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Contributed Paper

# NONLINEAR DYNAMICAL DISCONTINUOUS FEEDBACK CONTROLLED DESCENT ON A NON ATMOSPHERE-FREE PLANET: A DIFFERENTIAL ALGEBRAIC APPROACH*

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***Abstract.*** A new approach is proposed for the feedback controlled descent of a thrusted vehicle on the surface of a planet which exhibits non-negligible atmospheric resistance. A nonlinear Pulse-Width-Modulated (PWM) dynamical feedback controller is synthesized on the basis of a suitably defined nonlinear average model. Using Fliess' Generalized Observability Canonical Form, of the average model, a linearizing dynamical feedback strategy is synthesized by exact linearization. The smooth controlled behavior is then approximated, arbitrarily close, by means of the discontinuous PWM control scheme. A simulation example is presented.

***Key Words***—Smooth landing maneuvers, dynamical feedback controller design, pulse-width-modulation control, differential algebraic systems.

## 1. Introduction

The problem of soft controlled descent on the surface of a planet has been traditionally studied from the perspective of the Optimal Control theory. An early solution, using Calculus of Variations, was given by Miele (1960). Over the years, the problem has gained both theoretical and practical interest. In the mid 60's, Meditch (1964) proposed a minimum-time approach using Pontryaguin's Minimum Principle. In his work, Meditch has shown that the minimum-time and the minimum-fuel landing problems are equivalent. The optimal control approach was also used by Flemming and Rishel in their book (Flemming and Rishel, 1975) to illustrate Pontryaguin's Minimum Principle. Significant contributions were given later on by Cantoni and Finzi (1980) which further modified the solution proposed by Meditch. Recently, a Sliding Mode Control approach was proposed by Sira-Ramírez (1990), with various practical alternatives for the final touchdown stage. In Sira-Ramírez (1990), a suitable sliding manifold is synthesized which induces an exponentially stable behavior in the ideal sliding trajectories associated with average height and average vertical speed variables. Under the assumptions of Amplitude Modulation capabilities on the thrusters braking action, it has been proposed in Sira-Ramírez (1991), a dynamical feedback controller for the smooth descent problem based on exact linearization and pole placement.

Recent contributions to the understanding of nonlinear controlled systems

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have been given by Professor M. Fliess in a series of outstanding articles (Fliess, 1986; 1989; 1990 a; Fliess and Messager, 1990). In Fliess's work, a new and general approach, based on *Differential Algebra,* (see Kolchin, 1973) has been proposed for the study of linear and nonlinear lumped, or distributed, controlled dynamical systems. A powerful, most elegant, and general characterization has been contributed by Fliess's work on a number of long standing problems in automatic control theory. Among such problems, one can list: feedback decoupling, right and left invertibility, realization, cascade decomposition, feedback linearization and model matching. Sliding mode controller design has also been approached from the differential algebraic viewpoint by Fliess and Messager (1990).

Fliess's key observation is that a large class of lumped nonlinear dynamical controlled systems, in state space form, are more naturally described as a set of *implicit* first order differential equations (some of which could degenerate into algebraic equations) relating the state variables to a finite number of time derivatives of the several input components. Different state space representations are then related by input dependent state coordinate transformations, possibly involving a finite number of input component time derivatives. This setup not only allows for a unified view of regular and singular systems (i.e., systems with algebraic constraints, or *descriptor systems*), but it also explicitly exhibits some of the limitations found in standard state space descriptions of some physical systems. Namely, it points to the *local* validity of the state variable representation *á la Kalman.* Central in the developments is the theorem of *the differential primitive element* (Kolchin, 1973). This theorem accounts for the existence of a *Generalized Controller Canonical Form* (GCCF) from which dynamical feedback linearization can be trivially achieved. The approach duly requires, however, account of possible singularities, such as *impasse points* and assessment of the stability properties of the proposed linearizing controller. From an input-output viewpoint, physical systems are viewed as implicit scalar differential equations, relating a finite number of time derivatives of the input and output vector components under some proper constraints. Based on such representations, one can naturally define *generalized phase variable components* (see Conte et al., 1988; Diop, 1989), which immediately lead to a local *Generalized Observability Canonical Form* (GOCF). From this canonical form, input-output dynamical feedback linearization is also trivially feasible, modulo some singularities such as *nonminimum phase domains.* It turns out that, for scalar systems, if the given output variable also qualifies as a differential primitive element, then the dynamics of the GOCF correspond to the GCCF.

In this article, we extend the work in Sira-Ramírez (1991) to propose, based on Fliess's GOCF results, dynamical discontinuous feedback control actions of the Pulse-Width-Modulation (PWM) type for soft controlled landing maneuvers. The dynamical feedback controller design proposed in Sira-Ramírez (1991) is utilized as an *average* feedback controller, synthesizing the *duty ratio* function associated with the PWM strategy. A non-optimal, dynamical feedback solution is hence proposed for the problem of soft controlled landing on the surface of a non atmosphere-free planet. The nonlinear Pulse-Width-Modulation (PWM) dynamical feedback controller is deemed to be a more realistic control alternative, entirely possible with the available spacecraft technology. The smooth average solution is then translated into an actual discontinuous (PWM) controlled scheme which approximates, arbitrarily closely, the smooth designed behavior.

Section 2 presents some general results and derivations about dynamical feedback linearizing controllers, using Fliess's GOCF. Section 3 presents the main results of this article in relation to the soft landing maneuver via a dynamical feedback linearizing controller of the average PWM regulated system. Simulations are presented that illustrate the performance of the proposed controller under the assumption of a constant atmospheric resistance coefficient. The controller is also evaluated in the presence of significant unmodelled changes in such a coefficient. The Appendix contains some background material on PWM control of Nonlinear Systems and their design-oriented average models.

## 2. Dynamical Input Output Linearization via the GOCF

In this section, we present some background material on Fliess's GOCF (See Fliess, 1986) obtained through a state elimination procedure as proposed by Conte et al. (1988).

