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On the sliding mode control of nonlinear systems *

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Abstract: Implications of Fliess's differential algebraic approach for the study of systems dynamics are explored in the context of sliding regimes of nonlinear systems. This viewpoint directly leads to the possibilities of dynamically generated sliding mode controllers, characterizing sliding motions which are generically devoid of undesirable input chattering. A definite connection between controllability and the possibility of creation of 'higher order' sliding regimes is also established using differential algebra.

Keywords: Nonlinear systems; sliding regimes; differential algebraic systems.

1. Introduction

Sliding mode control of dynamical systems has undegone a wealth of theoretical and practical developments over the last 50 years. Recently, a thorough chronological collection of journal articles, and conference presentations, has been gathered by Professor S.V. Emelyanov [6]. Detailed and informative surveys have also been produced by Utkin over the years (see, for instance, [30]). Background on the subject may be acquired from the books written by Emelyanov [4], Utkin [31,32], Itkis [16] or Bühler [2].

Recent developments in nonlinear systems include the use of *Differential Algebra* for the formulation, understanding, and conceptual solution of long standing problems in automatic control. Contributions in this area are fundamentally due to Prof. M. Fliess [7,8]. Some other pioneering works were also independently presented by Pommaret [18,19].

Sliding mode control, and discontinuous feedback control, in general, have also received the influence of the differential algebraic approach. A seminal contribution, in the use of differential algebraic results for sliding mode control, was given by Fliess and Messager [14]. In that article, a system was presented for which no continuous feedback controller can achieve asymptotic stability of the motion to the origin, while a discontinuous feedback controller, based on sliding modes, does result in asymptotically stable behavior, modulo some small chattering. These basic results were later extended, and used, in several case studies, by Sira-Ramírez and his colleages in [20–23] where smoothed sliding regimes were proposed. A most interesting article dealing with multivariate linear systems and the possibilities of regulation of non-minimum phase systems is that of Fliess and Messager [15]. Extensions to pulse-width-modulation and pulse-frequency-modulation control strategies have been contributed by Sira-Ramírez [24–26].

This article presents a reapproachment to sliding mode control theory, from the perspective of Differential Algebra. Section 2 presents some of the implications of this new trend in sliding mode controller synthesis. In particular, a connection is established between 'higher order' sliding motions and controllability. The issue of robustness is also explored in some detail. Some directions for further research are suggested at the end, in the conclusion section.

2. A differential algebraic approach to sliding mode control of nonlinear systems

2.1. Nonlinear controlled dynamics and sliding regimes

In this section, for the basic definitions and results, we closely follow the contents of Fliess's contributions [7,8].

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Definition 2.1. Consider an ordinary differential field k of characteristic zero. A *system* is any finitely generated differential field extension of k, denoted by K/k.

Let u be a *differential transcendence element* of the system K/k. u is then a *differential indeterminate* representing the input to the system. By itself, u is then assumed not to satisfy any algebraic differential equation with coefficients in k. We say that u qualifies as a *differential transcendence basis*, of K/k.

The field extension $k\langle u\rangle$ denotes the smallest differential field containing both k and u. The field extension $k\langle u\rangle$ is also referred to as the field *generated* by k and u.

Definition 2.2. A *dynamics* is defined as a *finitely generated differentially algebraic extension* $K/k\langle u\rangle$ of the differential field $k\langle u\rangle$.

It is well known that if u is a differential transcendence basis of K/k, then the extension $K/k\langle u\rangle$ is differentially algebraic.

Proposition 2.3. *Suppose* $x=(x_1, x_2, \ldots, x_n)$ *is a* nondifferential transcendence basis *of* $K/k\langle u\rangle$. *Then the derivatives* $\mathrm{d}x_i/\mathrm{d}t$ $(i=1,\ldots,n)$ *are* $k\langle u\rangle$*-algebraically dependent on the components of* x.

The proof is immediate.

One of the consequences of the last result, obtained in [8], is that a more general and natural representation of nonlinear systems requires *implicit algebraic differential equations*. Indeed, from the above proposition, it follows that there exist exactly n polynomial differential equations with coefficients in k, of the form

$$P_i(\dot{x}_i, x, u, \dot{u}, \ldots, u^{(\alpha)})=0, \quad i=1,\ldots,n, \tag{2.1}$$

implicitly describing the controlled dynamics.

