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Adaptive learning in perceptrons: a sliding mode control approach*

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Abstract. A dynamical sliding mode control approach is proposed for robust adaptive learning in analog Adaptive Linear Elements (Adalines), constituting basic building blocks for perceptron-based feedforward neural networks. The zero level set of the learning error variable is regarded as a sliding surface in the space of learning parameters. A sliding mode trajectory can then be induced, in finite time, on such a desired sliding manifold. Neuron weights adaptation trajectories are shown to be of continuous nature, thus avoiding bang-bang weight adaptation procedures. Sliding mode invariance conditions determine a least squares characterization of the adaptive weights average dynamics whose stability features may be studied using standard time varying linear systems results. Robustness of the adaptive learning algorithm, with respect to bounded external perturbation signals, and measurement noises, is also demonstrated. The article presents some simulation examples dealing with applications of the proposed algorithm to forward and inverse plant dynamics identification.

Mathematics Subject Classifications. 93B12, 93B35

1. Introduction

The adjustment of learning parameters in perceptron based feedforward neural networks has been mainly explored form a discrete time viewpoint. The celebrated Widrow-Hoff *Delta Rule* (see Widrow and Lehr, [12]) constitutes a least mean square learning error minimization algorithm by which an

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asymptotically stable linear convergence dynamics is imposed on the underlying discrete time error dynamics. Using *quasi-sliding mode control* ideas (see Sira-Ramírez [8]) a modification of the Delta Rule was proposed by Sira-Ramínez and Żak in [9], and in [13], whereby a switching weight adaptation strategy is shown to also impose a discrete time asymptotically stable linear learning error dynamics. This algorithm is at the basis of recently proposed identification and control schemes based on feedforward neural networks, (see Colina-Morles and Mort [1], and Kuschewsky *et al*, [4]). To our knowledge, design of learning strategies in adaptive perceptrons, from the viewpoint of sliding mode control in continuous time, has not been addressed in the existing literature. However, the relevance of ordinary differential equations with discontinuous right hand sides, or Variable Structure Systems (see Utkin, [10]), was analyzed in the work of Li *et al*, in [5], in the context of Analog Neural Networks of the Hopfield type with infinite gain nonlinearities. In that work, it is established under what circumstances sliding mode trajectories do not appear in such a class of neurons.

In this article a *continuous time* sliding mode control approach is proposed for the robust adaptation of variable weights in *Adaptive Linear Elements*, also known as *Adalines*, so that its scalar output variable tracks an arbitrarily specified reference signal. The zero level set of the *learning error* variable is regarded as the sliding surface coordinate function and a discontinuous law of adaptive weight variation is proposed which induces, in finite time, a sliding motion which sustains the zero error condition. All the advantageous features of sliding mode controlled performance are then found in the behaviour of the different variables associated to the adaptation algorithm. The sliding mode controlled weight adaptation trajectories are shown to be *continuous*, rather than bang-bang signals. As a consequence, a rather smooth neuron output response is also obtained. The sliding mode invariance conditions, describing the average (ideal) sliding mode behaviour, provide with a least squares characterization of the dynamical features of the average evolution of the vector of adaptive weights. A study of the requirements for the stability of the linear time-varying average weight dynamics establishes essential connections with adaptive control issues, such as the convenience of the assumption of *persistency of extcitation* conditions.

A unique feature of the sliding mode approach lies in the enhanced insensitivity of the proposed adaptive learning algorithm, with respect to bounded external perturbation signals and measurement noises. These features are demonstrated using conventional sliding mode control properties, under rather standard assumptions. Some of the geometric features of the proposed sliding mode algorithm are naturally linked to the well known *matching conditions*, for the case of performance under the influence of neuron input, and output, measurement noises.

Section 2 contains the fundamental definitions, assumptions and derivations of the main characteristics of a sliding mode control approach to weight adaptation in Adalines. In this section, the robustness of the algorithm, with respect to bounded external perturbation inputs, and bounded measurement noises, is also demonstrated and a derivation of the required matching condition is provided. Section 3 contains some basic examples of relevant significance in the potential applications of the proposed adaptive learning strategy in automatic control applications. The treated examples include, both, identification of forward and inverse dynamics of unknown externally perturbed, nonlinear plants. The learning error signal is shown to converge to zero, in finite time, without noticeable chattering, and in spite of all possible external perturbations. Section 4 contains the conclusions and suggestions for further research.