Consider the following $n$-dimensional *state space realization* of a single-input single-output nonlinear analytic system written in the Kalman form:

$$\left.\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned}\right\}. \tag{2.1}$$

According to Conte et al.'s results (Conte et al., 1988), under mild conditions, there exists a non-uniquely defined, input-dependent state coordinate transformation, which eliminates the state vector $x$ from a representation of the form (2.1) and allows the finding of a possibly implicit, input-output representation for this system in the form,

$$C(y^{(d)}, \cdots, \dot{y}, y, u, \dot{u}, \cdots u^{(a)}) = 0, \tag{2.2}$$

where $d$ is defined as the smallest integer satisfying the following rank condition:

$$\operatorname{rank} \frac{\partial(h;\ \dot{h};\ \cdots;\ h^{(d-1)})}{\partial x} = \operatorname{rank} \frac{\partial(h;\ \dot{h};\ \cdots;\ h^{(d)})}{\partial x}. \tag{2.3}$$

By defining $y^{(i-1)} = \eta_i$; $i = 1, \cdots, d$, and, under the assumption that $\partial C / \partial y^{(d)}$ is non-identically zero, one locally obtains the following explicit *Generalized Observability Canonical Form* (GOCF) for the given system:

$$\left.\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \eta_3 \\ &\cdots \\ \dot{\eta}_{d-1} &= \eta_d \\ \dot{\eta}_d &= c(\eta, u, \dot{u}, \cdots u^{(a)}) \\ y &= \eta_1 \end{aligned}\right\}, \tag{2.4}$$

with $\eta \triangleq (\eta_1, \cdots, \eta_d)$ and where $a \triangleq d-r$, is assumed to be a positive integer, with $r$ being the *relative degree* of the output function $y$ with respect to the scalar control input $u$. The integer $r$ may be defined as the minimum number of times the output signal $y$ has to be differentiated, with respect to time, for the control input $u$ to appear explicitly in the output derivative expression (see Isidori, 1989, p. 145). Note that if $d<n$, then the state realization (2.1) is non-minimal. We henceforth assume, for the sake of argument and simplicity, that $d=n$.

The required input dependent state coordinate transformation, taking system (2.1) into the GOCF (2.4), is given by

$$\eta = \Phi(x, u, \dot{u}, \cdots, u^{(a-1)}) = \begin{bmatrix} h(x) \\ \dot{h}(x) \\ \cdots \\ h^{(n-a)}(x, u) \\ \cdots \\ h^{(n-1)}(x, u, \dot{u}, \cdots, u^{(a-1)}) \end{bmatrix}. \tag{2.5}$$

Suppose that system (2.1) exhibits, for a given constant control input $u=U$, a constant equilibrium state $x=X(U)$, for which the corresponding output $y$ is given by $y=h(X(U))=Y$. In terms of the representation (2.4), if we let $H \triangleq \mathrm{col}(Y, 0, \cdots, 0)$ denote the corresponding equilibrium value for the vector $\eta$, then the autonomous differential equation,

$$c(H, u, \dot{u}, \cdots, u^{(a)}) = 0, \tag{2.6}$$

exhibits, necessarily, as a local equilibrium for $u$ and its time derivatives, the constant value $(U, 0, \cdots, 0)$. It has been shown by Fliess (1990 b), using arguments from *differential specializations*, that the autonomous system (2.6) corresponds to the *zero dynamics* when $Y=0$. The stability properties of the dynamical system (2.6) around its equilibrium state $(U, 0, \cdots, 0)$ are crucial in all considerations regarding exact dynamical feedback linearization.

**Definition 2.1.** Under the assumption that $d$ is exactly equal to $n$, we say that the nonlinear system (2.1) is locally *minimum phase* at the given equilibrium point $(U, Y)$, if the **linearization** of the autonomous differential equation (2.6) around $(U, 0, \cdots, 0)$ is asymptotically stable to zero.

*Remark* 2.1: Note that in case the integer $d$ is smaller than $n$, the above definition of a minimum phase system has to be substantially modified. In such a case, asymptotic stability of (2.6) around its equilibrium point is not sufficient to guarantee a corresponding stable unobservable dynamics in the representation (2.1).

Imposing an asymptotically stable closed loop linear time invariant dynamics on (2.4), towards its equilibrium point, of the form,

$$\left.\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \eta_3 \\ &\cdots \\ \dot{\eta}_{n-1} &= \eta_n \\ \dot{\eta}_n &= -\gamma_1(\eta_1 - Y) - \gamma_2\eta_2 - \cdots - \gamma_n\eta_n \\ y &= \eta_1 \end{aligned}\right\}, \tag{2.7}$$

for a suitably chosen set of constant parameters $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, one immediately obtains an implicit expression for the required dynamical linearizing controller,

$$c(\eta, u, \dot{u}, \cdots, u^{(a)}) = -\gamma_1(\eta_1 - Y) - \gamma_2\eta_2 - \cdots - \gamma_n\eta_n. \tag{2.8}$$

Either in terms of the transformed state coordinates $\eta$, or the original state coordinates $x$, the nonlinear dynamical controller (2.8) evidently requires full state feedback. Moreover, the practical feasibility of the closed loop system depends exclusively on the stability characteristics of the dynamical controller (2.8). It follows from (2.8) and the fact that the closed loop system (2.7) has the vector $H$ as a constant equilibrium point, that the controlled system is locally asymptotically stable toward its equilibrium point $(U, Y)$, if and only if the system (2.1) is locally minimum phase around such an equilibrium point.

*Remark* 2.2: It should be pointed out that, in general, the GOCF approach for the synthesis of a dynamical controller in the form (2.8) does suffer from difficulties related to the existence of *impasse points*, *non-minimum phase regions*, and other singularities. The impasse points arise from the impossibility of explicitly solving for the highest derivative of the control, in the linearizing controller equation (2.8), at some set of singular points. A second major difficulty is usually represented by having the equilibrium point in a *non-minimum phase region*, or else, when the controlled trajectories visit such instability regions. The usual remedy for these situations has been extensively explored by Fliess and his coworkers from the perspective of discontinuous control actions (See Fliess et al., 1990; Abu el Ata-Doss and Fliess, 1989). The application discussed below corresponds to the case in which the system is minimum phase around the equilibrium point but it lies on a surface of singular points. As in Fliess et al. (1990), the proposed solution becomes feasible by introducing a suitable discontinuity in the control action.

