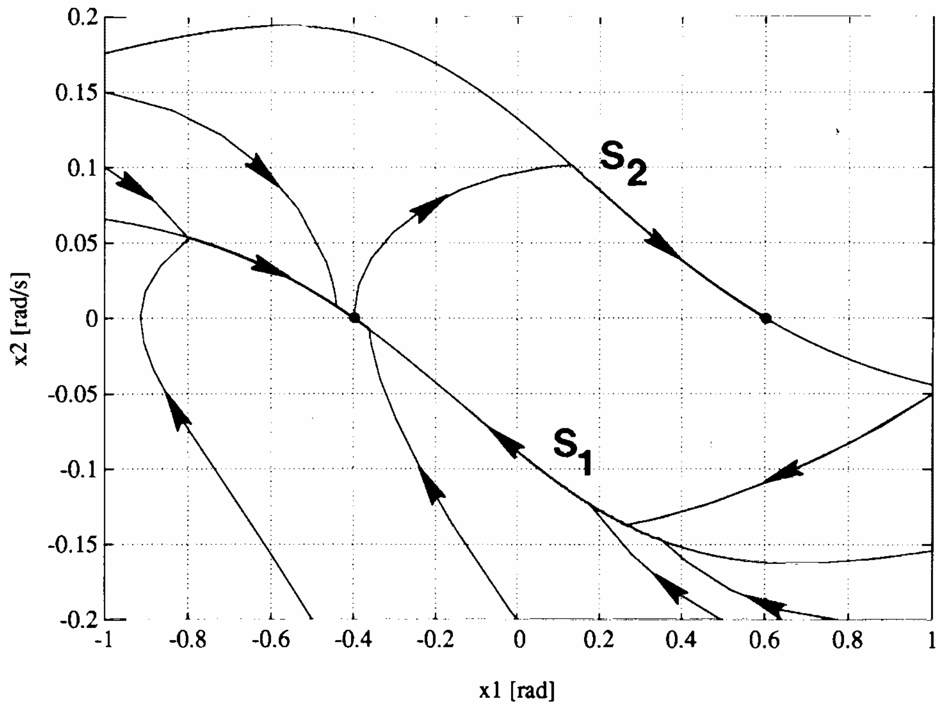

Figure 3. State trajectories of sliding-mode controlled spacecraft under a sudden change of the orientation parameter operating point (from $X_1 = -0.4$ to $X_1 = 0.6$).

3.3.4. Linear sliding surface and ideal sliding dynamics in transformed and original coordinates

In transformed coordinates:

$$\sigma_\delta(z_\delta) = z_{2\delta} + c_1 z_{1\delta} = 0; \; c_1 > 0 \tag{62}$$

$$\dot{z}_{1\delta} = - c_1 z_{1\delta} \tag{63}$$

In original coordinates:

$$s_\delta(x_\delta) = \left[c_1 - \frac{U}{\sqrt[4]{1 + U^2}} \right] x_{1\delta} + 2x_{2\delta} = 0 \tag{64}$$