2. A sliding mode control approach to weight adaptation in Adalines

In this section we establish the fundamental adaptation algorithm, based on continuous time sliding modes, for turning variable adaptive weights in adaptive linear combiner elements, or Adaline units.

2.1. Definitions and basic assumptions

Consider the perceptron model depicted in Figure 1 where $x(t) = (x_1(t), \ldots, x_n(t))$ represents a vector of bounded time-varying *inputs*, assumed also to exhibit bounded time derivatives, i.e.

$$\begin{aligned} \| x(t) \| &= \sqrt{x_1^2(t) + \ldots + x_n^2(t)} \leq V_x \ \forall t \\ \| \dot{x}(t) \| &= \sqrt{\dot{x}_1^2(t) + \ldots + \dot{x}_n^2(t)} \leq V_{\dot{x}} \ \forall t \end{aligned} \tag{1}$$

where V_x and $V_{\dot{x}}$ are known positive constants.

We denote by $\tilde{x}(t)$ the vector of *augmented inputs*, which includes a constant input of value $B > 1$, affecting the *bias*, or *threshold* weight w_{n+1} in the perceptron model, i.e.

$$\tilde{x}(t) = \text{col}\,(x_1(t), \ldots, x_n(t), B) = \text{col}(x(t), B) \tag{2}$$

REMARK 2.1. *The scalar product $\tilde{x}^T(t)\tilde{x}(t) = B^2 + x^T(t)x(t) = B^2 +$ $+ \| x(t) \|^2$ is bounded away from zero for all times.* □

The vector $\omega(t) = \text{col}(\omega_1(t), \ldots, \omega_n(t))$ represents the set of time-varying *weights*. It will be assumed that, due to physical constraints, the magnitude of the vector $\omega(t)$ is bounded $\| \omega(t) \| \leq W \; \forall t$, for some constant W. We also define the vector of *augmented weights* by including the bias weight component

$$\tilde{\omega}(t) = \text{col}(\omega_1(t), \ldots, \omega_n(t), \omega_{n+1}(t)) = \text{col}(\omega(t), \omega_{n+1}(t)) \tag{3}$$

Similarly, $\tilde{\omega}(t)$ is assumed to be bounded at each instant of time t by means of

$$\| \tilde{\omega}(t) \| = \sqrt{\omega_1^2(t) + \cdots + \omega_n^2(t) + \omega_{n+1}^2(T)} \leq \tilde{W} \; \forall t \tag{4}$$

for some constant $\tilde{W}$.

The scalar signal $y_d(t)$ represents the time-varying *desired output* of the perceptron. It will be assumed that $y_d(t)$ and $\dot{y}_d(t)$ are also bounded signals, i.e.

$$\begin{aligned} | y_d(t) | &\leq V_y \; \forall t \\ | \dot{y}_d(t) | &\leq V_{\dot{y}} \; \forall t \end{aligned} \tag{5}$$

The output signal $y(t)$ is a scalar quantity defined as:

$$\begin{aligned} y(t) &= \sum_{i=1}^{n} \omega_i(t)x_i(t) + \omega_{n+1}(t) = \omega^T(t)x(t) + \omega_{n+1}(t)B \\ &= \tilde{\omega}^T(t)\tilde{x}(t) \end{aligned} \tag{6}$$

We define the *learning error* $e(t)$ as the scalar quantity obtained from

$$e(t) = y(t) - y_d(t) \tag{7}$$

The nonlinear function $\Gamma(y)$ is generally assumed to be an *odd* function of y, i.e. $\Gamma(y) = -\Gamma(-y)$, known as the *activation* function. The activation functions are relevant to the analysis of networks involving several leyers of neurons, which we will not be considering here.

2.2. Problem formulation and main results

Using the theory of *Sliding Mode Control of Variable Structure Systems* (see [11]) we propose to consider the zero value of the learning error coordinate $e(t)$ as a time-varying *sliding surface*, i.e.

$$s(e(t)) = e(t) = 0 \tag{8}$$

Condition (8) is, therefore, deemed as a *desired* condition for the learning error signal $e(t)$ and one which guarantees that the perceptron output $y(t)$ coincides with the desired output signal $y_d(t)$ for all times $t > t_h$ where t_h is addressed as the *hitting time*.