## 3. A Dynamical Feedback Solution for Soft Controlled Landing of a PWM Thrusted Spacecraft

**3.1 The dynamical model of a soft PWM controlled landing including atmospheric resistance** Consider the nonlinear dynamical model describing the vertical descent, including the spacecraft mass behavior, of a thrust controlled vehicle attempting a regulated landing on the surface of a planet of gravity acceleration $g$ and non-negligible atmospheric resistance force opposing the vertical downwards motion (See Arnol'd, 1988 a, p. 4).

$$\left.\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= g - \left(\frac{\gamma}{x_3}\right)x_2^2 - \left(\frac{\sigma\alpha}{x_3}\right)u \\ \frac{dx_3}{dt} &= -\alpha u \end{aligned}\right\}, \tag{3.1}$$

where $x_1$ is the position (height) on the vertical axis, chosen here to be positively oriented downwards (i.e., $x_1<0$, for actual positive height), $x_2$ is the downwards velocity (See Fig. 1), and $x_3$ represents the combined mass of the vehicle and the residual fuel. The function $u$ is a binary-valued control function with values in the set $\{0, 1\}$, regulating, in a pulsed or bang-bang manner, the constant rate of ejection per unit time $\alpha$ and effectively acting as a control parameter. The constant $\sigma$ represents the relative ejection velocity of the gases in the thruster. Thus, $\sigma\alpha$ is the maximum thrust of the braking engine, while $\gamma$ is a positive quantity representing the atmospheric resistance coefficient.

The binary-valued control signal $u$ is assumed to be synthesized on the basis of a PWM control strategy (See Skoog and Blankenship, 1970; Sira-Ramírez, 1989 a; b; to appear) specified by

$$u = \begin{cases} 1 \text{ for } t_k < t \le t_k + \mu[x(t_k)]T \\ 0 \text{ for } t_k + \mu[x(t_k)]T < t \le t_k + T \end{cases}; \quad k = 0, 1, 2, \cdots, \tag{3.2}$$

where $\mu(x(t))$ is the *duty ratio* function generated in a feedback manner from knowledge of the sampled state vector $x(t)$ at time $t_k$. The sampling interval $T$, also known as the *duty cycle*, is assumed to be a small constant. As a function of the state $x$, the duty ratio function $\mu$ is a continuous piece-wise smooth function constrained within the bounds $0<\mu(x(t))<1$. The feedback synthesis problem is then defined as the problem of specifying a suitable duty ratio function $\mu$, in a feedback manner. The synthesis problem thus entitles a "hybrid" solution

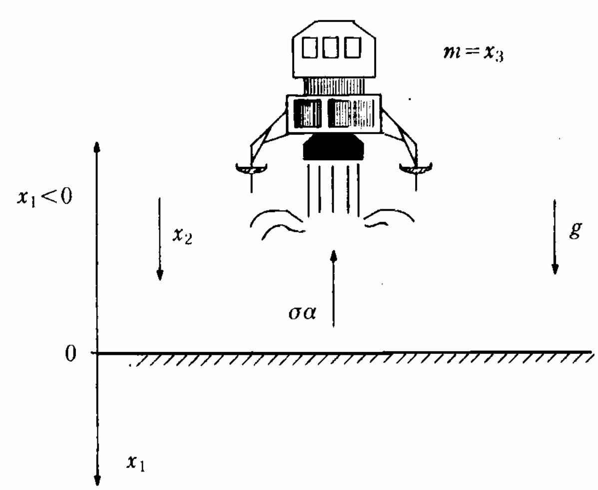


Fig. 1. Vertically controlled descent on the surface of a planet.

scheme, comprising the specification of a sampled piece-wise constant feedback control action to be exercised on a continuous time nonlinear system. Such a hybrid problem is, in general, extremely difficult, if one insists on an exact analysis or design technique. The design problem can thus be solved on the basis of an approximation scheme. We shall base our approximation scheme on an *average model* of **continuous nature** for the PWM feedback controlled system (3.1), (3.2) as developed in the Appendix.

A soft landing on the surface $x_1=0$ may be seen as a particular case of a controlled descent toward a sustained hovering about certain pre-specified height $x_1=K$. Usually, the landing maneuver entitles a regulated descent toward a small height (typically 1 [mt], or so, i.e., $K=-1$), on which a short hovering takes place before the main thruster is safely shut off. The final touchdown stage is actually a free fall toward the surface from the small hovering height. Taking the output function of the system as $y=h(x)=x_1-K$, the problem of sustained hovering is translated into the problem of zeroing the output $y$ that one can associate with the nonlinear system (3.1).

According to the results of the Appendix, the *average PWM controlled model* of the vertical descent of the controlled spacecraft is given by

$$\left.\begin{aligned} \frac{dz_1}{dt} &= z_2 \\ \frac{dz_2}{dt} &= g - \left(\frac{\gamma}{z_3}\right)z_2^2 - \left(\frac{\sigma\alpha}{z_3}\right)\mu \\ \frac{dz_3}{dt} &= -\alpha\mu \\ y &= z_1 - K \end{aligned}\right\}, \tag{3.3}$$

where $\mu$ is the *duty ratio* function, satisfying the limiting constraints $0<\mu<1$, acting as the piece-wise smooth control parameter to be designed in a dynamical feedback manner.

*Remark* 3.1: It is evident from the dynamical system equations (3.3) that the maximum value of the downwards velocity $z_2$ takes place only under free fall (i.e., uncontrolled) conditions ($\mu=0$, $z_3=\text{constant}=M$). This maximum velocity value is precisely given by $(gM/\gamma)^{1/2}$. In such a case, the downwards spacecraft acceleration is zero. A braking maneuver toward a sustained hovering, starting from free fall conditions, entitles a negative controlled acceleration until reaching zero downwards velocity at the pre-specified hovering height $z_1=K$. At this point the controlled acceleration should also become zero. It follows that, during the controlled descent, the downwards acceleration is always bounded above by zero.

**3.2 Nonlinear dynamic feedback controller design for the average PWM soft controlled landing of a thrusted spacecraft** We proceed to specify the *Generalized Observability Canonical Form* (See Fliess, 1986) of the average system (3.3) which allows us to derive a nonlinear dynamical feedback controller for the average slow descent maneuver.

It is easy to verify that for system (3.3), with output equation $y=z_1-K$, the rank condition (2.3) holds true for $d=3$. Since the relative degree of the system is $r=2$, it follows that $a=1$.