It has been shown by Fliess and Hassler [12] that such implicit representations are not entirely unusual in physical examples. The more traditional representation of the state equations, known as *normal forms* is recovered, in a local fashion, under the assumption that such polynomials locally satisfy the following rank condition:

$$\operatorname{rank}\begin{bmatrix} \dfrac{\partial P_1}{\partial \dot{x}_1} & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \dfrac{\partial P_n}{\partial \dot{x}_n} \end{bmatrix} = n. \tag{2.2}$$

The time derivatives of the x_i's may then be, locally, solved for as

$$\dot{x}_i = p_i(x, u, \dot{u}, \ldots, u^{(\alpha)})=0, \quad i=1,\ldots,n. \tag{2.3}$$

The representation (2.3) is now known as the *generalized state representation* of a nonlinear dynamics.

Consider a (nonlinear) dynamics $K/k\langle u\rangle$. Let, furthermore, $\zeta=(\zeta_1,\ldots,\zeta_n)$ be a *non-differential transcendence basis* for K, i.e., the (non-differential) *transcendence degree* of $K/k\langle u\rangle$ is, then, assumed to be n.

Definition 2.4. A *first order sliding surface* is any element σ of the dynamics $K/k\langle u\rangle$ such that its time derivative $\mathrm{d}\sigma/\mathrm{d}t$ is not k-algebraic and it is $k\langle u\rangle$-algebraically dependent on ζ. That is, there exists a polynomial S over k such that

$$S(\dot{\sigma}, \zeta, u, \dot{u}, \ldots, u^{(\beta)})=0. \tag{2.4}$$

Remark. A more traditional definition of sliding surface coordinate function is related to the fact that no input signals, nor any of its time derivatives, were customarily allowed to be part of the expression defining such a sliding surface candidate. In this unnecessarily restricted sense, the sliding surface was allowed to be an (algebraic) function of the state components only. Moreover, for systems in 'Kalman form', described by a state vector x, the time derivative of the sliding surface was required to be algebraically dependent only on x and u. Hence, all the resulting sliding mode controllers were, necessarily, of static nature. The differential algebraic approach naturally points to the possibilities of dynamical sliding mode controllers, specially in nonlinear systems where elimination of input derivatives may not be possible at all (see Fliess et al. [13], for a physical example of this nature).

One generalizes the above definition by considering 'higher order' sliding surface candidates.

Definition 2.5. A *p-th order sliding surface* is any element σ of the dynamics $K/k\langle u\rangle$ such that its p-th order time derivative is $k\langle u\rangle$-algebraically dependent on ζ and its lower order derivatives. That is, there exists a polynomial Σ over k such that

$$\Sigma(\sigma^{(p)}, \ldots, \sigma, \zeta, u, \dot{u}, \ldots, u^{(\gamma)}) = 0. \tag{2.5}$$

This definition gives rise to the possibilities of a smoothed asymptotic approach to the zero 'level set' of the sliding surface σ through a discontinuous feedback policy. The implications will be explored in detail in Section 2.3, below. Notice that the integer p is not necessarily the first higher order time derivative of σ for which a $k\langle u\rangle$-algebraic dependence on ζ may be established. Thus, a p-th order sliding surface candidate might have also qualified as a lower order sliding surface candidate.

Suppose σ is a first order sliding surface candidate. Imposing on σ a discontinuous sliding dynamics of the form

$$\dot{\sigma} = -W \operatorname{sign}(\sigma), \tag{2.6}$$

one obtains, from (2.4), an *implicit dynamical sliding mode controller* given by

$$S(-W \operatorname{sign}(\sigma), \zeta, u, \dot{u}, \ldots, u^{(\beta)}) = 0 \tag{2.7}$$

which is to be viewed as an implicit, time-varying, discontinous ordinary differential equation for the control input u.

The two 'structures' associated to the underlying variable structure control system are represented by the pair of implicit dynamical controllers:

$$S(-W, \zeta, u, \dot{u}, \ldots, u^{(\beta)}) = 0 \quad \text{for } \sigma > 0, \tag{2.8a}$$

$$S(W, \zeta, u, \dot{u}, \ldots, u^{(\beta)}) = 0 \quad \text{for } \sigma < 0, \tag{2.8b}$$

each one valid, respectively, on one of the 'regions': $\sigma > 0$ and $\sigma < 0$. Precisely on the condition $\sigma = 0$ neither one of the control structures is valid.

