INT. J. SYSTEMS SCI., 1991, VOL. 22, NO. 8, 1477–1494

Dynamical variable structure systems approach to switched capacitor circuit models

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A new model, of a dynamic nature, is proposed for the basic analysis of switched capacitor circuits. The model represents a significant conceptual departure from the traditional frequency-dependent equivalent resistor model. Using Filippov's concept of solution for systems governed by right-hand side discontinuous differential equations, such as those regulated by a pulse-width modulation control strategy, a general and more realistic dynamical model is proposed for switched capacitor circuits. The traditional model is easily rederived and reinterpreted as the outcome of a sequential singular perturbation procedure carried on an idealized version of the proposed dynamical model.

1. Introduction

The outstanding feature of switched capacitor circuits (SCC) is the possibility of replacing integrated resistors by a suitable combination of capacitors and frequency controlled switched arrangements. This possibility is largely responsible for the overwhelming development, in modern electronics, of resistorless integrated circuits using only MOS controlled switches, capacitors, and operational amplifiers as the basic constitutive elements. For a reasonable account of the history and background about SCC the reader is referred to the many articles appearing in the Special Issue of *I.E.E.E. Circuits and Systems Magazine* (1984), the I.E.E.E. Press book by Moschytz (1984) and the excellent text book written on the subject by Allen and Sanchez-Sinencio (1984).

For years, SCC have been studied from a purely discrete-time viewpoint based on the static voltage–charge conversion model of the involved capacitors (Caves *et al.* 1977, Hostika *et al.* 1977, Liou and Kuo 1979, Scanlan 1981, Tsividis 1979, 1983). Thanks to the very small capacitor-switch time-constants associated with the non-zero resistances present in non-ideal MOS switches, discrete-time models of SCC have perfectly suited the analysis and design needs for this outstanding class of circuits. The approach bears, however, some intrinsic limitations related to the frequency-dependent nature of the equivalent resistor model in the elementary SCC. Other limitations include the demands of highly involved discrete-time analysis schemes, which usually resort to not well-justified approximations.

From a dynamical viewpoint, SSC constitute a special class of variable structure systems (VSS) (Utkin 1978, Sira, Ramírez 1989 a) in which the structural changes are periodically executed by a switching process with, possibly, frequency modulation capabilities. From a different but related perspective, and assuming a fixed duty cycle frequency, the traditional structural changes in SCC can be viewed as the

Received 8 November 1989. Revised 20 August 1990.

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0020-7721/91 $3.00

outcome of a pulse-width modultation (PWM) scheme with fixed duty ratio (Skoog and Blankenship 1970, Sira-Ramírez 1989 b). By discarding the fixed duty ratio assumption, and allowing it to become a smooth feedback function of the circuit state, or even a constant externally regulated value, new properties of SCC can be studied on a more rigorous and exact basis using the theory of VSS. The crucial theoretical advantages of this approach not only reside in the possibilities of using the well-developed theory of dynamical VSS and its wealth of results, but also in the possibilities of practical analysis and design schemes entirely based on smooth continuous-time average models for which a vast amount of regulatory schemes are readily available (Sira-Ramírez 1987, 1990). VSS undergoing sliding motions can be effectively studied by means of averaging techniques. Such techniques are genuinely represented by Utkin's method of the equivalent control (Utkin 1978) and by Filippov's geometic averaging (Filippov 1988). These average models have been shown to capture all relevant qualitative features of the discontinuous controlled system.

In this article we propose the use of the thoery of VSS in establishing the most salient features of dynamical SCC controlled by a general PWM switching strategy. Our treatment includes both ideal and non-ideal switching devices. The proposed discontinuous dynamical models of several basic elementary configurations will also allow us to rederive the known useful static properties of SCC as a limiting case.

It will be first shown in full generality, that on a variable structure system undergoing structural changes, regulated according to a pulse-width modulation (PWM) scheme, an infinite duty cycle frequency assumption results, precisely, in Filippov's geometric average model of the discontinuous dynamical system. The associated Filippov scalar convex combination functions are shown to be coincident with the prescribed smooth duty ratio function of the original PWM scheme. It immediately folows that a corresponding sliding regime is exhibited by the actual PWM controlled system trajectories about integral manifolds of the infinite duty cycle frequency average (Filippov) PWM model (simply referred to as the average PWM model). Hence, the average PWM model has the primordial characteristic of entirely capturing all the relevant qualitative features of the actual discontinuously PWM controlled system, much in the same manner that the ideal sliding dynamics (Utkin 1978) capture the essential qualitative features of the actual (chattering) sliding motions about the sliding surface.

Using the general results stated above, dynamical models of elementary configurations in SCC are proposed and studied. Such models result from Filippov's averaging technique when applied to the PWM controlled dynamics representing the SCC. The steady-state characteristics of the proposed average models indicate that the resulting resistor-equivalent model is duty ratio-dependent, rather than frequency-dependent as in traditional SCC models. This bestows a greater flexibility in special SCC applications where controlled equivalent resistor values of a frequency-independent nature may be needed. As added bonuses, our model (*a*) circumvents the need for discrete-time approximations and (*b*) makes available a vast amount of continuous-time linear dynamical system results for the analysis and design tasks in the area of SCC (either from the state-space, or the frequency domain techniques viewpoint). The model is also sufficiently general, and realistic enough, to rederive the traditional (static) resistor-equivalent model as an idealized model which requires (*a*) a two-step sequential singular perturbation approximation carried on our proposed dynamical model and (*b*) the hypothesis of ideal switches (i.e. infinite open-switch resistance) (see Koehler 1969, Hirano and Nishimura 1972, Orfei and Pallotino 1973).

Section 2 of this article contains a general theory of variable structure systems (Sira-Ramírez 1989) of the PWM type. In this section it is shown that Filippov's geometric averaging technique is totally equivalent to an infinite frequency duty cycle assumption on the PWM controlled switch regulating the system's structural changes. Section 3 views SCC as dynamical VSS. The consequences of applying the relevant results of § 2 to such a class of dynamical systems are thoroughly reported in that section. Section 4 contains the conclusions. Appendix A contains general background material on sliding regimes associated with variable structure systems. Appendix B presents program listings for the SIMNON package (Elmquist *et al.* 1987), used in the simulations presented in this article.

2. Discontinuous dynamical systems

2.1. *General background results on PWM systems*

Consider the non-linear dynamical system locally defined on an open set $\mathbb{N}$ of $\mathbb{R}^n$:

$$\frac{dx}{dt} = f(x) \tag{2.1}$$

with structural changes regulated according to the PWM policy

$$f(x) = \begin{cases} f_1(x), & \text{for } t_k \leqslant t < t_k + \tau(x(t_k))T \\ f_2(x), & \text{for } t_k + \tau(x(t_k))T \leqslant t < t_k + T \end{cases} \tag{2.2}$$

where $f_1(x)$ and $f_2(x)$ are smooth (C^∞) vector fields with $f_1(x) \neq f_2(x)$ locally in $\mathbb{N}$, while $\tau(x)$ is, in general, a smooth scalar function of x, known as the duty ratio function. The scalar smooth function $\tau(x)$ takes values in the open interval (0, 1). The duty ratio values are usually determined at the beginning of the duty circle interval, t_k, on the basis of the sampled vector value $x(t_k)$. The parameter T is the sampling period, assumed to be sufficiently small with respect to the system dynamics and $F := 1/T$ is addressed as the sampling frequency.