The control-dependent state coordinate transformation (2.5) of the average PWM controlled system (3.3) is given by (see Conte et al., 1988)

$$\left.\begin{aligned} \eta_1 &= z_1 - K, \quad \eta_2 = z_2, \quad \eta_3 = g - \frac{\gamma z_2^2+\sigma\alpha\mu}{z_3} \\ z_1 &= \eta_1 + K, \quad z_2 = \eta_2, \quad z_3 = \frac{\gamma\eta_2^2+\sigma\alpha\mu}{g-\eta_3} \end{aligned}\right\}. \tag{3.4}$$

This transformation takes the system (3.3) into the following GOCF:

$$\left.\begin{aligned} \dot\eta_1 &= \eta_2 \\ \dot\eta_2 &= \eta_3 \\ \dot\eta_3 &= -(g-\eta_3)\left[\frac{2\gamma\eta_2 g+\sigma\alpha\dot\mu}{\gamma\eta_2^2+\sigma\alpha\mu}\right] + (2\gamma\eta_2-\alpha\mu)\left[\frac{(g-\eta_3)^2}{\gamma\eta_2^2+\sigma\alpha\mu}\right] \\ y &= \eta_1 \end{aligned}\right\}. \tag{3.5}$$

Thus, the input output representation of the average PWM controlled system is immediately obtained as

$$\dddot y + (g-\ddot y)\left[\frac{2\gamma\dot y g+\sigma\alpha\dot\mu}{\gamma(\dot y)^2+\sigma\alpha\mu}\right] - (2\gamma\dot y-\alpha\mu)\left[\frac{(g-\ddot y)^2}{\gamma(\dot y)^2+\sigma\alpha\mu}\right] = 0. \tag{3.6}$$

Note that the transformation (3.4) is not defined on the set of points where $z_3=0$. This singularity corresponds to the physical impossibility of having the total spacecraft mass (fuel mass plus spacecraft "dead" mass) as an non-existing quantity. In Sec. 3.2, we show, however, that the average closed loop dynamically controlled system has a globally asymptotically stable equilibrium point precisely at $z_3=0$.

It is also easy to verify that, in this case, the output $\eta_1=z_1-K$ qualifies as a *differential primitive element* which allows one to write the average dynamics of the PWM model (3.3) in GCCF (see also Sira-Ramírez, 1991).

The desirable equilibrium point for the average PWM system, in transformed coordinates $\eta$, is given by $\eta_1=\eta_2=\eta_3=0$. i.e., according to the notation of Sec. 2, the set point $Y=0$. An implicit expression for the required dynamical linearizing controller described in (2.8) is, hence, given by

$$\begin{aligned} -(g-\eta_3)\left[\frac{2\gamma\eta_2 g+\sigma\alpha\dot\mu}{\gamma\eta_2^2+\sigma\alpha\mu}\right] + (2\gamma\eta_2-\alpha\mu)\left[\frac{(g-\eta_3)^2}{\gamma\eta_2^2+\sigma\alpha\mu}\right] \\ = -\gamma_1\eta_1 - \gamma_2\eta_2 - \gamma_3\eta_3. \end{aligned} \tag{3.7}$$

In other words, by suitably choosing the constant coefficients $\gamma_1$, $\gamma_2$ and $\gamma_3$, the dynamical feedback controller synthesizing the *computed duty ratio* (henceforth denoted by $m$) accomplishes, within non-saturating conditions for the actuator's duty ratio values, any desirable exponential rate of decay on the relative height, the vertical velocity and the vertical acceleration variables. Such a dynamical feedback controller, yielding the computed duty ratio $m$, is immediately obtained from (3.7), in explicit form, as

$$\frac{dm}{dt} = \frac{\gamma\eta_2^2+\sigma\alpha m}{\sigma\alpha(g-\eta_3)}\left\{\gamma_1\eta_1+\gamma_2\eta_2+\gamma_3\eta_3+(2\gamma\eta_2-\alpha m)\left[\frac{(g-\eta_3)^2}{\gamma\eta_2^2+\sigma\alpha m}\right]\right\} - \frac{2\gamma\eta_2 g}{\sigma\alpha}. \tag{3.8}$$

Note that no singularity is implied by the presence of the factor $(g-\eta_3)^{-1}$ in (3.8) due to the established negativity of the vertical acceleration $\eta_3$ during the descent maneuver. In original average coordinates, the dynamical feedback controller is given by

$$\frac{dm}{dt} = \frac{z_3}{\sigma\alpha}\left[\gamma_1(z_1-K)+\gamma_2 z_2+\gamma_3\left(g-\frac{\gamma z_2^2+\sigma\alpha m}{z_3}\right) + (2\gamma z_2-\alpha m)\left(\frac{\gamma z_2^2+\sigma\alpha m}{z_3^2}\right)\right] - \frac{2\gamma z_2 g}{\sigma\alpha}. \tag{3.9}$$

The actual duty ratio function $\mu$ is obtained by properly limiting between 0 and 1 the values of the computed duty ratio function $m$, obtained as a solution of the nonlinear time-varying differential equation (3.9), i.e.,

$$\mu = \begin{cases} 1 & \text{if } m > 1, \\ m & \text{if } 0 < m < 1, \\ 0 & \text{if } m < 0. \end{cases} \tag{3.10}$$

A block diagram depicting the complete nonlinear PWM feedback scheme for the dynamically controlled vertical descent is shown in Fig. 2.

**3.3 Stability considerations about a sustained hovering condition** As seen before, a hovering condition on $y=Y=0$, implies a zero equilibrium point for the relative position coordinate $\eta_1$, the vertical velocity $\eta_2$ and vertical acceleration $\eta_3$ in (3.5). As it can be seen from the last state equation in (3.5),

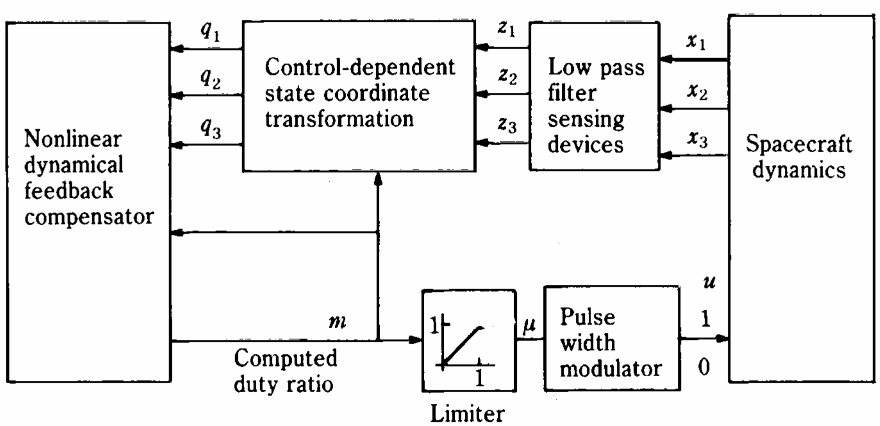


Fig. 2. Nonlinear dynamical feedback control scheme for regulation of non-linear PWM controlled system.

the hovering condition: $\eta_1 = \eta_2 = \eta_3 = 0$, entitles an exponentially stable autonomous trajectory for the duty ratio function $\mu$, governed by

$$\frac{d\mu}{dt} = -\left(\frac{g}{\sigma}\right)\mu; \tag{3.11}$$

i.e., the *zero dynamics*, as defined in (2.6), of the average PWM system is therefore globally asymptotically stable to zero.