We formally define the *equivalent control dynamics* as the dynamical state feedback control law obtained by letting $d\sigma/dt$ become zero in (2.4), and considering the resulting implicit differential equation for u:

$$S(0, \zeta, u_{EQ}, \dot{u}_{EQ}, \ldots, u_{EQ}^{(\beta)}) = 0. \tag{2.9}$$

Suppose now that in (2.4), $\partial S/\partial(d\sigma/dt) \neq 0$, then one locally obtains

$$\dot{\sigma} = s(\zeta, u, \dot{u}, \ldots, u^{(\beta)}) \tag{2.10}$$

and the corresponding dynamic sliding mode controller, complying with (2.6), is given by

$$s(\zeta, u, \dot{u}, \ldots, u^{(\beta)}) = -W \operatorname{sign}(\sigma). \tag{2.11}$$

If, furthermore, $\partial s/\partial u^{(\beta)}$ is non-zero, one locally obtains an explicit time-varying state space representation for the dynamical sliding mode controller (2.11), in the form

$$\dot{u}_1 = u_{i+1}, \quad i = 1, 2, \ldots, \beta - 1, \tag{2.12a}$$

$$\dot{u}_\beta = \theta(u_1, \ldots, u_\beta, \zeta, W\operatorname{sign}(\sigma)), \tag{2.12b}$$

$$u = u_1. \tag{2.12c}$$

All discontinuities arising from the bang–bang control policy are seen to be confined to the highest derivative of the control input through the nonlinear function θ. The output u of the dynamical controller is clearly the outcome of β integrations performed on such discontinuous time derivative of u and, for this reason, the signal u, emerging from the controller, is sufficiently smoothed out.

2.2. *Dynamical sliding regimes based on Fliess's generalized controller canonical form*

The following theorem plays a fundamental role in the study of systems dynamics from the differential algebraic approach [8].

Theorem 2.6. *Let $K/k\langle u\rangle$ be a dynamics. Then, there exists an element $\xi \in K$ such that $K = k\langle u, \xi\rangle$, i.e., such that K coincides with the smallest field generated by the indeterminates u and ξ.*

The (nondifferential) transcendence degree n of $K/k\langle u\rangle$ is the smallest integer n such that

$\xi^{(n)}$ is $k\langle u\rangle$-algebraically dependent on

$\xi, \mathrm{d}\xi/\mathrm{d}t, \ldots, \mathrm{d}^{(n-1)}\xi/\mathrm{d}t^{(n-1)}$.

We let

$$q_1 = \xi, \quad q_2 = \mathrm{d}\xi/\mathrm{d}t, \ \ldots, \ q_n = \mathrm{d}^{(n-1)}\xi/\mathrm{d}t^{(n-1)}.$$

It follows that $q = (q_1, \ldots, q_n)$ also qualifies as a (non-differential) transcendence basis of $K/k\langle u\rangle$. One, hence, obtains a nonlinear generalization of the controller canonical form, known as the Global Generalized Controller Canonical Form (GGCCF) [8]:

$$\frac{\mathrm{d}}{\mathrm{d}t} q_i = q_{i+1}, \quad i = 1, 2, \ldots, n-1, \tag{2.13a}$$

$$C(\dot{q}_n, q, u, \dot{u}, \ldots, u^{(\alpha)}) = 0, \tag{2.13b}$$

where C is a polynomial with coefficients in k. If one can locally solve for the time derivative of q_n in the last equation, one locally obtains an explicit system of first order differential equations, known as the *Local Generalized Controller Canonical Form* (LGCCF):

$$\frac{\mathrm{d}}{\mathrm{d}t} q_i = q_{i+1}, \quad i = 1, 2, \ldots, n-1, \tag{2.14a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} q_n = c(q, u, \dot{u}, \ddot{u}, \ldots, u^{(\alpha)}). \tag{2.14b}$$

The element $q_1 = \xi$ is known as the *differential primitive element* [8]. Its physical meaning should be adscribed to that of the particular element in the dynamics which is of utmost interest to regulate. An output of the system, or a particular tracking error, often, but not always, constitute natural candidates for differential primitive elements. A simple rank test on the gradients (with respect to the state) of the first $n-1$ time derivatives of a particular candidate may be used to establish the $k\langle u\rangle$-independence of these time derivatives. Such a simple test reveals whether or not the chosen element qualifies as a differential primitive element (see [20–23] for details and examples).