System (2.1), (2.2) may be equivalently represented in terms of an ideal discrete switching functions $u \in \{0, 1\}$, as follows:

$$\frac{dx}{dt} = uf_1(x) + (1 - u) f_2(x) \tag{2.3}$$

where

$$u = \begin{cases} 1, & \text{for } t_k \leqslant t < t_k + \tau(x(t_k))T \\ 0, & \text{for } t_k + \tau(x(t_k))T \leqslant t < t_k + T \end{cases} \tag{2.4}$$

The following lemma is a straightforward consequence of the fundamental theorem of calculus.

Let f be a smooth vector field and let

$$I_f(t_k) := \int_0^{t_k} f(\sigma)\, d\sigma$$

Then, for any smooth strictly positive function $\tau(x)$,

$$\lim_{T \to 0,\, t_k \to t} \{I_f(t_k + \tau[x(t_k)]T) - I_f(t_k)\}/T = \tau[x(t)] f(x(t))$$

Theorem 1

As the sampling frequency F tends to infinity in system (2.3), (2.4), the discontinuous model (2.3), (2.4) is substituted by Filippov's average model (see Appendix A) with a corresponding convex combination function represented by the duty ratio $\tau(x)$. Moreover, a sliding regime is exhibited by the actual PWM controlled system (2.3), (2.4) on integral manifolds of the average PWM controlled dynamics.

Proof

From (2.3), (2.4), the state x at time $t_k + T$ is exactly computed as

$$x(t_k + T) = x(t_k) + \int_{t_k}^{t_k+\tau[x(t_k)]T} f_1[x(\sigma)]\,d\sigma + \int_{t_k+\tau[x(t_k)]T}^{t_k+T} f_2[x(\sigma)]\,d\sigma$$

$$= x(t_k) + \int_{t_k}^{t_k+\tau[x(t_k)]T} f_1[x(\sigma)]\,d\sigma + \int_{t_k}^{t_k+T} f_2[x(\sigma)]\,d\sigma - \int_{t_k}^{t_k+\tau[x(t_k)]T} f_2[x(\sigma)]\,d\sigma$$

that is, using the result of Lemma 1,

$$\lim_{T\to 0, t_k \to t}\left[\frac{x(t_k + T) - x(t_k)}{T}\right]$$

$$= \lim_{T\to 0, t_k\to t}\left[\frac{x(t_k) + \int_{t_k}^{t_k+\tau[x(t_k)]T} f_1[x(\sigma)]\,d\sigma + \int_{t_k}^{t_k+T} f_2[x(\sigma)]\,d\sigma - \int_{t_k}^{t_k+\tau[x(t_k)]T} f_2[x(\sigma)]\,d\sigma}{T}\right]$$

$$= \tau(x(t))\,f_1(x(t)) + [1 - \tau(x(t))]\,f_2(x(t))$$

or

$$\frac{dx}{dt} = \tau(x)\,f_1(x) + [1 - \tau(x)]f_2(x) \qquad (2.5)$$

i.e. that is the infinite frequency model of (2.3), (2.4) coincides with Filippov's geometric average model (see Appendix A) in which the convex combination functions are taken, precisely, as the smooth duty ratio function $\tau(x)$. By definition, the duty ratio is bounded in the open interval (0, 1). From the results of Theorem A.2, it follows that a sliding regime locally exists on the manifold S for the variable structure system (2.3), (2.4). The equivalent control $u^{EQ}(x)$ associated with such a sliding regime is simply obtained from the ideal sliding dynamics of (2.3) on S:

$$\frac{ds}{dt} = \langle ds, u^{EQ}f_1(x) + [1 - u^{EQ}]f_2(x)\rangle$$

$$= u^{EQ}\langle ds, f_1(x)\rangle + [1 - u^{EQ}]\langle ds, f_2(x)\rangle = 0$$

or

$$u^{EQ} = -\frac{\langle ds, f_2(x)\rangle}{\langle ds, f_1(x) - f_2(x)\rangle}$$

It follows from the uniqueness of the equivalent control and (A 7) that

$$u^{EQ}(x) = \tau(x) \qquad (2.6)$$

that is, the equivalent control of the sliding motion associated with (2.3), (2.4) is precisely constituted by the duty ratio associated with the PWM control scheme. The corresponding ideal sliding dynamics is then represented by

$$\frac{dx}{dt} = u^{EQ}(x) f_1(x) + [1 - u^{EQ}(x)] f_2(x) = \tau(x) f_1(x) + [1 - \tau(x)] f_2(x)$$

which is just the infinite frequency average dynamics (2.5).

The region of the existence of a sliding motion is, according to the results of Theorem A.1, determined by the region interesected with S where

$$0 < \tau(x) = u^{EQ}(x) < 1$$

which is, by definition of duty ratio, globally satisfied along the integral manifold S in all regions of the state space. Hence, the sliding motion on S does not only locally exist but it exists globally about S.

3. Average dynamic analysis of switch capacitor circuit models

3 1. *A basic example*

Consider the elementary switch regulated circuit of Fig. 1, where the switch position function u is prescribed according to a PWM control law of the form (2.2) with constant duty ratio t. The non-ideal switch is assumed to have resistance R when it is closed and resistance kR when it is open (evidently, $k \gg 1$). The differential equations governing the system behaviour in each case are given by

$$\begin{aligned} \text{Switch closed:} \quad & \frac{d(v_1 - v_3)}{dt} = \frac{v_3 - v_2}{CR} \\ \text{Switch open:} \quad & \frac{d(v_1 - v_3)}{dt} = \frac{v_3 - v_2}{CkR} \end{aligned} \tag{3.1}$$

A single differential equation may be written in terms of the switch position function $u \in \{0, 1\}$ as

$$\frac{d(v_1 - v_3)}{dt} = u\left[\frac{v_3 - v_2}{CR}\right] + (1 - u)\left[\frac{v_3 - v_2}{CkR}\right] \tag{3.2}$$

The results of §2 indicate that an infinite frequency PWM average model, or Filippov's average model, of the discontinuous system is obtained formally by substituting u by the duty ratio function τ. Rewriting the resulting expression as

$$\frac{d(v_1 - v_3)}{dt} = \left[\frac{v_3 - v_2}{CR_{eq}(\tau, k)}\right] \tag{3.3}$$

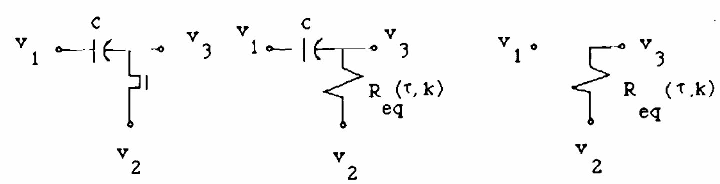

Figure 1. (*a*) Elementary switch capacitor circuit. (*b*) Average PWM dynamic equivalent model. (*c*) Steady-state resistor equivalent model.