$$\dot{x}_{1_\delta} = - c_1 x_{1\delta} \tag{65}$$

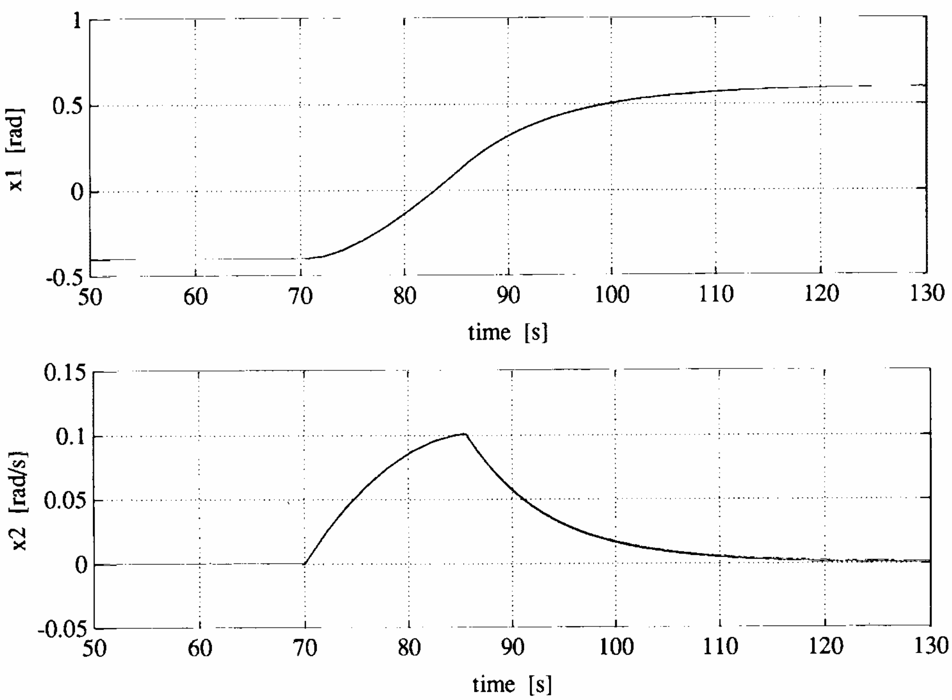

Figure 4. State responses of sliding-mode controlled spacecraft under a sudden change of the orientation parameter operating point.

3.3.5. Nonlinear sliding surface, ideal sliding dynamics, and nonlinear sliding-mode controller

$$s(x, U) = \frac{-\cos x_1 + x_2^2}{x_2} + c_1 [x_1 + \tan^{-1}(U)] = 0 \tag{66}$$

$$\dot{x}_1 = -c_1 [x_1 + \tan^{-1}(U)] \tag{67}$$

$$u^{EQ}(x, U) = -\frac{1}{\cos x_1 + x_2^2} \left[c_1 \left(x_2 - \frac{\cos x_1}{x_2} \right) - \frac{\sin 2x_1}{x_2^2} \right] - \kappa \operatorname{sgn} x(s, U) \tag{69}$$

Remark 1. Notice that in this case the factor $g\varphi_2$ (of equation (35), equal here to $-\cos x_1 - x_2^2$), is negative in the range $-\pi/2 \leq x_1 \leq \pi/2$. Hence κ is chosen to be negative. ■

Remark 2. Again, it is possible to obtain a different sliding manifold whose linearization coincides with equation (64) and contains the equilibrium point. For instance, using the *integration formula* of equation (17), one obtains:

$$s(x, U) = \int_U^{-tg(x_1)} \left[c_1 - \frac{\vartheta}{\sqrt[4]{1+\vartheta^2}} \quad 2 \right] \begin{bmatrix} \dfrac{-1}{1+\vartheta^2} \\ \dfrac{-\vartheta}{2\sqrt[4]{(1+\vartheta^2)^5}} \end{bmatrix} d\vartheta + 2\left(x_2 - \sqrt{\cos x_1}\right)$$

i.e.,

$$s(x, U) = c_1 \left[x_1 + \tan^{-1}(U)\right] + 2\left(x_2 - \sqrt{\cos x_1}\right) = 0 \tag{70}$$

which, as can be easily verified, does not result in a linear ideal sliding dynamics as in equation (67).

3.3.6. Simulation. Computer simulations were carried out for the synthesized sliding-mode controller (66), (69) on a glider with c_1 chosen as 1.0 $\sec^{-1}$, and $\kappa = 0.1$. Figure 5 shows the time responses of the state variables x_1 and x_2 of the sliding-mode controlled system when the reference operating point, for the control input u, abruptly changes from $U = 1.0$ to $U = 1.5$ at $t = 8.0$ sec. This change in the control-input operating point corresponds to a change from -0.78 rad to -0.98 rad, in the operating point for the angle x_1.