DEFINITION 2.1. A sliding motion *is said to exist on a sliding surface* $s(e(t)) = e(t) = 0$, *after time* t_h, *if the condition* $s(t)\dot{s}(t) = e(t)\dot{e}(t) < 0$ *is satisfied for all* t *in some nontrivial semi-open subinterval of time of the form* $[t, t_h) \subset (-\infty, t_h)$.

Basic Problem Formulation

It is desired to devise a dynamical feedback *adaptation mechanism*, or *adaptation law*, for the augmented vector of variable weights $\tilde{\omega}(t)$ such that the sliding mode condition of definition 2.1 is enforced.

2.2.1 Zero adaptive learning error in finite time

Let "sign $e(t)$" stand for the *signum* function, defined as:

$$\text{sign}\, e = \begin{cases} +1 & \text{for } e(t) > 0 \\ 0 & \text{for } e(t) = 0 \\ -1 & \text{for } e(t) < 0 \end{cases} \tag{9}$$

We then have the following result

THEOREM 2.1. *If the adaptation law for the augmented weight vector* $\tilde{\omega}(t)$ *is chosen as*

$$\begin{aligned} \dot{\tilde{\omega}}(t) &= -\left(\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right) k \; sign\, e(t) \\ &= -\left(\frac{\begin{bmatrix} x(t) \\ B \end{bmatrix}}{B^2 + x^T(t)x(t)} \right) k \; sign\, e(t) \end{aligned} \tag{10}$$

with k *being a sufficiently large positive design constant satisfying*

$$k > \tilde{W} V_{\dot{x}} + V_{\dot{y}} \tag{11}$$

then, given an arbitrary initial condition $e(0)$, *the learning error* $e(t)$ *converges to zero in finite time,* t_h, *estimated by*

$$t_h \leq \frac{|\, e(0) \,|}{k - \tilde{W} V_{\dot{x}} - V_{\dot{y}}} \tag{12}$$

and a sliding motion is sustained on $e = 0$ *for all* $t > t_h$.

Proof Consider a Lyapunov function candidate given by

$$V(e(t)) = \frac{1}{2}e^2(t) \tag{13}$$

The time derivative of $V(e(t))$ is given by

$$\begin{aligned} \dot{V}(e(t)) &= e(t)(\dot{\tilde{\omega}}^T(t)\tilde{x}(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t)) \\ &= -k \mid e(t) \mid + e(t)(\tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t)) \\ &\leq -k \mid e(t) \mid + (\tilde{W}V_{\dot{x}} + V_{\dot{y}}) \mid e(t) \mid \\ &= (-k + \tilde{W}V_{\dot{x}} + V_{\dot{y}}) \mid e(t) \mid \leq 0 \end{aligned} \tag{14}$$

Thus, the controlled trajectories of the learning error converge to zero. We may actually show that such a convergence takes place in finite time.

Indeed, the differential equation satisfied by the regulated error trajectories $e(t)$ is simply given by

$$\dot{e}(t) = -k \text{ sign } e(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t) \tag{15}$$

For all times $t \leq t_h$, the solution, $e(t)$, of such a differential equation, with initial condition $e(0)$ at $t = 0$, satisfies

$$e(t) - e(0) = -kt \text{ sign } e(0) + \int_0^t (\tilde{\omega}^T(\sigma)\dot{\tilde{x}}(\sigma) - \dot{y}_d(\sigma))d\sigma \tag{16}$$

at time $t = t_h$ the solution takes the value zero and, hence,

$$- e(0) = -kt_h \text{ sign } e(0) + \int_0^{t_h} (\tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t))dt \tag{17}$$

Multiplying both sides of the equality by $-\text{sign}\, e(0)$ one immediately obtains the estimate in equation (12) of t_h by means of the following inequality

$$\begin{aligned} \mid e(0) \mid &= kt_h - (\int_0^{t_k} (\tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t))dt) \text{ sign } e(0) \\ &\geq [k - (\tilde{W}V_{\dot{x}} + V_{\dot{y}})]t_h \end{aligned} \tag{18}$$