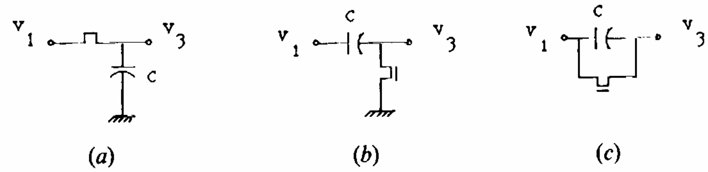

Figure 2. (*a*) Grounded capacitor. (*b*) Grounded switch. (*c*) Parallel switch-capacitor combination.

with

$$R_{eq}(\tau, k) = \left[\frac{kR}{\tau k + (1 - \tau)}\right] \tag{3.4}$$

one obtains the average PWM controlled model of (3.2) (see Fig. 1 (*b*)).

The infinite frequency average model of the original PWM controlled SCC is then obtained by formally replacing the switch by the finite duty ratio-dependent resistance $R_{eq}(\tau, k)$ given by (3.4). In steady-state conditions the capacitor is fully charged and its corresponding branch acts as an open branch. The circuit is then simply replaced by the voltage sources and the resistor $R_{eq}(\tau, k)$ (Fig. 1(*c*)).

If the switch has infinite resistance when it is open, then one lets $k \to \infty$ and the equivalent resistor, replacing the switch in the average dynamical model, has the value

$$R_{eq}(\tau) = \lim_{k \to \infty} R_{eq}(\tau, k) = \frac{R_1}{\tau} \tag{3.5}$$

The above 'series' switch model of a switch-capacitor combination is sufficiently general to generate, as particular cases, several important basic configurations commonly appearing in SCC topologies (see Fig. 2 (*a*)–(*c*)). We briefly summarize those particular cases below.

$$\text{Grounded capacitor:} \quad v_1 = 0, -v_3 = v_c \tag{3.6}$$

$$\text{Grounded switch:} \quad v_2 = 0, v_1 - v_3 = v_c \tag{3.7}$$

$$\text{Parallel switch:} \quad v_2 = v_1, v_1 - v_2 = v_c \tag{3.8}$$

In either of the above cases, the equivalent resistor value of the average PWM dynamical network, obtained in (3.4) or (3.5), is exactly the same, as can be easily verified. The equivalent resistor value substituting the controlled switch is hence a topological invariant for the three basic configurations contained in the basic switch arrangement described by (3.1).

Thus the basic rule in obtaining the average PWM model of SCC simply consists of a direct, formal, substitution of the controlled switches by its resistor equivalents (3.5) (or (3.6), if an infinite open switch resistance assumption is valid) while leaving the capacitor branches untouched.

It is important to stress that, in the above rule, the duty ratio t is the one precisely corresponding to the substituted switch. In arrangements of switches acting in a complimentary fashion, some duty ratios have the value τ while in their complements the duty ratio is $(1 - \tau)$. In such cases the non-ideal switches are substituted respectively by $R_{eq}(\tau, k) = kR/[\tau k + (1 - \tau)]$ and $R_{eq}(\tau, k) = kR/[(1 - \tau)k + \tau]$.

Although the above rule could be systematically used throughout the paper in the derivation of the equivalent dynamical circuits of the basic arrangements to be studied, we prefer to derive, from scratch, the equivalent average PWM circuit of each basic SCC arrangement treated in this paper, namely; the parallel switched capacitor circuit, the series switched capacitor circuit, the series–parallel combination, the bilinear arrangement (or series inverter) and the parallel inverter (Allen and Sanchez-Sinencio 1984).

3.2. *An average PWM parallel switched capacitor circuit model with non-ideal switches*

Consider the parallel switched capacitor circuit model (Allen and Sanchez-Sinencio 1984) shown in Fig. 3(*a*). The model includes finite (but large) open switch resistances with synchronous operation of the switches acting in a sequentially complementary fashion. The constant k represents the ratio of open-switch resistances. It will be assumed that the switches are governed by a PWM scheme with a duty ratio function that may, in general, depend upon the capacitor voltage (closed-loop duty ratio) or may be externally controlled to a constant value (open-loop duty ratio). We refer to the left-hand side switch as the 'input' switch and to the right-hand side switch as the 'output' switch.

The differential equations governing the capacitor voltage on each one of the two intervening structures are given by

Input switch closed and output switch open:

$$\frac{dv_c}{dt} = \frac{(v_1 - v_c)}{R_1 C} + \frac{(v_2 - v_c)}{kR_2 C} \tag{3.9}$$

Input switch open and output switch closed:

$$\frac{dv_c}{dt} = \frac{(v_1 - v_c)}{kR_1 C} + \frac{(v_2 - v_c)}{R_2 C} \tag{3.10}$$

Using the convention established in (2.3), (2.4), one may represent the underlying variable structure system in terms of a single differential equation with discontinuous right-hand side determined by the value of an ideal switch position function u, taking values on the discrete set $\{0, 1\}$. Such a dynamical model is described by

$$\frac{dv_c}{dt} = u\left[\frac{(v_1 - v_c)}{R_1 C} + \frac{(v_2 - v_c)}{kR_2 C}\right] + (1 - u)\left[\frac{(v_1 - v_c)}{kR_1 C} + \frac{(v_2 - v_c)}{R_2 C}\right] \tag{3.11}$$

with the switch position function regulated by means of a PWM scheme with duty ratio $\tau(v_c)$ given by

$$u = \begin{cases} 1, & \text{for } t_k \leqslant t < t_k + \tau(v_c)T \\ 0, & \text{for } t_k + \tau(v_c)T \leqslant t < t_k + T \end{cases} \tag{3.12}$$

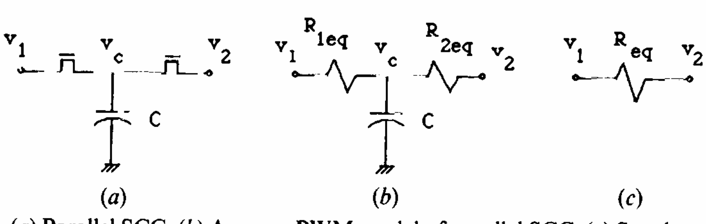

Figure 3. (*a*) Parallel SCC. (*b*) Average PWM model of parallel SCC. (*c*) Steady-state resistor equivalent model.