From (3.4), it follows that, under such a hovering condition, the total mass behavior is governed by

$$z_3 = \left(\frac{\sigma\alpha}{g}\right)\mu. \tag{3.12}$$

An autonomous differential equation describing the behavior of the total mass $z_3$ is immediately obtained by differentiating (3.12) and using both (3.11) and (3.12),

$$\frac{dz_3}{dt} = -\left(\frac{g}{\sigma}\right)z_3; \tag{3.13}$$

i.e., the total spacecraft mass $z_3$ (residual fuel mass plus spacecraft "dead" mass) obeys the same linear differential equation obeyed by the duty ratio function and it asymptotically converges to zero.

This reveals the consequences of sustaining a hovering condition in an indefinite manner. Since the total spacecraft mass $z_3$ converges to zero, the equilibrium point is by no means physically meaningful. As a matter of fact, since the residual fuel mass is depleted in finite time, the control model (3.3) becomes unrealistic after the fuel mass has been exhausted. In spite of this unrealistic fact, the controlled descent toward the surface can still be practically performed at the expense of sustained fuel mass expenditure within an allowable safety limit in the hovering condition. The final touchdown maneuver, from the hovering position, may be accomplished via **switching off** of the main engine, thus allowing a free fall from the small hovering height, or, alternatively, resorting to a bang-bang time-optimal maneuver (Meditch, 1964). These maneuvers must be performed so as to guarantee enough residual fuel for the ascending stage, if any, later on. The feasibility of the proposed average solution, thus, entitles introducing a **discontinuity** in the control action at a convenient moment to avoid singularities. This scheme for singularity avoidance is pretty much in the same spirit of the techniques used by Fliess et al. (1990).

We may also establish the stability characteristics of the hovering condition by resorting to considerations about the *normal canonical form* and the associated *zero dynamics* of the average PWM controlled system (3.3) (see Isidori, 1989; also Sira-Ramírez, 1991).

*Remark* 3.2: An alternative, and possibly simpler, approach[†] to the analysis of the zero dynamics discussed in this section can be carried out by considering the *extended affine system* (see Nijmeijer and Van der Schaft, 1990, p. 190) of (3.3) obtained by just placing an integrator before the average input $\mu$. Consideration of the *normal canonical form* (Isidori, 1989) and the associated zero dynamics of the resulting fourth order system leads, precisely, to the same

[†] This approach was kindly suggested, in full detail, by an anonymous reviewer.

results obtained above.

**A Simulation Example.** Simulations were performed for both the average and the discontinuous controlled landing models discussed above, with the following constant parameters:

$$\sigma = 200\ [\text{mt/sec}], \qquad \alpha = 50\ [\text{kg/sec}],$$
$$g = 3.72\ [\text{mt/sec}^2], \qquad \gamma = 1\ [\text{kg/mt}],$$
$$K = -1\ [\text{mt}].$$

The three poles of the exactly linearized closed loop system were located at $-1.2$ [sec$^{-1}$]. The sampling frequency for the PWM actuator was set at 5 samples per second, i.e., $T=0.2$ [sec]. On a planet with the given physical constants, the free fall limit velocity is 51.03 [mt/sec]. Figure 3 shows the evolution of the controlled state variables $x_1$ and $x_2$ (height and vertical velocity) in comparison with their average values $z_1$ and $z_2$. Figure 4 depicts the average and actual behavior of the spacecraft mass under the designed control policy. Figure 5 represents the time evolution of the average computed duty ratio $m$ and the actual (i.e., limited) duty ratio function $\mu$ during the controlled descent maneuver. Initial states were chosen, from a free fall condition, at

$$x_1(0) = -500\ [\text{mt}], \quad x_2 = 51.03\ [\text{mt/sec}], \quad x_3(0) = 700\ [\text{kg}].$$

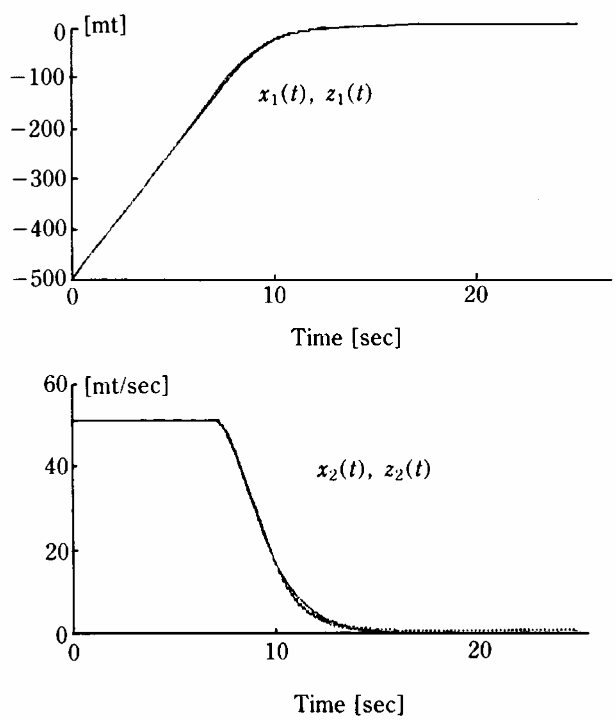


Fig. 3. Actual and average PWM controlled trajectories for dynamically feedback controlled position and vertical velocity state variables.

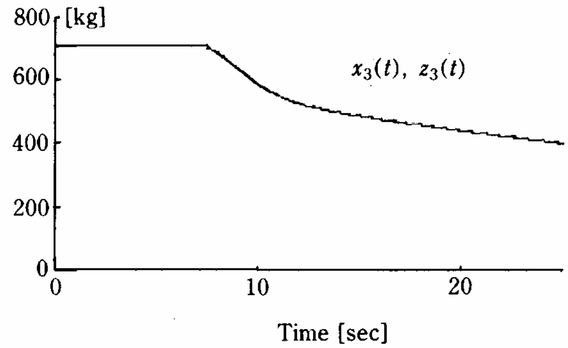


Fig. 4. Controlled behavior of combined spacecraft and residual fuel mass.