Evidently, for any $t < t_h$ and for the chosen sliding mode controller gain k, in (11), one has from (15)

$$\begin{aligned} e(t)\dot{e}(t) &= -k \mid e(t) \mid + (\tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t))e(t) \\ &\leq (-k + \tilde{W}V_{\dot{x}} + V_{\dot{y}}) \mid e(t) \mid < 0 \end{aligned} \tag{19}$$

and a sliding mode exists on $e(t) = 0$ for $t > t_h$. □

REMARK 2.2. *Note that the proposed dynamical feedback adaptation law for the vector of weights in (10) results in a* continuous regulated evolution *of the vector of variable weights $\tilde{\omega}(t)$. The discontinuous feedback strategy (10) actually represents a least squares solution, with respect to $\dot{\tilde{\omega}}(t)$ of the following linear time-varying equation*

$$\dot{\tilde{\omega}}^T(t)\tilde{x}(t) = -k\ sign\,[y(t) - y_d(t)] \tag{20}$$

which yields the following suggestive regulated dynamics for the perceptron output signal $y(t)$

$$\dot{y} = \tilde{\omega}^T(t)\dot{\tilde{x}}(t) - k\ sign\,(y(t) - y_d(t)) \tag{21}$$

where the signal $\tilde{\omega}^T(t)\dot{\tilde{x}}(t)$ acts as a bounded perturbation signal. □

Note that if the quantity $\dot{\tilde{x}}(t)$ is measurable, one can obtain a more relaxed variable structure feedback control strategy than the one obtained in (10). Generally speaking, such an adaptive feedback strategy for the variable weights requires smaller design gains k to obtain a corresponding sliding motion. Since such a case is of some practical importance, we summarize its details in the following theorem

THEOREM 2.2. *If the adaptation law for the augmented weight vector $\tilde{\omega}(t)$ is chosen as*

$$\begin{aligned} \dot{\tilde{\omega}}(t) &= -\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}(k\ sign\, e(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t)) \\ &= -\left(\frac{\tilde{x}(t)\dot{\tilde{x}}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\tilde{\omega}(t) - \left(\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)k\ sign\, e(t) \end{aligned} \tag{22}$$

with k being a positive design constant satisfying

$$k > V_{\dot{y}} \tag{23}$$

then, given an arbitrary initial condition $e(0)$, the learning error $e(t)$ converges to zero in finite time t_h satisfying

$$t_h \leq \frac{|\ e(0)\ |}{k - V_{\dot{y}}} \tag{24}$$

and a sliding motion is sustained on $e = 0$ for all $t > t_h$.

Proof. The proof proceeds along similar lines of that of theorem 2.1 after realizing that the controlled learning error satisfies the following differential equation with discontinuous right hand side

$$\dot{e}(t) = -k \operatorname{sign} e(t) - \dot{y}_d(t) \tag{25}$$

□

REMARK 2.3. *As before, the proposed dynamical feedback adaptation law for the vector of weights in (22) results in a* continuous weight evolution. *Such a law of variation actually represents a minimum square error solution, with respect to $\dot{\tilde{\omega}}(t)$ of the following linear time-varying equation*

$$\dot{y} = \dot{\tilde{\omega}}^T(t)\tilde{x}(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t) = -k \operatorname{sign}\,[y(t) - y_d(t)] \tag{26}$$

□

The proposed solution for $\dot{\tilde{\omega}}(t)$ in (26) is, necessarily, aligned with the augmented vector of inputs $\tilde{x}(t)$. The total disregard for the effect of the scalar signal $\dot{y}_d(t)$ in the above adaptation schemes, (10) and (22), arises from the implicit assumption that such a signal is not, generally speaking, measurable in practise, nor can it be estimated with sufficient precision. On the contrary, it will be shown in the next section that there is a large class of problems for which $\dot{\tilde{x}}(t)$ may be assumed to be measurable. The previous theorem shows that as long as $\dot{y}_d(t)$ is bounded, the adaptation policy always manages to bring the learning error to zero in finite time. A similar remark can be made about the control law in (10). Figure 2 depicts the (instantaneous) geometric features at the basis of the proposed algorithms.

2.2.2 Average features of the proposed adaptation mechanisms

In order to assess the qualitative features of the adaptive discontinuous feedback algorithm governing the evolution of the variable weights we proceed, as it is customary in sliding mode control theory, to investigate the *average* behaviour of the involved controlled variables. Such an analysis involves the consideration of the following *invariance conditions*,

$$e(t) = 0\,; \quad \dot{e}(t) = 0 \tag{27}$$

which are ideally satisfied after the sliding motion starts on the sliding surface and is indefinitely sustained theorem. Consideration of such invariance conditions naturally leads to propose the substitution of the discontinuous (bang-bang) input signals by a smooth input signal, known as the *equivalent control input*. This method has been rigorously validated in [10] as the *Method of the Equivalent Control*.