Remark. We assume throughout that $\alpha \geqslant 1$. The case $\alpha = 0$ corresponds to that of exactly linearizable systems under state coordinate transformations and static state feedback. One may still obtain the same smoothing effect of the dynamical sliding mode controllers that we propose in this article, by considering a suitable *prolongation* of the input space. This is accomplished by successively considering the 'extended system' (see Nijmeijer and Van der Schaft [17]) of the original one, and proceeding to use the same differential primitive element yielding the Generalized Controller Canonical Form of the given smaller dimensional system.

The preceding general results on canonical forms for nonlinear systems have an immediate consequence in the definition of sliding surfaces for stabilization and tracking problems in nonlinear systems.

Consider the following *sliding surface coordinate function*, expressed in the generalized phase coordinates q previously defined:

$$\sigma = c_1 q_1 + \cdots + c_{n-1} q_{n-1} + q_n \tag{2.15}$$

where the scalar coefficients c_i $(i = 1, \ldots, n-1)$ are chosen in such a manner that the following polynomial, $p(\lambda)$, in the complex variable λ, is Hurwitz:

$$p(\lambda) = c_1 + c_2\lambda + \cdots + c_{n-1}\lambda^{n-2} + \lambda^{n-1}. \tag{2.16}$$

Imposing on the sliding surface coordinate function σ the discontinuous dynamics

$$\dot{\sigma} = -W \operatorname{sign}(\sigma), \tag{2.17}$$

then the trajectories of σ are seen to exhibit, in finite time T given by $T = W^{-1}\,|\sigma(0)|$, a sliding regime on $\sigma = 0$. Substituting on (2.17) the expression (2.15) for σ, and using (2.14), one obtains, after some straightforward algebraic manipulations, the following dynamical implicit sliding mode controller:

$$\begin{aligned} c(q, u, \dot{u}, \ddot{u}, \ldots, u^{(\alpha)}) &= -c_1 q_2 - \cdots - c_{n-1} q_n \\ &\quad - W \operatorname{sign}[c_1 q_1 + \cdots + c_{n-1} + q_n]. \end{aligned} \tag{2.18}$$

Evidently, under ideal sliding conditions $\sigma = 0$, the variable q_n no longer qualifies as a state variable for the system since it is expressible as a linear combination of the remanining states and, hence, q_n is no longer a non-differentially transcendental element of the field extension K. The ideal (autonomous) closed loop dynamics may then be expressed in terms of a reduced non-differential transcendence basis K/k which only includes the remaining $n-1$ phase coordinates associated to the original differential primitive element. This leads to the following *ideal sliding dynamics*:

$$\frac{\mathrm{d}}{\mathrm{d}t} q_i = q_{i+1}, \quad i = 1, 2, \ldots, n-2, \tag{2.19a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} q_{n-1} = -c_1 q_1 - \cdots - c_{n-1} q_{n-1}. \tag{2.19b}$$

The characteristic polynomial of (2.19) is evidently given by (2.16) and, hence, the (reduced) autonomous closed loop dynamics is asymptotically stable to zero. Notice that by virtue of (2.15), the condition $\sigma = 0$, and the asymptotic stability of (2.19), q_n also tends to zero in an asymptotically stable fashion.

The *equivalent control*, denoted by u_{EQ} is a *virtual* feedback control action achieving ideally smooth evolution of the system on the constraining sliding surface $\sigma = 0$, provided initial conditions are precisely set on such a switching surface. The equivalent control is formally obtained from the condition $\mathrm{d}\sigma/\mathrm{d}t = 0$. After some simple algebraic manipulations one obtains from (2.15), (2.18) and $\sigma = 0$:

$$\begin{aligned} c(q, u_{\mathrm{EQ}}, \dot{u}_{\mathrm{EQ}}, \ldots, u_{\mathrm{EQ}}^{(\alpha)}) &= c_1 c_{n-1} q_1 + (c_2 c_{n-1} - c_1) q_2 + \cdots \\ &\quad + (c_{n-2} c_{n-1} - c_{n-3}) q_{n-2} \\ &\quad + (c_{n-1} c_{n-1} - c_{n-2}) q_{n-1}. \end{aligned} \tag{2.20}$$

Since q asymptotically converges to zero, the solutions of the above time-varying implicit differential equation, describing the evolution of the equivalent control, asymptotically approach the solutions of the following autonomous implicit differential equation:

$$c(0, u, \dot{u}, \ldots, u^{(\alpha)}) = 0. \tag{2.21}$$

Equation (2.21) constitutes the *zero dynamics* (see [9]) associated to the problem of zeroing the differential primitive element, considered now as an (auxiliary) output of the system. Notice that (2.20) may also be regarded as the *zero dynamics* associated with zeroing of the sliding surface coordinate function σ. If (2.21) locally asymptotically approaches a constant equilibrium point $u = U$, then the system is said to be locally *minimum phase* around such an equilibrium point, otherwise the system is said to be *non-minimum phase*. The equivalent control is, thus, locally asymptotically stable to U, whenever the underlying input–output system is minimum phase.