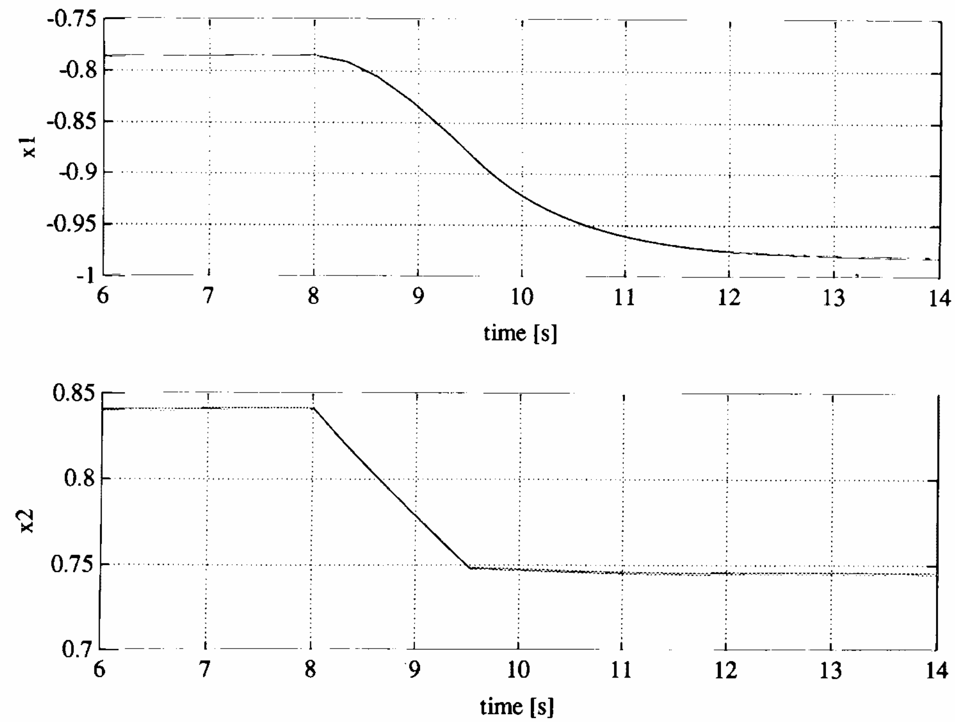

Figure 5. State responses of sliding-mode controlled glider under a sudden change of the control-input operating point (from $U = 1.0$ to $U = 1.5$).

Remark. In the previous second-order examples, the nonlinear systems were already in *regular canonical form.* Hence, in accordance with the results of section 3.1, the linearizing sliding manifold could have been obtained *directly* from the systems equations. However, it should be remarked that this is not the case for (single-input) higher-order systems, nor for systems that are not in regular canonical form (even if they are affine in the control). For the last class of systems, obtaining a linearizing sliding manifold is by no means a trivial task. Moreover, transformation to regular canonical form of a nonlinear system involves a quite complicated procedure dealing with the solution of certain associated *Pfaffian systems* (see Luk'yanov and Utkin [13]). The method of extended linearization therefore provides us with an alternative synthesis approach for such particular cases and, more importantly, for the general case represented by systems of the form (1). The next example deals with a second-order control-affine (bilinear) system that is not in regular canonical form. ■

3.4. Input-current scheduled sliding-mode control of angular velocity in a DC motor

Consider a DC, field-controlled motor provided with separate excitation. Let V_a be the constant armature voltage and let u be the field current, acting as a control parameter. The set of bilinear differential equations describing the dynamics of such a controlled system, acting on a load that exhibits a nonnegligible damping reaction, is given by (see Rugh [15], pp. 98–99).

$$\begin{aligned} \dot{x}_1 &= -\frac{R_a}{L_a}x_1 - \frac{K}{L_a}x_2 u + \frac{V_a}{L_a} \\ \dot{x}_2 &= -\frac{B}{J}x_2 + \frac{K}{J}x_1 u, \end{aligned} \tag{71}$$

where x_1 is the armature current, x_2 is the motor-shaft angular velocity, L_a and R_a are, respectively, the inductance and the resistance in the armature circuit, K is the torque constant, and J and B are, respectively, the load's moment of inertia and the associated viscous damping coefficient.

It is required to maintain a fixed nominal angular velocity W by suitable discontinuous control actions generated by the field-circuit input current u. We summarize next the nonlinear sliding-mode controller synthesis.