Consider the adaptation law (10) and the associated error equation (15) and substitute the discontinuous signal k sign $e(t)$ by its smooth equivalent value $v_{eq}(t)$.

$$\dot{e}(t) = -v_{eq}(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t) \tag{28}$$

The second condition in (27) implies that

$$v_{eq}(t) = \tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t) \ \forall t > t_h \tag{29}$$

Upon use of (29), a *virtual*, or *equivalent variable weight adaptation law* can also be associated to the actual discontinuous (bang-bang) policy described by (10). We denote such an *equivalent* adaptive weight vector by $\tilde{\omega}_{eq}(t)$. One obtains, for all $t > t_h$,

$$\begin{aligned} \dot{\tilde{\omega}}_{eq}(t) &= -\frac{\tilde{x}}{\tilde{x}^T(t)\tilde{x}(t)}(\tilde{\omega}_{eq}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t)) \\ &= -\left(\frac{\tilde{x}(t)\dot{\tilde{x}}^T(t)}{\tilde{x}^T(t)\tilde{x}(T)}\right)\tilde{\omega}_{eq}(t) + \left(\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\dot{y}_d(t) \end{aligned} \tag{30}$$

i.e., the average variable weight vector trajectory satisfies a *linear* time-varying vector differential equation with forcing function represented by the bounded function $\dot{y}_d(t)$. Note that $\tilde{\omega}_{eq}(t)$ itself does not, necessarily, lie in the *range* of $\tilde{x}(t)$. The obtained expression (30) describes the projection, along the range of the vector of augmented inputs $\tilde{x}(t)$, of the derivative of the average regulated evolution of $\tilde{\omega}(t)$. We formalize this result in the following paragraphs after the following related remark.

REMARK 2.4. *The first invariance condition* $e(t) = \tilde{\omega}^T(t)\tilde{x}(t) - y_d(t) = 0$ *also leads to some minimum norm solution for the weight adaptation trajectory* $\tilde{\omega}(t)$. *Such a solution is given by*

$$\hat{\omega}_{eq}(t) = \frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}y_d(t) \tag{31}$$

The solution (31) is, evidently, aligned with $\tilde{x}(t)$ *for all* t. *It is easy to verify that, in general, the time varying vector* $\hat{\omega}_{eq}(t)$ *of equation (31)* is not *a solution of the linear time-varying forced differential equation (30), but only its instantaneous projection onto the range of* $\tilde{x}(t)$. □

DEFINITION 2.2. *A matrix $M(t)$ is said to be a time-varying* projection operator, *along the range space $\mathcal{V}(t)$ of a nonzero vector function $v(t)$, onto its (instantaneous) perpendicular hyperplane, if $M(t)$ satisfies*

1. $M(t)z(t) = 0 \; \forall \; z(t) \in \mathcal{V}(t)$
2. $M(t)\zeta(t) = \zeta(t) \; \forall \zeta(t)$ *s.t.* $v^T(t)\zeta(t) = 0$

PROPOSITION 2.1. *Let $\tilde{\mathcal{X}}(t)$ denote, one dimensional, time-varying range space of the vector function $\tilde{x}(t)$. The matrix*

$$M(t) = \left(I - \frac{\tilde{x}(t)\tilde{x}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right) \tag{32}$$

is a time-varying projection operator *along $\tilde{\mathcal{X}}(t)$ onto its instantaneous orthogonal hyperplane.*

Proof. Immediate upon verification of the two given conditions in definition 2.2 for $v(t) = \tilde{x}(t)$. □

PROPOSITION 2.2. *The projection of the vector $\dot{\tilde{\omega}}_{eq}(t)$, onto the hyperplane representing the ideal sliding condition $e(t) = 0$, is zero, i.e. the projection of the vector $\tilde{\omega}(t)$, onto such a time-varying hyperplane, remains constant.*