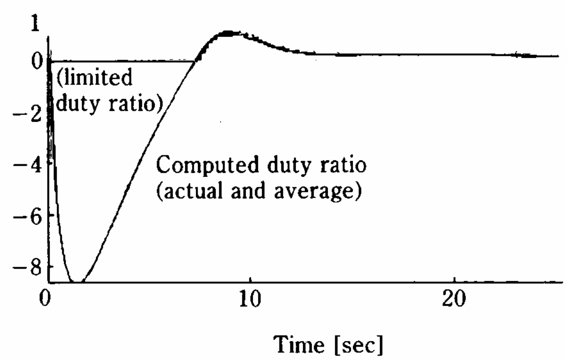


Fig. 5. Average computed and actual duty ratio functions for soft landing maneuver.

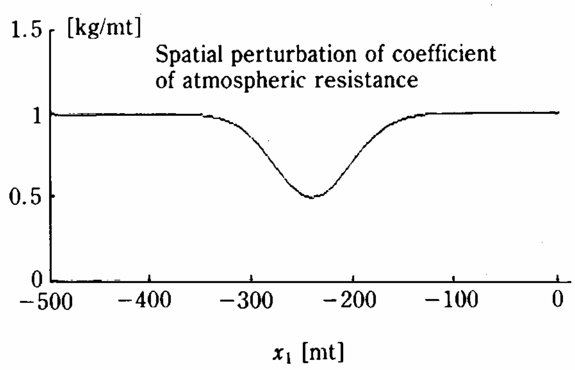


Fig. 6. Spatial variation of atmospheric resistance coefficient.

In order to evaluate the controller performance in the presence of unmodeled time-varying perturbations in the coefficient of atmospheric resistance, the value of $\gamma$ in the dynamical system model was assumed to be a function of the height coordinate $x_1$ of the form $\gamma = \gamma_0 + \gamma_\delta(x_1)$, as shown in Fig. 6, with $\gamma_0 = 1$

[kg/mt] taken as the nominal value of the coefficient $\gamma$ to be used only in the dynamical controller equations. Figures 7 to 10 depict the behavior of the average and actual state and control input variables during the perturbed descending maneuver.

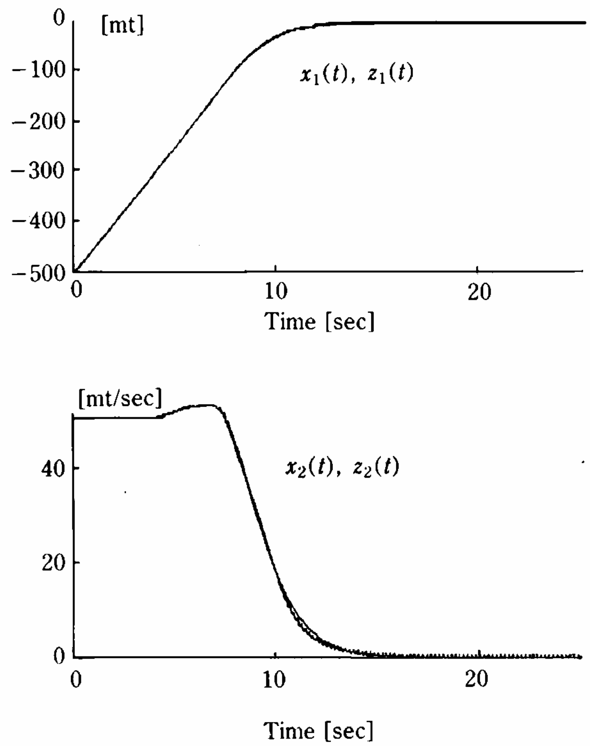


Fig. 7. Actual and average PWM controlled position and vertical velocity state variable trajectories for perturbed landing maneuver.

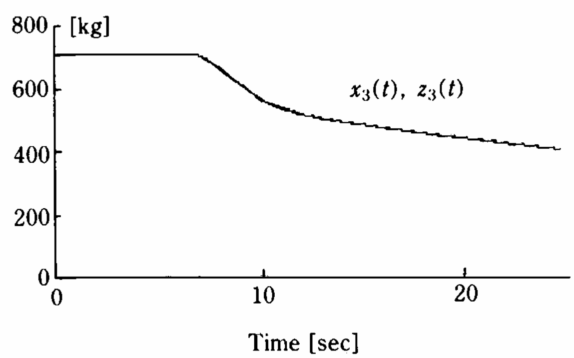


Fig. 8. Controlled behavior of combined spacecraft and residual fuel mass for perturbed landing maneuver.

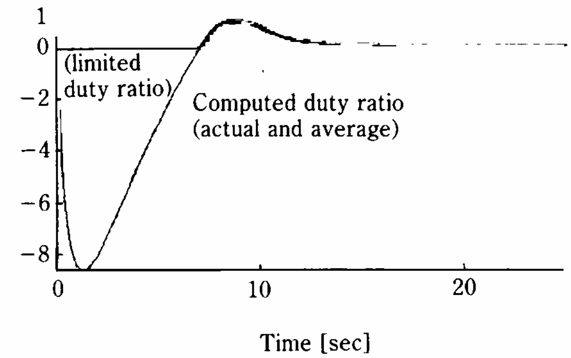


Fig. 9. Average computed and actual duty ratio functions for perturbed landing maneuver.

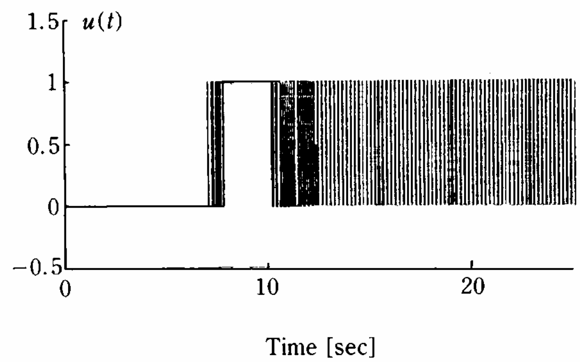


Fig. 10. Actual PWM thrust control input for perturbed descent maneuver.