3.4.1. Family of constant equilibrium points

$$u = U,\ x_1 = X_1(U) = \frac{BV_a}{R_aB + K^2U^2},\ x_2 = X_2(U) = W(U) = \frac{V_aKU}{R_aB + K^2U^2} \tag{72}$$

3.4.2. Family of linearizations parametrized by constant equilibrium point

$$\begin{bmatrix} \dot{x}_{1\delta} \\ \dot{x}_{2\delta} \end{bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{KU}{L_a} \\ \frac{KU}{J} & -\frac{B}{J} \end{bmatrix} \begin{bmatrix} x_{1\delta} \\ x_{2\delta} \end{bmatrix} + \begin{bmatrix} -\frac{K^2 V_a U}{L_a(R_a B + K^2U^2)} \\ \frac{KBV_a}{J(R_a B + K^2 U^2)} \end{bmatrix} u_\delta \tag{73}$$

3.4.3. State-coordinate transformation to controllable canonical form

$$\begin{bmatrix} z_{1\delta} \\ z_{2\delta} \end{bmatrix} = \begin{bmatrix} \frac{B}{J\,\eta(U)} & \frac{KU}{L_a\,\eta(U)} \\ \frac{K^2U^2 - BR_a}{L_a\,J\,\eta(U)} & -\frac{2BKU}{L_a\,J\,\eta(U)} \end{bmatrix} \begin{bmatrix} x_{1\delta} \\ x_{2\delta} \end{bmatrix} \tag{74}$$

with

$$\eta(U) = \frac{K^2V_a\,U(-\,2B^2L_a + R_aJB - K^2U^2\,J)}{L_a^2J^2[R_aB + K^2U^2]} \tag{75}$$

3.4.4. Parametrized linear sliding surface and ideal sliding dynamics in transformed and original coordinates

In transformed coordinates:

$$\sigma_\delta(z_\delta) = z_{2\delta} + c_1 z_{1\delta} = 0;\ c_1 > 0 \tag{76}$$

$$\dot{z}_{1\delta} = -\,c_1 z_{1\delta}. \tag{77}$$

In original coordinates

$$s_\delta(x_\delta,\ U) = -\,(c_1BL_a - R_aB + K^2U^2)\,x_{1\delta} + (2B - Jc_1)KU\,x_{2\delta} = 0 \tag{78}$$

$$\dot{x}_{1\delta} = -\,c_1x_{1\delta};\ \dot{x}_{2\delta} = -\,c_1x_{2\delta}. \tag{79}$$

3.4.5. Nonlinear sliding surface, ideal sliding dynamics, and nonlinear sliding-mode controller

$$s(x,\ U) = -\,(c_1L_a - 2R_a)\left(x_1 - \frac{BV_a}{R_aB + K^2U^2}\right) - V_a\ln\left(\frac{R_aB + K^2U^2}{BV_a}\,x_1\right) + \frac{(2B - Jc_1)}{2x_1}\left[(x_2)^2 - \left(\frac{V_aKU}{R_aB + K^2U^2}\right)^2\right] = 0 \tag{80}$$

$$\dot{x}_2 = -c_1 \left[x_2 - \frac{V_a K U}{R_a B + K^2 U^2} \right] \tag{81}$$

$$u^{EQ}(x, U) = \tag{82}$$

$$\frac{J[2(c_1L_a - 2R_a)x_1^2 + 2V_aX_1 + (2B - Jc_1)(x_2^2 - x_2^2(U))](-R_ax_1 + V_a) + 2BL_a(2B - Jc_1)x_2^2x_1}{j[2(c_1L_a - 2R_a)x_1^2 + 2V_ax_1 + (2B - Jc_1)(x_2^2 - x_2^2(U))]Kx_2 + 2L_a(2B - Jc_1)KX_1^2x_2}$$

$$u(x, U) = -k \mid u^{EQ}(x, U) \mid \operatorname{sgn} s(x, u) \tag{83}$$

3.4.6. Simulations. Computer simulations were carried out for the synthesized sliding-mode controller (80), (83) on a loaded DC motor with moment of inertia $J = 1.06 \times 10^{-6}$N-m-sec²/rad, $B = 6.04 \times 10^{-6}$N-m-sec/rad, and $L_a = 120$ mH, $K = 1.41 \times 10^{-2}$N-m/A, $R_a = 7\Omega$, $V_a = 5$ V. c_1 was chosen as 5.0 sec^{-1}, and $k = 1.05$. Figure 6 shows the time responses of the state variables x_1 and x_2 for the sliding-mode controlled system when the reference operating point for the motor-shaft angular velocity abruptly changes for $W_1 = 159.25$ rad/sec to $W_2 = 280.69$ rad/sec at $t = 0.1$ sec. The corresponding change in the input-current nominal value is from $U = 0.1$ A to $U = 0.2$ A.