Proof. Consider again equation (30), along with the condition $\dot{e}(t) = 0$, i.e. with $\dot{y}_d(t) = \dot{y}(t)$. We rewrite equation (30) as

$$\begin{aligned} \dot{\tilde{\omega}}_{eq}(t) &= -\left(\frac{\tilde{x}(t)\dot{\tilde{x}}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\tilde{\omega}_{eq}(t) + \left(\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\dot{y}(t) \\ &= \left(\frac{\tilde{x}(t)\tilde{x}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\dot{\tilde{\omega}}_{eq}(t) \end{aligned} \tag{33}$$

rearranging (33) one obtains

$$\left(I - \frac{\tilde{x}(t)\tilde{x}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\dot{\tilde{\omega}}_{eq}(t) = 0 \tag{34}$$

□

The same proposition holds valid for the actual (discontinuous) sliding mode controlled trajectories of the adaptation weights, given by (10) and (22).

According to the results of proposition 2.2, the equivalent weight adaptation velocity vector satisfies the property : $\dot{\tilde{\omega}}(t) \in \tilde{\mathcal{X}}(t)$. This is in full accordance with the form of the proposed actual discontinuous adaptation law represents by equations (10) and (22). This result has an important bearing on the stability features of the adaptive algorithm. Namely, the boundedness of the vector of variable weights, after sliding occurs, is exclusively dependent upon the variations of the input $\tilde{x}(t)$ and that of the desired output signal $y_d(t)$. Before stating general results to this respect, several particular cases, regarding constant inputs, $\tilde{x}(t) = \tilde{x}$ and $y_d(t) = y_d$, can be briefly considered.

Example 2.1. If $\tilde{x}(t) = \tilde{x}$ and $y_d(t) = y_d$ are constant, then the equivalent adaptation law (30) satisfies $\dot{\tilde{\omega}}_{eq}(t) = 0$ and $\tilde{\omega}_{eq}(t) = \omega_{eq} =$ constant, while the output derivative signal $\dot{y}$ is zero. The discontinuous adaptation law takes the form

$$\dot{\tilde{\omega}}(t) = \frac{\tilde{x}}{\tilde{x}^T\tilde{x}}\, k \,\mathrm{sign}\, e(t)$$

Example 2.2. If $\tilde{x}(t) = \tilde{x}$, is constant, then the equivalent adaptation law satisfies

$$\dot{\tilde{\omega}}_{eq}(t) = \frac{\tilde{x}}{\tilde{x}^T\tilde{x}}\dot{y}_d(t)$$

In this case

$$\tilde{\omega}_{eq}(t) = \frac{\tilde{x}}{\tilde{x}^T\tilde{x}}y_d(t)$$

i.e. the minimum norm solution $\tilde{\omega}_{eq}(t)$ of $e(t) = \tilde{x}^T\tilde{\omega}_{eq}(t) - y_d(t) = 0$, is also a solution of the differential equation defining $\tilde{\omega}(t)$

Example 2.3. If the weight adaptation process is started from the initial condition $\tilde{\omega}(t_0) = 0$, or from any vector $\tilde{\omega}(t_0) \in \tilde{\mathcal{X}}(t_0)$, then for all $t > t_0$, the evolution of $\tilde{\omega}(t)$ is entirely confined to the one dimensional time varying subspace $\tilde{\mathcal{X}}(t)$, containing the vector $\tilde{x}(t)$. Since the minimum possible norm of $\tilde{x}(t)$ is B, the weight vector $\tilde{\omega}(t)$ is bounded if, and only if, the output $y_d(t)$ and its time derivative $\dot{y}_d(t)$ are bounded.

The following proposition follows readily from the fact that for the discontinuous strategy (30), the equivalent input v_{eq} is obtained from the invariance condition $\dot{e}(t) = 0$, and the error equation (25), as

$$\dot{e}(t) = -v_{eq}(t) - \dot{y}_d(t)$$

i.e., $\forall\ t > t_h$

$$v_{eq}(t) = -\dot{y}_d(t)$$

PROPOSITION 2.3. *The equivalent adaptation law corresponding to the discontinuous strategy (22) results in the same expression as in (30).*

2.2.3 Requirements for the stability of the average controlled weights dynamics

In this section we shall establish some results related to the stability requirements of the average feedback regulated adaptation dynamics, as represented by equation (30). This requirements guarantee boundedness of solutions for the variable weight trajectories and comprise fundamental relations with well known areas of adaptive control.