2.3. Higher order sliding regimes

Chattering reduction has been a major concern in sliding mode controller design. A standard technique consists in replacing the switching element by a high gain, saturation amplifier (see Slotine [28]). Some other interesting and effective variations of this idea have also been proposed in Slotine and Li [29]. In recent times, however, some efforts have been devoted to non-traditional smoothing of sliding regimes through the so called 'higher order' sliding regimes (see Chang [3] for a second order sliding mode controller example). The ideas behind 'binary control systems', as applied to variable structure control, are geared towards obtaining asymptotic convergence in the direction of the sliding surface, in a manner that avoids control input chattering through integration (see Emelyanov [5]). These two developments are also closely related to the differential algebraic approach presented here. In the following paragraphs we explain how the same ideas may be formally derived from differential algebra, in all generality.

Let σ be a p-th order sliding surface candidate, i.e.

$$\Sigma(\sigma^{(p)}, \ldots, \sigma, \zeta, u, \dot{u}, \ldots, u^{(\gamma)}) = 0 \tag{2.22}$$

for some polynomial function Σ. Let us assume that (2.22) may be locally expressed as

$$\sigma^{(p)} = \eta(\sigma^{(p-1)}, \ldots, \sigma, \zeta, u, \dot{u}, \ldots, u^{(\gamma)}). \tag{2.23}$$

Let M be a positive constant. Moreover, let the set of coefficients $\{m_1, \ldots, m_{p-1}\}$ be such that the following polynomial in the complex variable λ,

$$q(\lambda) = \lambda^p + m_{p-1}\lambda^{p-1} + \cdots + m_2\lambda + m_1,$$

is Hurwitz. The following dynamical implicit sliding mode controller achieves an asymptotic approach to the zero level set of the sliding surface σ:

$$\eta(\sigma^{(p-1)},\dots,\sigma,\zeta,u,\dot u,\dots,u^{(\gamma)}) = -m_1\dot\sigma - m_2\ddot\sigma - \cdots - m_{p-1}\sigma^{(p-1)} - M\operatorname{sign}\left[m_1\sigma + m_2\dot\sigma + \cdots + m_{p-1}\sigma^{(p-2)} + \sigma^{(p-1)}\right]. \quad (2.24)$$

Since, generally speaking, the time derivatives of σ are $k\langle u\rangle$-algebraically dependent on ζ, the dynamical sliding mode controller (2.24) may be ultimately expressed in terms of the (time-varying) state components.

Remark. A differential primitive element of the finitely generated dynamics $K/k\langle u\rangle$, with (non-differential) transcendence degree n, always qualifies as a candidate for an n-th order sliding regime.

An additional possibility of creating higher order sliding regimes is represented by the consideration of the input-sliding surface system as an input–output system.

Consider the differential field extension $k\langle u,\sigma\rangle/k\langle u\rangle$ as an input–output system. Evidently since $k\langle u,\sigma\rangle$ is finitely generated over $k\langle u\rangle$, then $k\langle u,\sigma\rangle$ is differentially algebraic over $k\langle u\rangle$. The sliding surface candidate σ satisfies then an implicit algebraic differential equation with coefficients in $k\langle u\rangle$, i.e.,

$$\Pi(\sigma,\dot\sigma,\dots,\sigma^{(\nu)},u,\dot u,\dots,u^{(\mu)}) = 0. \quad (2.25)$$