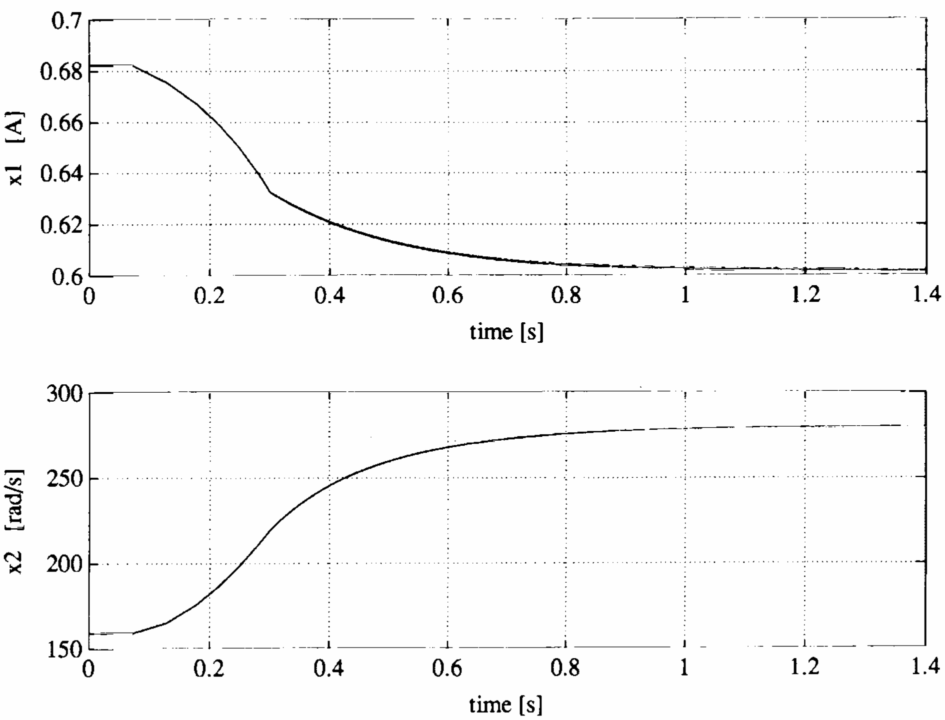

Figure 6. State responses of sliding-mode controlled DC motor under a sudden change of the shaft's angular-velocity operating point (from $W = 159.25$ to $W = 280.69$).

4. Conclusions and suggestions for further work

A general systematic approach has been proposed for the synthesis of sliding-mode control regulators for a rather wide class of nonlinear systems, possessing no particular control-input structure and exhibiting a continuous family of constant operating points. The method makes use of the extended linearization technique for the specification of the nonlinear switching manifold, the associated equivalent control, and the required switching strategy. As demonstrated by a general second-order example, in which the required linearizing sliding surface is readily apparent, the method appears to be a natural one, since it yields the intuitively obvious solution. The self-scheduling properties of the proposed controller were demonstrated in three physically motivated simulation examples

The fundamental advantages of the proposed design scheme are as follows:

1. The approach benefits from an extensive list of well-known theoretical contributions for design of linear sliding modes, including efficient computer packages already developed for such design tasks.
2. The possibilities of nontrivial applications can be greatly enhanced, and carried out, by means of existing algebraic manipulation systems.
3. The method naturally enjoys rather useful self-scheduling properties when nominal operating conditions are abruptly changed. This is particularly important in the field of control of mechanical manipulators, aerospace systems, and other practical nonlinear control application areas.
4. For approximate linearization of nonlinear systems, the method developed in this article also constitutes an alternative approach to that developed by Barolini and Zolezzi [16].

As a topic for future work, the multivariable sliding-mode control case needs some special attention and careful examination from the perspective of the extended linearization approach. Also, automation of the design process via computational algebra packages, such as MACSYMA, REDUCE, or MAPLE, is strongly encouraged.

Acknowledgments

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