We begin by providing some standard definitions (see Brockett [2])

DEFINITION 2.3. *Denote by $F(t)$ the time varying matrix*

$$F(t) = -\frac{\tilde{x}(t)\dot{\tilde{x}}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)} \tag{35}$$

The differential equation $\dot{\tilde{\omega}}_{eq}(t) = F(t)\tilde{\omega}_{eq}(t)$ is said to be uniformly stable *if there exists a positive constant γ such that, for all t_0 and all $t > t_0$, the state transition matrix $\Phi(t, t_0)$, corresponding to the matrix $F(t)$, satisfies*

$$\| \Phi(t, t_0) \| < \gamma \tag{36}$$

This definition allows us to formulate the following proposition

PROPOSITION 2.4. *Suppose the system $\dot{\tilde{\omega}}_{eq}(t) = F(t)\tilde{\omega}_{eq}(t)$ is uniformly stable and let $\dot{y}_d(t)$ be absolutely integrable. Then, the solution of (30) are bounded*

Proof. Consider the inequalities

$$\int_{t_0}^{\infty} \frac{| \dot{y}_d(t) |}{\| \tilde{x}(t) \|} \mathrm{d}t \leq \int_{t_0}^{\infty} | \dot{y}_d(t) | \, \mathrm{d}t = \beta \tag{37}$$

and assume that the initial states, $\tilde{\omega}_{eq}(t_0)$, of the weight adaptation trajectories are bounded by a constant $\tilde{W}_0$.

From the *variation of constants formula*, the solutions of the linear time-varying differential equation (30) are written as

$$\tilde{\omega}_{eq}(t) = \Phi(t, t_0)\tilde{\omega}_{eq}(t_0) + \int_{t_0}^{t} \Phi(t, \sigma) \frac{\tilde{x}\sigma}{\tilde{x}^T(\sigma)\tilde{x}(\sigma)} \dot{y}_d(\sigma) \mathrm{d}\sigma$$

By the virtue of (37), the norm of $\tilde{\omega}_{eq}(t)$ satisfies

$$\| \tilde{\omega}_{eq}(t) \| \leq \| \Phi(t, t_0) \| \| \tilde{\omega}_{eq}(t_0) \| + \| \int_{t_0}^{t} \Phi(t, \sigma) \frac{\tilde{x}(\sigma)}{\tilde{x}^T(\sigma)\tilde{x}(\sigma)} \dot{y}_d(\sigma) \mathrm{d}\sigma \|$$

$$\leq \| \Phi(t,t_0) \| \| \tilde{\omega}_{eq}(t_0) \| + \int_{t_0}^{t} \| \Phi(t,\sigma) \| \frac{| \dot{y}_d(\sigma) |}{\| \tilde{x}(\sigma) \|} d\sigma$$

$$\leq \gamma(\tilde{W}_0 + \beta) ; \quad \forall\, t > t_0 \tag{38}$$

the result follows. □

DEFINITION 2.4. *The system* $\dot{\tilde{\omega}}_{eq}(t) = F(t)\tilde{\omega}_{eq}(t)$, *is* exponentially stable *if there exists positive constants* γ *and* λ *such that, for all* $t > t_0$, *the state transition matrix* $\Phi(t,t_0)$, *associated to* $F(t)$, *satisfies*

$$\| \Phi(t,t_0) \| \leq \gamma e^{-\lambda(t-t_0)} \tag{39}$$

PROPOSITION 2.5. *The matrix* $F(t)$ *is bounded if* $\dot{\tilde{x}}(t)$ *is bounded*

Proof. We take as definition of the matrix norm the *induced norm*

$$\| F(t) \| = \max_{\|z\|=1} \| F(t)z \| \tag{40}$$

Evidently, from the definition of $F(t)$ in (35), it readily follows that

$$\| F(t) \| \leq \max_{\|z\|=1} \frac{\| \dot{x}(t) \| \| z \|}{\| \tilde{x}(t) \|} \leq \| \dot{\tilde{x}}(t) \|$$

□

It is well known [2] that if the matrix $F(t)$ is bounded, then exponential stability is equivalent to the uniform integrability, over arbitrary interval of times, of the norm of the corresponding transition matrix. We then have

THEOREM 2.3. *Let* $\dot{\tilde{x}}(t)$ *be bounded on* $(-\infty, +\infty)$ *and let* M *be a constant, independent of* t_0 *and* t_1, *then, the system* $\dot{\tilde{\omega}}_{eq}(t) = F(t)\tilde{\omega}_{eq}(t)$ *is exponentially stable if and only if*

$$\int_{t_0}^{t_1} \| \Phi(t,t_0) \| \, dt \leq M \quad \forall\, t_1 > t_0 \tag{41}$$

Proof. The proof of this theorem can be found in [2].