We may rewrite such an implicit differential equation as the following Global Generalized Observability Canonical Form (GGOCF) (see [7]):

$$\dot\sigma_i = \sigma_{i+1},\quad i=1,2,\dots,\nu-1, \quad (2.26a)$$

$$\Pi(\sigma_1,\dots,\sigma_\nu,\dot\sigma_\nu,u,\dot u,\dots,u^{(\mu)}) = 0, \quad (2.26c)$$

$$\sigma = \sigma_1 \quad (2.26c)$$

where

$$\sigma_i := \mathrm{d}^{i-1}\sigma/\mathrm{d}^{i-1}\quad (i=1,2,\dots,\nu).$$

As before, an explicit Local Generalized Observability Canonical Form (LGOCF) can be obtained for the element σ whenever $\partial\Pi/\partial(\mathrm{d}\sigma_\nu/\mathrm{d}t) \neq 0$:

$$\dot\sigma_i = \sigma_{i+1},\quad i=1,2,\dots,\nu-1, \quad (2.27a)$$

$$\dot\sigma_\nu = p(\sigma_1,\dots,\sigma_\nu,u,\dot u,\dots,u^{(\mu)}), \quad (2.27b)$$

$$\sigma = \sigma_1. \quad (2.27c)$$

One takes as a *higher order stabilizing sliding surface* a suitable (algebraic) function of σ and its time derivatives, up to $(\nu-1)$-st order. For obvious reasons, the most convenient type of function is represented by a stabilizing linear combination of σ and its time derivatives:

$$\mathrm{s} = m_1\sigma_1 + m_2\sigma_2 + \cdots + m_{\nu-1} + \sigma_\nu. \quad (2.28)$$

A first-order sliding motion is then imposed on such a linear combination of generalized 'phase variables', by means of the discontinuous sliding mode dynamics:

$$\dot{\mathrm{s}} = -M\operatorname{sign}(\mathrm{s}),\quad M>0. \quad (2.29)$$

This results in the following implicit dynamical higher order sliding mode controller:

$$p(\sigma_1,\dots,\sigma_\nu,u,\dot u,\ddot u,\dots,u^{(\mu)}) = -m_1\sigma_2 - \cdots - m_{\nu-1}\sigma_\nu - M\operatorname{sign}[m_1\sigma_1 + \cdots + m_{\nu-1}\sigma_{\nu-1} + \sigma_\nu]. \quad (2.30)$$

As previously discussed, s goes to zero in finite time and, provided the coefficients in (2.28) are properly chosen, an ideally asymptotically stable motion can then be obtained for σ, as it is ideally governed by the following autonomous linear dynamics:

$$\dot\sigma_i = \sigma_{i+1},\quad i=1,2,\dots,\nu-2, \quad (2.31a)$$

$$\dot\sigma_{\nu-1} = -m_1\sigma_1 - m_2\sigma_2 - \cdots - m_{\nu-1}\sigma_{\nu-1}, \quad (2.31b)$$

$$\sigma = \sigma_1. \quad (2.31c)$$

Remark. The implicit dynamical controllers given by (2.18), or by (2.30), constitute a substantial departure from traditional sliding mode control ideas. The validity of such controllers is necessarily constrained to those regions where, respectively, $\partial c/\partial u^{(\alpha)} \neq 0$ and $\partial p/\partial u^{(\mu)} \neq 0$. Otherwise, singularities, known as *impasse points* [12], arise for the underlying differential equation describing the controller. The avoidance of such singularities, through possibly discontinuous feedback and 'jump' strategies, has been a major concern in the work of Fliess and his coworkers. The reader is refered to Abu el Atta Dos et al. [1], and references therein, for further details.

2.4. Sliding regimes and the controllability of nonlinear systems

The *differentially algebraic closure* of the ground field k in the dynamics K is defined as the differential field κ, where $K \supseteq \kappa \supseteq k$, consisting of the elements of K which are differentially algebraic over k. The field k is *differentially algebraically closed* if, and only if, $\kappa = k$.

The following definition is taken from Fliess [10] (see also [19]):

Definition 2.7. The dynamics $K/k\langle u \rangle$ is said to be *algebraically controllable* if, and only if, the ground field k is differentially algebraically closed in K.

Algebraic controllability implies, then, that any element of K is necessarily influenced by the input u, since such an element satisfies a differential equation which is not independent of u and of, possibly, some of its time derivatives.