According to the results in § 2, the infinite frequency average model is readily obtained by using Filippov's geometric average model of system (3.11), (3.12), in which the discontinuous control function u is formally replaced by the prescribed duty ratio function τ. One then obtains

$$\frac{dv_c}{dt} = \tau(v_c)\left[\frac{(v_1 - v_c)}{R_1 C} + \frac{(v_2 - v_c)}{kR_2 C}\right] + [1 - \tau(v_c)]\left[\frac{(v_1 - v_c)}{kR_1 C} + \frac{(v_2 - v_c)}{R_2 C}\right] \tag{3.13}$$

The equivalent dynamical circuit leading to the average model (3.13) may be deduced by rewriting (3.13) as

$$\frac{dv_c}{dt} = \left[\frac{(v_1 - v_c)}{R_{1\text{eq}}(\tau, k)C} + \frac{(v_2 - v_c)}{R_{2\text{eq}}(\tau, k)C}\right] \tag{3.14}$$

with

$$R_{1\text{eq}}(\tau, k) = \left[\frac{kR_1}{\tau k + (1 - \tau)}\right], \quad R_{2\text{eq}}(\tau, k) = \left[\frac{kR_2}{\tau + k(1 - \tau)}\right] \tag{3.15}$$

where the argument v_c in the duty ratio function has been dropped just for convenience.

The resulting dynamical model (3.14), (3.15), shown in Fig. 3(*b*), constitutes an infinite frequency average PWM dynamical model, or simply Filippov's model, of the parallel SCC of Fig. 3(*a*). Notice that the basic rule derived in (3.1) is directly applicable.

Under the assumption of a constant duty ratio τ (traditionally this value is fixed at 0·5), the steady-state conditions of the circuit imply that the capacitor C is fully chrged and its branch may be considered as open. One then obtains the steady-state resistor equivalent of the parallel SCC (se Fig. 3(*c*)) with $R_{\text{eq}}(\tau, k)$ as given by

$$R_{1\text{eq}}(\tau, k) = R_{1\text{eq}}(\tau, k) + R_{2\text{eq}}(\tau, k) = \frac{kR_1}{\tau k + (1 - \tau)} + \frac{kR_2}{\tau + k(1 - \tau)} \tag{3.16}$$

If the duty ratio function t is a non-linear smooth function of the capacitor voltage, i.e., $\tau = \tau(v_c)$, then the equivalent steady-state model is still constituted by a resistor equivalent. Its value must, however, be deduced from the solution of (3.6), (3.7), with respect to v_c, and $dv_c/dt = 0$.

Example

Figure 4(*a*) shows several simulated average PWM trajectories of system (3.14), (3.15), on an appropriately scaled time axis, for several constant values of the duty ratio function τ. In thse simulations, standard switch resistance values and capacitance values were used (open-switch resistance 100 MΩ, closed-switch resistance 10 kΩ, C was taken as 1 pF).

The nature of the actual PWM controlled response of the dynamical model (3.11), (3.12) is, according to the results of § 2, constituted by a sliding regime which chatters above the average PWM controlled trajectories of system (3.14), (3.15). The existence of such a sliding motion is clearly portrayed in Fig. 4(*b*) where a simulated trajectory of the actual PWM controlled dynamical system (3.11), (3.12) is superimposed on the time response of the average PWM controlled system (3.14), (3.15), for a constant

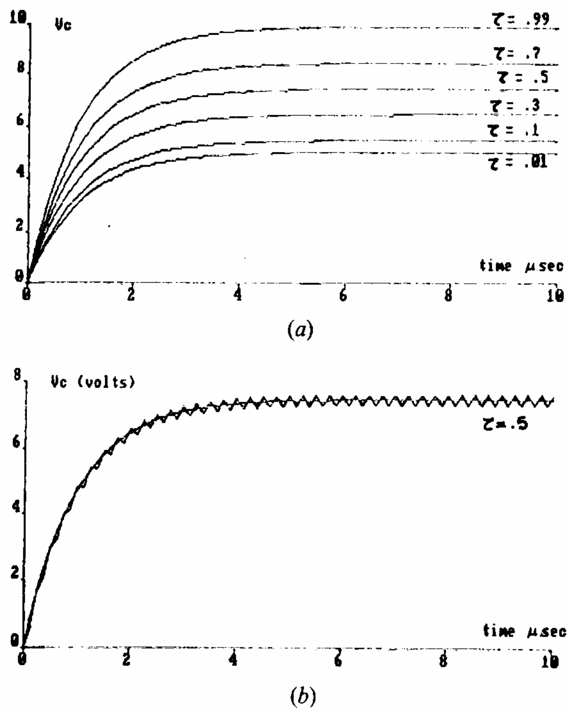

Figure 4. (*a*) Transient and steady-state response of parallel SCC average PWM dynamical model for several constant duty ratio values. (*b*) Superimposed actual and average PWM responses of parallel SCC.

duty ratio $\tau = 0{\cdot}5$. The computer programs for the PC version of the SIMNON package, used in obtaining these graphs, are presented in Apprendix B.

An ideal infinite open switch resistance model can be obtained by simply letting the parameter k be infinitely large in (3.15). The equivalent resistor values in the average PWM dynamical model of the parallel SCC are given by

$$\left.\begin{aligned} R_{1\text{eq}}(\tau) &= \lim_{k\to\infty} R_{1\text{eq}}(\tau, k) = R_1/\tau \\ R_{2\text{eq}}(\tau) &= \lim_{k\to\infty} R_{2\text{eq}}(\tau, k) = R_2/(1-\tau) \end{aligned}\right\} \qquad (3.17)$$

The dynamical model associated with this case exhibits an equivalent resistor of value R_1/τ in place of the controlled input switch and a resistor of value $R_2/(1-\tau)$ in place of the controlled output switch. This dynamical model makes perfect sense in the limits $\tau \to 0$ or $\tau \to 1$ since, in each case, the corresponding equivalent resistor on each side of the capacitor branch acts as an open resistance when its corresponding switch is assumed to be open and it acts as a pure resistance when its corresponding switch is closed. This is certainly a capability that the traditional SCC model simply does not have.

The steady-state resistor equivalent model for the parallel SCC is obtained from (3.16) as

$$\lim_{k\to\infty} R_{1\text{eq}}(\tau, k) = R_{1\text{eq}}(\tau) + R_{2\text{eq}}(\tau) = R_1/\tau + R_2/(1-\tau) \qquad (3.18)$$

Notice that this (static) resistor equivalent model precisely coincides with the idea of having a variable resistor smoothly changing its value according to the duty ratio. The fact that the model is frequency independent constitutes a significant departure from the traditional model.

3.3. *An average PWM series switched capacitor circuit model with non-ideal switches*

Consider the elementary series SCC model (Allen and Sanchez-Sinencio 1984) shown in Fig. 5(*a*). As before, the left-hand side switch is referred to as the 'input' switch and the right-hand side switch is referred to as the 'output' switch.