## 4. Conclusions

A dynamical feedback control scheme of the PWM type has been presented for the soft landing of a vertically controlled vehicle on the surface of a planet provided with an atmosphere. An exact dynamical feedback linearization using Fliess's Generalized Observable Canonical Form was shown to allow an ideal (average) exponentially controlled descent trajectory toward a preselected small hovering height with asymptotically stable vertical velocity and vertical acceleration variables. The derived nonlinear dynamical controller, governing the landing maneuver, was shown to be asymptotically stable, thanks to the minimum phase character of the average PWM system. However, an imposed asymptotically stable behavior in the average controlled state vector not only implies infinite time reachability of the proposed hovering height but it also entitles total residual fuel mass exhaustion and a nonphysically meaningful asymptotic equilibrium point for the total controlled spacecraft mass. In order to handle this situation, the discontinuous PWM control policy must be necessarily combined with either a free fall or, alternatively, a bang-bang time-optimal strategy for the final touchdown stage of the landing maneuver from the pre-specified hovering height (see also Sira-Ramírez, 1990). In the appendix of this article, it is shown in full generality that, for nonlinear single-input single-output PWM systems, an average designed behavior can be approximated arbitrarily close by a suitable discontinuous PWM strategy, provided a sufficiently high sampling frequency is allowed.

The differential algebraic approach and its associated GOCF has been shown to be of great practical use in the design of nonlinear dynamical compensators. Its use in sliding mode controller design has also enormous potential in actually yielding chattering-free feedback control inputs and outputs. This avenue is currently being explored with preliminary results reported in Sira-Ramírez et al. (1990).

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## Appendix

### Generalities about Pulse-width-modulation Control of Nonlinear Systems

Consider a single input nonlinear dynamical system defined on an open set of $R^n$ described by

$$\dot{x} = f(x, v) \tag{A.1}$$

with $v$ a discontinuous feedback control strategy of the PWM type, given by

$$v = \begin{cases} v^+(x) & \text{for } t_k < t \le t_k + \mu[x(t_k)]T \\ v^-(x) & \text{for } t_k + \mu[x(t_k)]T < t \le t_k + T \end{cases}; \quad k = 0, 1, 2, \cdots, \tag{A.2}$$

where $T$ is a fixed sampling period also known as the *duty cycle*, $t_k$ is the $k$th sampling instant and $\mu(x(t))$ is a continuous piece-wise smooth feedback function known as the *duty ratio* function determining the variable structure feedback control pulse width during the ongoing inter-sampling interval $[t_k, t_k+T]$. The pulse width $\mu(x(t_k))T$ is determined at the beginning of each sampling interval $t_k$ on the basis of the value of the state vector at such instant (schemes on the basis of an output function or output error are also possible). The continuous piece-wise smooth duty ratio function is assumed to be bounded by $0<\mu(x(t))<1$ for all $t$.

The effect of such a discontinuous feedback strategy on the controlled state trajectories is to produce a zig-zag motion, very much reminiscent of actual Sliding Mode Controlled trajectories (Utkin, 1978). The analysis and design of such class of hybrid systems (A.1)–(A.2) is extremely difficult and can only be carried out in an approximate manner. However, the inconveniences of the nonlinear discrete-time approximations can be eliminated, if some smooth continuous average model is adopted as an approximation for the actual PWM controlled system. Such a smooth average behavior may be considered on the basis of a **high sampling frequency** for systems which are relatively slow as compared with such fast control changes. In the following paragraphs we justify the use of an average continuous model based on an *infinite sampling frequency* assumption for (A.1)–(A.2). The advantages of such an averaging procedure, aside from some intimate connections with Sliding Mode Control (See Sira-Ramírez, 1989 a; b; to appear), lay in the possibility of using modern nonlinear feedback control design techniques for the synthesis of the duty ratio function. Furthermore, the smooth average designed behavior can be arbitrarily closely approximated by the actual discontinuous feedback controlled trajectories as the sampling frequency of the PWM actuator is suitably increased within finite bounds.

Let $f(x, v^+(x))=X^+(x)$ and $f(x, v^-(x))=X^-(x)$. It is easily seen that the discontinuously controlled model (A.1), (A.2) is equivalent to the following switch controlled model:

$$\frac{dx}{dt} = uX^+(x) + (1-u)X^-(x) = X^-(x) + [X^+(x)-X^-(x)]u$$

$$\triangleq \boldsymbol{f}(x) + \boldsymbol{g}(x)u \tag{A.3}$$

with

$$u = \begin{cases} 1 \text{ for } t_k < t \leq t_k + \mu[x(t_k)]T \\ 0 \text{ for } t_k + \mu[x(t_k)]T < t \leq t_k + T \end{cases}; \quad k = 0, 1, 2, \cdots. \tag{A.4}$$

**Definition A.1.** An *average PWM model* for the discontinuously controlled system (A.1)–(A.2) (or equivalently (A.3), (A.4)) is defined by the dynamical system formally obtained by letting the sampling frequency $1/T$ of the PWM actuator grow to infinity; i.e., letting the duty cycle $T\mapsto 0$. We shall denote the state of the averaged system by $z(t)$ to differentiate it from the state vector $x(t)$ of the discontinuously controlled system.

**Proposition A.1.** The average PWM model obtained by formally imposing an infinitely large sampling frequency, $1/T$, for the controlled system (A.3), (A.4) is given by

$$\frac{dz}{dt} = \mu X^+(z) + (1-\mu)X^-(z) = X^-(z) + [X^+(z)-X^-(z)]\mu$$
$$\triangleq \boldsymbol{f}(z) + \boldsymbol{g}(z)\mu. \tag{A.5}$$

*Proof.* See Sira-Ramírez (1989).

*Remark* A.1: The average PWM model (A.5) has a right hand side which coincides with the Filippov average vector field (See Filippov, 1988) of $X^-(z)$ and $X^+(z)$, when an infinitely fast switching strategy takes place around a discontinuity surface on which the resulting controlled trajectory can be locally sustained. The switching surface is then none other than ***an integral manifold*** for the closed loop system (A.5), and the ***equivalent control*** that induces the manifold invariance is just the duty ratio function $\mu$ (See Sira-Ramírez (1989) for more details and connections with sliding regimes of variable structure control). Note, furthermore, that (A.5) is a linear-in-the-control vector differential equation formally obtained from the original discontinuous model (A.3), (A.4) just by replacing the binary control parameter $\boldsymbol{u}$ by the continuous piecewise smooth duty ratio function $\mu$.

The following result states that under identical initial conditions, the controlled trajectories of the actual discontinuous feedback controlled system (A.3), (A.4) continuously tend toward the average PWM controlled trajectories generated by (A.5), as the sampling frequency associated to the PWM actuator (A.4) is increased without limit. Hence, to arbitrarily closely retain the qualitative and quantitative stability characteristics of the average PWM designed trajectories, a sufficiently high sampling frequency is required for the PWM actuator of the actual discontinuously controlled system. This is the key feature that allows an efficient design scheme based on the continuous average PWM model.