Theorem 2.8. *A higher order sliding regime can be created on any element σ of the dynamics $K/k\langle u \rangle$ if, and only if, $K/k\langle u \rangle$ is algebraically controllable.*

Proof. Sufficiency is obvious from the fact that algebraic controllability implies that σ satisfies a differential equation with coefficients in $k\langle u \rangle$. For the necessity of the condition, suppose, contrary to what is asserted, that $K/k\langle u \rangle$ is not algebraically controllable and yet a higher order sliding regime can be created on any element of the differential field extension $K/k\langle u \rangle$. Since k is not differentially algebraically closed, then, there are elements in K, which belong to a differential field κ containing k, which satisfy differential equations with coefficients found exclusively in k. Clearly, these elements are not related to the control input u through differential equations. It follows that a higher order sliding regime cannot be created on such elements. A contradiction is established. □

In this more relaxed notion of a higher order sliding regime, one may say that a sliding regime can be created on any element of the dynamic of the system, if, and only if, the system algebraically is controllable. This characterization of sliding mode existence through controlability is a direct consequence of the differential algebraic approach.

2.5. Robustness with respect to parametric and external perturbations

An important feature of sliding mode control is constituted by its traditionally outstanding robustness properties with respect to, both, unmodelled parametric variations and the influence of external (bounded) perturbations. The first aspect has been treated, from an *adaptive control* viewpoint, in Sira-Ramírez et al. [27], within the same framework of dynamical sliding mode control as presented in this article. The second aspect is briefly treated below, using results in Fliess [11]. Robustness results of dynamical sliding mode controllers for linear systems are available from [15].

Let w be a differential indeterminate representing an (unmeasurable) external perturbation input to the system. We assume, moreover, that w and u are, each, differentially transcendent over k and k-differentially algebraically independent. Consider the nonlinear dynamics $K/k\langle u, w \rangle$ and let ζ be a non-differential transcendence basis of $K/k\langle u, w \rangle$ with cardinality n. We consider *differential k-specializations* of the differential ring $k\{u, \zeta\}$, generated by u and ζ, into a universal differential extension field Φ, as differential homomorphisms $\nu^0 : k\{u, \zeta\} \to \Phi$, that respect time derivation. Let

$$Q(\nu^0(k\{u, \zeta\}))$$

denote the *quotient field* of $\nu^0(k\{u, \zeta\})$. If, furthermore, we assume that the differential transcendence degree of the extension,

$$Q(\nu^0(k\{u, \zeta\}))/k,$$

is zero then, $\nu^0 u$, $\nu^0 \zeta_1$, $\nu^0 \zeta_2, \ldots, \nu^0 \zeta_n$ are differentially algebraic over k. One usually takes ν^0 as the identity. Differential specializations thus, naturally, conform to the idea of dynamical state feedback (or output feedback) with, or, as in our case, without, the use of an additional external input (see [11] for further details).

Definition 2.9. A sliding surface candidate σ of K, is said to be *robust* with respect to w, if (1) σ is differentially transcendent over k, (2) σ is $k\langle\zeta, u\rangle$-algebraically independent of w, (3) $\mathrm{d}\sigma/\mathrm{d}t$ is either $k\langle u\rangle$ or $k\langle u, w\rangle$ algebraically dependent on ζ, and (4) there exists *differential k-specializations* ν^+, ν^- of $k\{\zeta, u\}$ over Φ, with

$$\begin{aligned}&\text{diff tr deg } Q(\nu^+(k\{\zeta, u\}))/k\\ &\quad = \text{diff tr deg } Q(\nu^-(k\{\zeta, u\}))/k\\ &\quad = 0,\end{aligned}$$

i.e., there exist w-independent dynamical feedback control laws, (abusively) denoted here by their closed form solutions: $u^+(\zeta)$, $u^-(\zeta)$, such that the closed loop behavior of $\mathrm{d}\sigma/\mathrm{d}t$ satisfies (2.6), possibly for a sufficiently large value of W, in a manner totally independent of the perturbation signal w.

This means that w need not be known, in any manner, to be able to synthesize the sliding surface candidate σ, but the possible influence of w on the impossed dynamics (2.6) of σ (which maintains the desirable sliding condition $\sigma = 0$) may be, somehow, counteracted by the control input u in our desire to impose the condition $\sigma = 0$, by means of dynamical feedback. Indeed, the $k\langle u, w\rangle$ algebraic dependence of $\mathrm{d}\sigma/\mathrm{d}t$ on ζ implies the existence of a polynomial S over k such that

$$S(\dot\sigma, \zeta, u, \dot u, \dots, u^{(\beta)}, w, \dot w, \dots, w^{(\gamma)}) = 0 \tag{2.32}$$

which is locally valid as

$$\dot\sigma = s(\zeta, u, \dot u, \dots, u^{(\beta)}, w, \dot w, \dots, w^{(\gamma)}). \tag{2.33}$$