The differential equations governing the capacitor voltage on each one of the two intervening structures are given by

Input switch closed and output switch open:

$$\frac{d(v_c - v_2)}{dt} = \frac{(v_1 - v_c)}{kR_1C} + \frac{(v_2 - v_c)}{R_2C} \tag{3.19}$$

Input switch open and output switch closed:

$$\frac{d(v_c - v_2)}{dt} = \frac{(v_1 - v_c)}{R_1C} + \frac{(v_2 - v_c)}{kR_2C} \tag{3.20}$$

The underlying variable structure system is represented in terms of a single differential equation, with discontinuous right-hand side, by

$$\frac{d(v_c - v_2)}{dt} = u\left[\frac{(v_1 - v_c)}{kR_1C} + \frac{(v_2 - v_c)}{R_2C}\right] + (1-u)\left[\frac{(v_1 - v_c)}{R_1C} + \frac{v_2 - v_c)}{kR_2C}\right] \tag{3.21}$$

with u regulated by means of a PWM scheme with duty ratio $\tau(v_c)$ given by (3.12).

Using the results in § 2, the infinite frequency average model, is readily obtained as

$$\frac{d(v_c - v_2)}{dt} = \tau(v_c)\left[\frac{(v_1 - v_c)}{kR_1C} + \frac{(v_2 - v_c)}{R_2C}\right] + [1 - \tau(v_c)]\left[\frac{(v_1 - v_c)}{R_1C} + \frac{(v_2 - v_c)}{kR_2C}\right] \tag{3.22}$$

The equivalent dynamical circuit leading to the average model (3.22) may be deduced by rewriting (3.22) as (see Fig. 5(*b*))

$$\frac{d(v_c - v_2)}{dt} = \left[\frac{(v_1 - v_c)}{R_{1\text{eq}}(\tau, k)C} + \frac{(v_2 - v_c)}{R_{2\text{eq}}(\tau, k)C}\right] \tag{3.23}$$

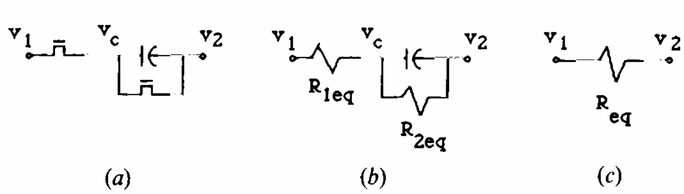

Figure 5. (*a*) Series SCC. (*b*) Average PWM model of series SCC. (*c*) Steady-state resistor equivalent model.

with

$$R_{1\text{eq}}(\tau, k) = \left[\frac{kR_1}{\tau k + (1 - \tau)}\right], \quad R_{2\text{eq}}(\tau, k) = \left[\frac{kR_2}{\tau + k(1 - \tau)}\right] \tag{3.24}$$

where the argument v_c in the duty ratio function has been dropped just for convenience. Notice once again that the basic rule of § 3.1 is still valid for this case.

The equivalent values replacing the non-ideal input and output switches precisely coincide with those derived for the parallel SCC model. This is as it should be, since, in steady-state conditions, both resistor-equivalent average models are topologically coincident.

In steady-state conditions and under the assumption of a constant duty ratio τ, the capacitor C is fully charged and its branch may be considered as open. One then obtains the steady-state resistor equivalent of the series SCC (see Fig. 5(*c*)) with $R_{\text{eq}}(\tau, k)$ as given by

$$R_{\text{eq}}(\tau, k) = R_{1\text{eq}}(\tau, k) + R_{2\text{eq}}(\tau, k) = \frac{kR_1}{\tau k + (1 - \tau)} + \frac{kR_2}{\tau + k(1 - \tau)} \tag{3.25}$$

As in the previous case, the same general remarks apply regarding non-linear smooth duty ratio functions and the sliding character of the actual PWM controlled response about the average PWM trajectories (similar computer programs to those of Appendix B can be used to illustrate this fact). Again, the ideal switch model (infinite open-switch resistance) can be obtained by simply letting the parameter k be infinitely large in (3.19), (3.20). The equivalent resistor values in the average PWM dynamical model and the steady-state resistor equivalent model for the series SCC are still given by equations (3.17) and (3.18), respectively.

3.4. *An average PWM parallel–series switched capacitor circuit model with non-ideal switches*

Consider the elementary parallel–series SCC model (Allen and Sanchez-Sinencio 1984) shown in Fig. 6(*a*). The left-hand side switch is the 'input' switch and the right-hand side switch is the 'output' switch. The differential equations, governing the capacitor voltage on each intervening structure, are given by

Input switch closed and output switch open:

$$\frac{d(v_1 - v_2)}{dt} = -\frac{(v_1 - v_2)}{R_1C_1} + \frac{C_2}{C_1}\frac{dv_2}{dt} + \frac{(v_2 - v_3)}{kR_2C_1} \tag{3.26}$$

Input switch open and output switch closed:

$$\frac{d(v_1 - v_2)}{dt} = -\frac{(v_1 - v_2)}{kR_1C_1} + \frac{C_2}{C_1}\frac{dv_2}{dt} + \frac{(v_2 - v_3)}{R_2C_1} \tag{3.27}$$

Directly using the results in § 2, the infinite frequency average PWM dynamical model is readily obtained as

$$\frac{d(v_1 - v_2)}{dt} = \tau\left[-\frac{(v_1 - v_2)}{R_1C_1} + \frac{(v_2 - v_3)}{kR_2C_1}\right] + (1 - \tau)\left[-\frac{(v_1 - v_2)}{kR_1C_1} + \frac{(v_2 - v_3)}{RC_1}\right] + \frac{C_2}{C_1}\frac{dv_2}{dt} \tag{3.28}$$

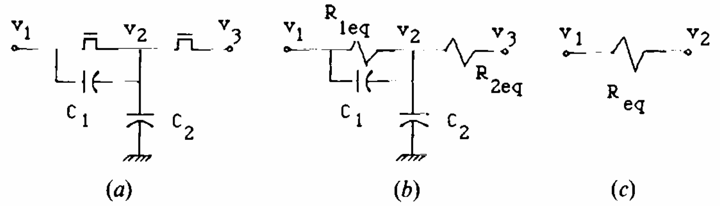

Figure 6. (*a*) Parallel–series SCC. (*b*) Average PWM model of parallel–series SCC. (*c*) Steady-state resistor equivalent model.

The equivalent dynamical circuit leading to the average model (3.28) may be deduced by rewriting this equation as (see Fig. 6(*b*))

$$\frac{d(v_1 - v_2)}{dt} = -\frac{(v_1 - v_2)}{R_{1\text{eq}}C_1} + \frac{(v_2 - v_3)}{kR_{2\text{eq}}C_1} + \frac{C_2}{C_1}\frac{dv_2}{dt} \tag{3.29}$$

with

$$R_{1\text{eq}}(\tau, k) = \left[\frac{kR_1}{\tau k + (1 - \tau)}\right], \quad R_{2\text{eq}}(\tau, k) = \left[\frac{kR_2}{\tau + k(1 - \tau)}\right] \tag{3.30}$$

Notice, once again, that the basic rule of § 3.1 is still valid for this case.