**Theorem A.1.** Let $\mu(t)$ be a given continuous piece-wise smooth duty ratio function bounded by $0<\mu(t)<1$. Under identical initial conditions for the actual and average PWM controlled models, the corresponding controlled state trajectories of the discontinuous PWM system (A.3), (A.4) continuously and globally converge toward those of the corresponding average PWM system (A.5), as the sampling frequency $1/T$ grows without bound.

*Proof.* Note that if, in (A.3) and (A.5), the smooth vector field $\boldsymbol{g}(x)=0$, the theorem is trivially true, and, as a matter of fact, both trajectories $x(t)$ and $z(t)$ coincide for all $t$. The same statement holds true for identical initial conditions and $\mu(t)\equiv 1$, or $\mu(t)\equiv 0$, on open intervals of time. Thus, assume $\boldsymbol{g}(x)\neq 0$. By virtue of the *Theorem of Rectifiability of Vector Fields* (See Arnol'd, 1988 b, p. 85), there exists a diffeomorphic state coordinate transformation $\Phi(\cdot)$: $R^n \to R^n$, yielding $\xi=\Phi(x)$ and $\zeta=\Phi(z)$ for the actual and average state coordinates, such that in the transformed coordinates the vector field $\boldsymbol{g}(x)$ is expressed as a constant vector of value, say, $\boldsymbol{b}$; i.e.,

$$\left[\frac{\partial \Phi(x)}{\partial x}\right]_{x=\Phi^{-1}(\xi)} \boldsymbol{g}(\Phi^{-1}(\xi)) = \boldsymbol{b}; \quad \left[\frac{\partial \Phi(z)}{\partial z}\right]_{z=\Phi^{-1}(\zeta)} \boldsymbol{g}(\Phi^{-1}(\zeta)) = \boldsymbol{b}.$$

Evidently, such a diffeomorphic state coordinate transformation is not expected to produce any particularly special structure on the *drift* vector fields $\boldsymbol{f}(x)$ and $\boldsymbol{f}(z)$ in (A.3) and (A.5). We denote the transformed drift vector fields respectively by $\phi(\xi)$ and $\phi(\zeta)$; i.e.,

$$\left[\frac{\partial \Phi(x)}{\partial x}\right]_{x=\Phi^{-1}(\xi)} \boldsymbol{f}(\Phi^{-1}(\xi)) = \phi(\xi); \quad \left[\frac{\partial \Phi(z)}{\partial z}\right]_{z=\Phi^{-1}(\zeta)} \boldsymbol{f}(\Phi^{-1}(\zeta)) = \phi(\zeta).$$

Then, the controlled dynamical systems (A.3) and (A.5) are expressed, in new coordinates, as

$$\left.\begin{aligned} \frac{d\xi}{dt} &= \phi(\xi) + \boldsymbol{b}u \\ \frac{d\zeta}{dt} &= \phi(\zeta) + \boldsymbol{b}\mu \end{aligned}\right\} \tag{A.6}$$

with identical initial conditions being assumed $(\xi(t_0) = \zeta(t_0) = \psi_0)$.

Let $\mathscr{T}$ be any finite time interval containing an integer number $N$ of sampling periods $T$; i.e., $\mathscr{T} = NT$, with $N$ being of order $[K/T]$, i.e., the order of $\mathscr{T}$ is independent of $T$. The differential equations (A.6) can be equivalently expressed as integral equations of the form

$$\xi(\mathscr{T}) = \psi_0 + \int_{t_0}^{\mathscr{T}} \phi(\xi(\sigma))d\sigma + \boldsymbol{b}\sum_{k=0}^{N} \int_{t_k}^{t_k+\mu(t_k)T} d\sigma, \tag{A.7}$$

$$\zeta(\mathscr{T}) = \psi_0 + \int_{t_0}^{\mathscr{T}} \phi(\zeta(\sigma))d\sigma + \boldsymbol{b}\sum_{k=0}^{N} \int_{t_k}^{t_k+T} \mu(\sigma)d\sigma. \tag{A.8}$$

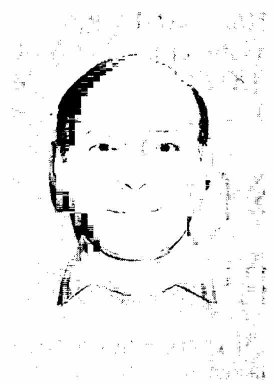
Evidently, the sum of integral terms in (A.7) represents a second order approximation to the sum of integral terms in (A.8). Hence, the integral equation in (A.8) is a *regular second order perturbation*, in terms of the sampling parameter $T$, of the integral equation (A.7). Indeed, using a Taylor series expansion of $\mu(\sigma)$ around $t_k$ on each sum in (A.8), one may rewrite Eqs. (A.7)–(A.8) as

$$\zeta(\mathscr{T}) = \psi_0 + \int_{t_0}^{\mathscr{T}} \phi(\zeta(\sigma))d\sigma + \boldsymbol{b}\sum_{k=0}^{N} \mu(t_k)T, \tag{A.9}$$

$$\xi(\mathscr{T}) = \psi_0 + \int_{t_0}^{\mathscr{T}} \phi(\xi(\sigma))d\sigma + \boldsymbol{b}\sum_{k=0}^{N}\left[\mu(t_k)T + \frac{1d\mu(\sigma)}{2d\sigma}\bigg|_{\sigma=t_k} T^2 + O(T^3)\right]. \tag{A.10}$$

It follows from well known results in the theory of perturbations of integral equations (See Miller, 1971, p. 273–285), that as the regular perturbation decreases to zero (i.e., as sampling period $T$ decreases to zero), the solution of the first integral equation, representing the actual discontinuously PWM controlled system, *continuously converges*, in a global manner, toward the solution of the second integral equation representing the average PWM system (See also, Tikhonov et al. (1988, p. 180–185) for the same basic result in the context of ordinary differential equations).

The final step in completing a design procedure based on the average PWM model consists in translating the average continuous stabilizing feedback controller design into a suitable ON-OFF (i.e., discontinuous) feedback controller of PWM nature. Such ON-OFF controller must retain the stabilizing features of the continuous average designed controller and, at the same time, it should yield actual discontinuous responses that remain arbitrarily close to the smooth designed responses. This is primarily accomplished by specifying a sufficiently high sampling frequency for the actual PWM actuator and, secondly, by suitably smoothing of the state variables before using them in the synthesis of the average stabilizing designed controller. The smoothing action may be accomplished by introducing low pass filtering effects on the state variables measurements. One then simply relies on the high-frequency rejection characteristics of most sensing devices.



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