Suppose there exist (dynamical) state-dependent feedback controllers with outputs denoted by $u^+(\zeta)$, $u^-(\zeta)$, with each controller acting, respectively, on the regions $\sigma > 0$, and $\sigma < 0$, and such that for all possible (bounded) values of w, and of its time derivatives, one may locally guarantee that

$$\begin{aligned}\dot\sigma &= s\big(\zeta, u^+(\zeta), (\dot u^+)(\zeta), \dots, (u^+)^{(\beta)}(\zeta),\\ &\qquad\qquad w, \dot w, \dots, w^{(\gamma)}\big)\\ &\leqslant -W,\end{aligned} \tag{2.34a}$$

for all

$$(w, \dot w, \dots, w^{(\gamma)}) \in \prod_{j=0}^{\gamma} \Omega^{(j)}$$

and for all

$$\zeta \in N(\sigma^1(0)) \cap \{\zeta : \sigma(\zeta) > 0\},$$

and that

$$\begin{aligned}\dot\sigma &= s\big(\zeta, u^-(\zeta), (\dot u^-)(\zeta), \dots, (u^-)^{(\beta)}(\zeta),\\ &\qquad\qquad w, \dot w, \dots, w^{(\gamma)}\big)\\ &\geqslant W,\end{aligned} \tag{2.34b}$$

for all

$$(w, \dot w, \dots, w^{(\gamma)}) \in \prod_{j=0}^{\gamma} \Omega^{(j)}$$

and for all

$$\zeta \in N(\sigma^{-1}(0)) \cap \{\zeta : \sigma(\zeta) < 0\},$$

where $\Omega^{(j)}$ is a compact set in R, bounding the j-th derivative of w, and $N(\sigma^{-1}(0))$ represents a small neighborhood of $\sigma = 0$ and $(u^+)^{(j)}(\zeta)$ (respectively $(u^-)^{(j)}(\zeta)$) represents the j-th time derivative of the dynamic controller output $u^+(\zeta)$ (respectively $u^-(\zeta)$) whose actions are valid only on $\sigma > 0$ (resp. $\sigma < 0$). Then, the sliding motion is robust with respect to w in the sense that the condition $\sigma = 0$ is locally achievable in finite time, and indefinitely sustained, in spite of w. Computation of the dynamical controllers satisfying (2.34) may be extremely difficult in the general nonlinear case. Properness of a dynamical controller directly computed from (2.32), specially in the linear case, demands that $\beta \geqslant \gamma$, which is a well known *matching condition* (see [29,32]). Evidently, robustness is guaranteed whenever $\mathrm{d}\sigma/\mathrm{d}t$ is only $k\langle u\rangle$-algebraically dependent on the state ζ. In such a case the controller is computed as in (2.11).

The robustness developments and characterizations given above can be greatly simplified if one takes an input–output viewpoint on the nonlinear system

$$k\langle\sigma, u, w\rangle / k\langle u, w\rangle$$

and resorts to higher order sliding regimes. In such a case, sliding on $\sigma = 0$ is easily shown to be robust if σ is differentially algebraic over $k\langle u\rangle$.

3. Conclusions and suggestions for further research

The use of the differential algebraic methods provides a firm theoretical basis to sliding mode control of nonlinear systems. The results are seen to point towards potential practical implications. More general classes of sliding surfaces, which include the presence of inputs and, possibly, their time derivatives, were shown to naturally allow for chattering-free sliding mode controllers of dynamical nature. The theoretical simplicity, and conceptual advantages, stemming from the differential algebraic approach, render new possibilities to the broader area of discontinous feedback control in general. Extensions of the theory, and its implications, to other classes of discontinuous feedback controlled systems, such as pulse-width-modulated control strategies, are entirely possible (see [24]). The less explored pulse-frequency-modulated control techniques may be shown to also benefit from this new approach [26]. For other classes of systems, such as nonlinear multivariable systems, infinite dimensional, discrete time and differential-difference systems, the extensions of the sliding mode control theory remain largely unexplored, from this new viewpoint. Robustness issues still deserve further research and developments, specially in the multivariable cases.

It has been shown, in a most elegant manner, in [15], that non-minimum phase linear systems can be asymptotically stabilized using *dynamical precompensators* in combination with dynamical sliding mode controllers. Such result could be extended to be nonlinear systems case with, possibly, some significant additional efforts.

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