The equivalent values replacing the non-ideal input and output switches precisely coincide with those derived for the previous models. Thus far all resistor-equivalent average models are steady-state topologically coincident.

In steady-state conditions and under the assumption of a constant duty ratio τ, the capacitors are fully charged and their branches may be considered as open. One then obtains the steady-state resistor equivalent of the parallel–series SCC (see Fig. 6 (*c*)) with $R_{\text{eq}}(\tau, k)$ given by

$$R_{\text{eq}}(\tau, k) = R_{1\text{eq}}(\tau, k) + R_{2\text{eq}}(\tau, k) = \frac{kR_1}{\tau k + (1 - \tau)} + \frac{kR_2}{\tau + k(1 - \tau)} \tag{3.31}$$

3.5. *An average PWM bilinear switched capacitor circuit model with non-ideal switches*

Figure 7 (*a*) shows the bilinear SCC model (Allen and Sanchez-Sinencio 1984). The relations describing such a model, including non-infinite open-switch resistances, are given by

Switches A and B closed, switches C and D open:

$$\frac{d(v_3 - v_4)}{dt} = \frac{(v_1 - v_3)}{R_1C} + \frac{(v_2 - v_3)}{kR_2C} = \frac{(v_4 - v_2)}{R_1C} + \frac{(v_4 - v_1)}{kR_2C} \tag{3.32}$$

Switches A and B open, switches C and D closed:

$$\frac{d(v_3 - v_4)}{dt} = \frac{(v_1 - v_3)}{kR_1C} + \frac{(v_2 - v_3)}{R_2C} = \frac{(v_4 - v_2)}{kR_1C} + \frac{(v_4 - v_1)}{R_2C} \tag{3.33}$$

Filippov's PWM average dynamical model (Fig. 7 (*b*)) is obtained in accordance with the results of § 2 and using the same general procedure presented in the three

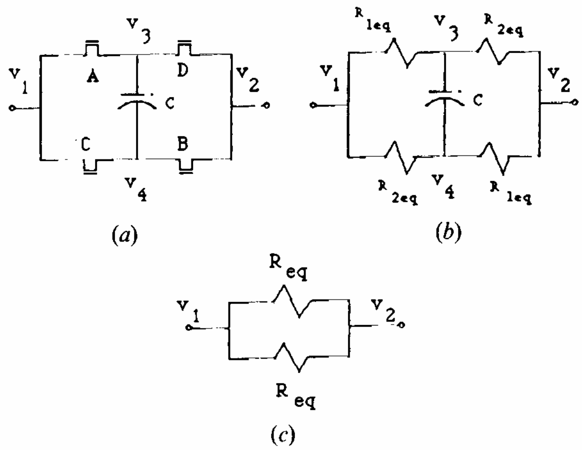

Figure 7. (*a*) Bilinear SCC. (*b*) Average PWM model of bilinear SCC. (*c*) Steady-state resistor equivalent model.

previous cases, as

$$\frac{d(v_3 - v_4)}{dt} = \frac{(v_1 - v_3)}{R_{1\mathrm{eq}}C} + \frac{(v_2 - v_3)}{R_{2\mathrm{eq}}C} = \frac{(v_4 - v_2)}{R_{1\mathrm{eq}}C} + \frac{(v_4\ v_1)}{R_{2\mathrm{eq}}C} \tag{3.34}$$

with

$$R_{1\mathrm{eq}}(\tau, k) = \left[\frac{kR_1}{\tau k + (1 - \tau)}\right], \quad R_{2\mathrm{eq}}(\tau, k) = \left[\frac{kR_2}{\tau + k(1 - \tau)}\right] \tag{3.35}$$

i.e., that is, the same general rule of § 3.1 applies to each controlled switch of the circuit.

Under steady-state conditions, the capacitor branch is open and the resistor equivalent model of the bilinear SCC model is constituted by the parallel of a resistor of value $R_{1\mathrm{eq}}(\tau, k) + R_{2\mathrm{eq}}(\tau, k)$ with itself (Fig. 7 (*c*)). In other words

$$\begin{aligned} R_{\mathrm{eq}}(\tau, k) &= 0{\cdot}5\,[R_{1\mathrm{eq}}(\tau, k) + R_{2\mathrm{eq}}(\tau, k)] \\ &= 0{\cdot}5\left[\frac{kR_1}{\tau k + (1 - \tau)} + \frac{kR_2}{\tau + k(1 - \tau)}\right] \end{aligned} \tag{3.36}$$

If an infinite open switch resistance can be assumed then the equivalent resistor model of the SCC arrangement is simply given by

$$R_{\mathrm{eq}}(\tau) = 0{\cdot}5\,[R_{1\mathrm{eq}}(\tau) + R_{2\mathrm{eq}}(\tau)] := 0{\cdot}5\,[R_1/\tau + R_2/(1 - \tau)] \tag{3.37}$$

3.6. *A rederivation of the traditional resistor equivalent model*

Here we rederive the traditional model using elementary concepts from singular perturbation theory of dynamical systems described by differential equations (Kokotovic *et al.* 1986), as applied to the basic parallel SCC configuration. The other configurations can be similarly treated.

Consider the basic parallel SCC configuration of Fig. 3 (*a*), with switching taking place once within a duty cycle of $T\,(=: 1/f)$ seconds (the duty ratio is hence assumed to be $T/2$). Under the assumption of ideal (infinite open-switch resistance), and idential input and output switches, the governing differential equations for each intervening structure can be written as (see (3.9), (3.10))

$$\varepsilon\, dv_c/dt = (v_1 - v_c) \tag{3.38}$$

$$\varepsilon\, dv_c/dt = (v_2 - v_c) \tag{3.39}$$

with $\varepsilon = R_1 C$ Assuming ε to be very small as compared with $T/2$, the first equation leads, in the slow time scale, to the equilibrium condition $v_1 = v_c$. It is easy to see that in the fast time scale, $t' = t/\varepsilon$, v_c asymptotically approaches v_1 with scaled time constant equals to 1. A singular perturbation of (3.38) (i.e. letting $\varepsilon = 0$) indicates that the capacitor C is already charged to voltage v_1 (total stored charge $q = Cv_1$) when the switch is about to change to its new position in the middle of the switching period T. Using the same procedue and assumptions, we conclude that v_c acquires voltage v_2 by the end of the switching cycle. The net transferred charge from the left-hand side terminal to the right-hand side terminal of the circuit is then $C(v_1 - v_2)$. The pulsed current begins and ends its flow, from the first to the second circuit terminal, in a period of T seconds. The average current is then approximated by $C(v_1 - v_2)/T$ that is, the capacitor acts as a resistor of value $R_{eq} = T/C = 1/fC$.

4. Conclusions

Fundamental SCC models have been examined from the viewpoint of variable structure systems and their associated sliding regimes using Filippov's average dynamics concept. A dynamical model approach for SCC analysis has been proposed which includes finite open-switch resistances. The structural changes were assumed to be operated under a suitable arrangment of synchronous complementary switches regulated by a pulse-width modulation strategy. The underlying duty ratio was considered either as a constant, or as state-dependent feedback function. Independently of the nature of the duty ratio, the results for the basic SCC models indicate that the general rule for building a dynamical infinite frequency average model of the discontinuously controlled network, consists of formally replacing the controlled switches by duty ratio-dependent resistors. The average models capture all essential transient and steady-state features of the actual discontinuous dynamical systems by identifying themselves with the ideal sliding dynamics related to the associated sliding motion exhibited by the actual PWM controlled system responses. This fact alone may represent a crucial advantage in the transient and steady-state analysis as well as in design tasks related to PWM controlled SCC. The steady-state behaviour of the basic SCC models (series, parallel, parallel-series and bilinear SCC) yield duty-ratio resistor equivalent models of a more flexible and, possibly, a more useful nature than the traditional model.

Acknowledgements

The first author is highly indebted to Dr K. K. D. Young of the National Livermore Laboratory for kindly suggesting to the author to conduct research in this field. The research was supported by the Consejo de Desarrollo Científico, Humanístico y Tecnológico of the Universidad de Los Andes under Research Grant I-325-90.

Appendix A

Consider the n-dimensional variable structure system

$$\frac{dx}{dt} = u f_1(x) + (1 - u) f_2(x) \tag{A 1}$$

with

$$u = \begin{cases} 1, & \text{for } s(x) > 0 \\ 0, & \text{for } s(x) < 0 \end{cases} \tag{A 2}$$

with $S = \{x: s(x) = 0\}$ being a smooth regular $(n - 1)$-dimensional manifold defined in the open set $\mathbb{N}$ of $\mathbb{R}^n$, with locally nowhere zero gradient $\partial s/\partial x$, denoted from now on by ds.

Definition A.1

A sliding regime is said to locally exist on the manifold S whenever the following conditions are satisfied (see Utkin 1978, Sira-Ramírez 1989 a):

$$\begin{aligned} \lim_{s \to +0} \frac{ds}{dt} &=: \lim_{s \to +0} \langle ds, f_1(x) \rangle < 0 \\ \lim_{s \to -0} \frac{ds}{dt} &=: \lim_{s \to -0} \langle ds, f_2(x) \rangle > 0 \end{aligned} \tag{A 3}$$

where $\langle ds, f_i \rangle$ is a shorthand notation for the chain rule $[\partial s/\partial x]^{\mathrm{T}} f_i(x)$.

Proposition A.1 (Sira-Ramírez 1989)

If a sliding regime locally exists on S then, necessarily, for all $x \in S$ where the sliding regime exists, the following transversality condition is satisfied:

$$\langle ds, f_1(x) - f_2(x) \rangle < 0 \tag{A 4}$$

Proof □

Obvious upon subtracting the expressions in (2.7) evaluated on S.

Being a necessary condition, (A 4) determines the extent of a region which properly contains the region of existence of a sliding regime on the surface S provided the switching policy (A 2) is used on (A 1).

If a sliding motion locally exists on S, the state trajectories undergo a chattering motion about the switching (sliding) manifold. An idealized version of such a motion is obtained by assuming that the trajectories smoothly evolve on the sliding manifold. To describe such an ideal sliding dynamics two general methods have been proposed: Utkin's method, based on the so-called method of the equivalent control and the method of Filippov's geometric averaging (Filippov 1988).

The method of the equivalent control is based on defining a smooth control function, called the equivalent control and denoted by $u^{\mathrm{EQ}}(x)$, locally defined along S, for which the following invariance conditions (Utkin 1978) are satisfied:

$$\frac{ds}{dt} = 0 \quad \text{on } s = 0 \tag{A 5}$$

Using our shorthand notation, these conditions are expressed as

$$\frac{ds}{dt} = \langle ds, u^{\mathrm{EQ}} f_1(x) + [1 - u^{\mathrm{EQ}}] f_2(x) \rangle = 0 \quad \text{on } s(x) = 0 \tag{A 6}$$

The geometric interpretation of (A 6) should be clear: the vector field $u^{EQ}(x)f_1(x) + [1 - u^{EQ}(x)]f_2(x)$ must be locally orthogonal to the surface gradient ds at every point $x \in S$ located in the region of existence of the sliding mode. From (A 6) one finds the unique value of the equivalent control, for $x \in S$, as

$$u^{EQ}(x) = -\frac{\langle ds, f_2(x)\rangle}{\langle ds, f_1(x) - f_2(x)\rangle} \tag{A 7}$$

To see that $u^{EQ}(x)$ is indeed unique (Sira-Ramírez 1989 a), assume $\mu(x)$ is a different smooth function also satisfying (A 6) that is, $\langle ds, \mu(x)f_1(x) + [1 - \mu(x)]f_2(x)\rangle = 0$. Subtracting from (A 6) the obtained expression with $\mu(x)$ one obtains $\langle ds, [u^{EQ}(x) - \mu(x)]f_1(x) - [u^{EQ}(x) - \mu(x)]f_2(x)\rangle = [u^{EQ}(x) - \mu(x)]\langle ds, f_1(x) - f_2(x)\rangle = 0$. Since necessarily $\langle ds, f_1(x) - f_2(x)\rangle < 0$, it follows that $u^{EQ}(x) = \mu(x)$ which is a contradiction.

When (A 7) is formally substituted in place of the discontinuous control u in (A 1), the obtained dynamics, constrained to evolve on S, is known as the ideal sliding dynamics (Utkin 1978). Its explicit expression is readily obtained as

$$\frac{dx}{dt} = \frac{[-\langle ds, f_2(x)\rangle f_1(x) + \langle ds, f_1(x)\rangle f_2(x)]}{\langle ds, f_1(x) - f_2(x)\rangle} \tag{A 8}$$

Theorem A.1 (Sira-Ramírez 1989 a)

Let the transversality condition (A 4) be locally satisfied on S. The necessary and sufficient condition for the local existence of a sliding regime of (a 1), (A 2) on S, is that the equivalent control $u^{EQ}(x)$ satisfies

$$0 < u^{EQ}(x) < 1 \quad \text{on } s(x) = 0 \tag{A 9}$$

Proof

Suppose (A 9) holds locally valid on S. Then inverting the expression in (A 7) and according to (A 9) one has

$$-\frac{\langle ds, f_1(x) - f_2(x)\rangle}{\langle ds, f_2(x)\rangle} > 1, \quad x \in S \tag{A 10}$$

that is,

$$-\frac{\langle ds, f_1(x)\rangle}{\langle ds, f_2(x)\rangle} > 0, \quad x \in S$$

Hence, $\langle ds, f_1(x)\rangle$ and $\langle ds, f_2(x)\rangle$ have opposite signs on S. Since the numerator of expression (A 10) is positive, according to the validity of the transversality condition (A 4), then $\langle ds, f_2(x)\rangle$ is necessarily positive and hence $\langle ds, f_1(x)\rangle < 0$, locally on S. Therefore there exists an open neighbourhood surrounding S where conditions (A 3) remain valid. A sliding regime locally exists on S.

To prove necessity, suppose a sliding regime locally exists on S and conditions (A 3) are locally valid on S. Then there exists a smooth positive function $0 < \mu(x) < 1$, such that $\mu(x)\langle ds, f_1(x)\rangle + [1 + \mu(x)]\langle ds, f_2(x)\rangle = 0$. Solving for $\mu(x)$ and by virtue of the uniqueness of the equivalent control then $\mu(x) = u^{EQ}(x)$ and the result follows. □

In Filippov's geometric averaging method of solution (Filippov 1988), the basic result can be phrased as in the following theorem.

Theorem A.2

A sliding regime locally exists on S for system (A 1), (A 2) if an only if there exists a smooth scalar function $0 < \mu(x) < 1$ defined on S such that $S(x)$ is a local integral manifold for the smooth dynamics

$$\frac{dx}{dt} = \mu(x) f_1(x) + [1 - \mu(x)] f_2(x) \tag{A 11}$$

Proof

Filippov's basic result (Filippov 1988) is now an immediate consequence of the previous theorem.

One immediately concludes that Filippov's convex combination function $\mu(x)$ is none other than the equivalent control and that therefore Filippov's average dynamics coincides with the ideal sliding dynamics. This result is not at variance with those of Utkin (1978). The crucial reason being the 'artificial' control-linearization procedure (elsewhere called pre-linearization) by which a general variable structure system (say, of the form $dx/dt = F(x, u)$) is written in the form (A 1), (A 2) (with $F(x, 1) = f_1(x)$ and $F(x, 0) = f_2(x)$). For the control-linearized system, both Utkin's and Filippov's approaches yield the same results (otherwise, it is obvious that $\mu(x)F(x, 1) + [1 - \mu(x)]F(x, 0) = F(x, u^{EQ}(x))$ does not necessarily imply $u^{EQ}(x) = \mu(x)$).

Appendix B

SIMNON program listing for average PWM controlled parallel SCC

```
Continuous system SCC
state vc                                   "State variable
der dvc                                    "Derivative definition
dvc=(v1-vc)/(C*R1EQ)+(v2-vc)/C*R2EQ)       "Average PWM model
R1EQ=k*R1/((1-tau)+k*tau))                 "Equivalent resistor R1 (W)
R2EQ=k*R2/(tau+k*(1-tau))                  "Equivalent resistor R2 (W)
v1:10                                      "Voltage V1 (Volts)
v2:5                                       "Voltage V2 (Volts)
R1:10E3                                    "Closed-switch resistance R1
R2:10E3                                    "Closed-switch resistance R2
k:10E3                                     "Open to closed-switch
                                            resistance ratio
C:100E-12                                  "Capacitance C (farads)
tau:0·5                                    "Duty ratio
end
```

SIMNON program listing for actual PWM controlled parallel SCC

```
Continuous system SCC
state vc                                   "State variable
time t                                     "Time definition
der dvc                                    "Derivative definition
st1=(v1-vc)/(R1*C)+(v2-vc)/(k*R2*C)        "Structure #1
st2=(v1-vc)/(k*R1*C)+(v2-vc)/R2*C)         "Structure #2
dvc=u*st1+(1-u)*st2                        "Actual PWM controlled model
u=if int(t*fr)+tau>t*fr then 1 else 0      "PWM control law
v1:10                                      "Voltage V1 (Volts)
```

```
v2:5              "Voltage V2 (Volts)
R1:10E3           "Closed-switch resistance R1 (W)
R2:10E3           "Closed-switch resistance R2 (W)
k:10E3            "Open to closed-switch
                   resistance ratio
C:100E-12         "Capacitance C (farads)
tau:0·5           "Duty ratio
fr:4E6            "Switching frequency
end
```

References

The origins and early work on switched capacitor networks, 1984, *I.E.E.E. Circuits Systems Magazine*, **6.**

Allen, P. E., and Sanchez-Sinencio, E., 1984, *Switched Capacitor Circuits* (New York: Van Nostrand Reinhold).

Caves, J. T., Copeland, M. A., Rahim, C. F., and Rosenbaum, S. D., 1977, *I.E.E.E. J. Solid-State Circuits*, **12,** 592.

Elmquist, H., Åström, K. J., and Schonthal, T., 1987, *Simnon; User's Guide for MS-DOS Computers.* Department of Automatic Control, Lund Institute of Technology, Student-litteratur, Lund, Sweden.

Filippov, A. F., 1988, *Differential Equations with Discontinuous Righthand side* (Dordrecht, The Netherlands: Kluwer).

Hirano, K., and Nishimura, S., 1972, *I.E.E.E. Trans. Circuit Theory*, **19,** 253.

Hostika, B. J., Brodersen, R. W., and Gray, P. R., 1977, *I.E.E.E. J. Solid-State Circuits*, **12,** 600.

Koehler, J. A., 1969, *I.E.E.E. J. Solid-State Circuits*, 225.

Kokotovic, P. V., Khalil, H. K., and O'Reilly, J., 1986, *Singular Perturbation Methods in Control: Analysis and Design* (New York: Academic Press).

Liou, M., and Kuo, Y. L., 1979, *I.E.E.E. Trans. Circuits Systems*, **26,** 213.

Moschytz, G. S., 1984, *MOS Switched-Capacitor Filters: Analysis and Design* (New York: I.E.E.E. Press).

Orfei, R., and Pallotino, G. V., 1973, *Int. J. Circuit Theory Applic.* **1,** 373.

Scanlan, S. O., 1981, *I.E.E.E. Trans. Circuits Systems*, **28,** 85.

Sira-Ramírez, H., 1987, *I.E.E.E. Trans. Circuits Systems*, **34,** 919; 1989 a, *Int. J. Control*, **48,** 1359; 1989 b, *I.E.E.E. Trans. autom. Control*, **34,** 184; 1990, Nonlinear pulse-width modulations controller design. *Variable Structure Control for Robotics and Aerospace Applications*, edited by K. K. D. Young (Amsterdam: Elsevier), to be published.

Skoog, R. A., and Blankenship, G. L., 1970, *I.E.E.E. Trans. autom. Control*, **15,** 300.

Tsividis, Y. P., 1979, *I.E.E.E. Trans. Circuits Systems*, **26,** 935; 1983, *Int. J. Circuit Theory Applic.* **11,** 342.

Utkin, V. L., 1978, *Slidng Regimes and Their Applications in Variable Structure Systems* (Moscow: Mir Publishers).