The next result touches upon a special form of the well known condition of *persistency of excitation*, of common occurrence in linear and nonlinear adaptive control schemes (see Sastry and Bodson [6] and Sastry and Isodori [7]).

THEOREM 2.4. *Let $\dot{\tilde{x}}(t)$ be bounded on $(-\infty, +\infty)$, moreover, assume that the following form of the persistency of excitation condition holds uniformly in t:*

There exists positive constants δ and ϵ, such that the following matrix condition is satisfied

$$\int_t^{t+\sigma} \Phi(t,\sigma)\left[\frac{\tilde{x}(\sigma)\tilde{x}^T(\sigma)}{(\tilde{x}^T(\sigma)\tilde{x}(\sigma))^2}\right]\Phi^T(t,\sigma)d\sigma \geq \epsilon I \quad \forall\ t > t_0 \tag{42}$$

Then, the equivalent adaptation law (30) uniformly yields a bounded trajectory for the vector of weights $\tilde{\omega}_{eq}(t)$, for every bounded signal $\dot{y}_d(t)$, if, and only if, the autonomous system $\dot{\tilde{\omega}}_{eq}(t) = F(t)\tilde{\omega}_{eq}(t)$ is exponentially stable.

Proof. It is easy to realize, from the definition of the augmented input vector $\tilde{x}(t)$ in (2), that the input channel matrix for the signal $\dot{y}_d(t)$, given by $\tilde{x}(t)/(\tilde{x}^T(t)\tilde{x}(t))$, is bounded for all $t \in (-\infty, +\infty)$. Moreover, according to proposition 2.5, the boundedness of $\dot{\tilde{x}}(t)$ implies the boundedness of $F(t)$. The proof of the theorem may now follow, quite closely, the proof found in pp. 167 of [2] □

Condition (42) admits the following scalar form ([6])

$$\begin{aligned} \int_t^{t+\delta} z^T\Phi(t,\sigma)\left[\frac{\tilde{x}(\sigma)\tilde{x}^T(\sigma)}{(\tilde{x}^T(\sigma)\tilde{x}(\sigma))^2}\right]\Phi^T(t,\sigma)z\,\mathrm{d}\sigma &= \int_t^{t+\sigma} \mid \frac{z^T\Phi(t,\sigma)\tilde{x}(\sigma)}{\tilde{x}^T(\sigma)\tilde{x}(\sigma)} \mid^2 \mathrm{d}\sigma \\ &\geq \epsilon\ \forall t > t_0,\ ;\ \| z \| = 1 \end{aligned} \tag{43}$$

which is a condition on the *energy*, averaged over all directions of a unit sphere, of the nonsingularly transformed input vector, $\tilde{\chi}(\tau) = \Phi(\tau,t)\tilde{x}(t)/(\tilde{x}^T(t)\tilde{x}(t))$. This means that the vector function $\tilde{\chi}(\tau)$ is quite an "active" time-varying vector, so that the integral of the matrix $\tilde{\chi}(t)\tilde{\chi}^T(t)$ is uniformly positive definite over any interval of finite length δ.

2.3. Robustness features with respect to external perturbations

One of the well known key characteristics of sliding mode control is related to the advantageous insensitivity of regulated variables with respect to parameter variations, and with respect to external bounded perturbations, affecting the underlying system behaviour. Here, we include in our analysis the presence of bounded external input perturbations, exhibiting also bounded time derivatives, for two important cases. The proposed sliding mode control algorithms are shown to be robust with respect to such perturbations.

2.3.1 Inputs with bounded additive noise

Consider a vector valued norm bounded *external perturbation inputs*, denoted by $\xi(t) = (\xi_1(t), \ldots, \xi_n(t))$, which additively affects the values of the input vector $x(t)$ to the perceptron. It is assumed that the perturbation input $\xi(t)$ is not "larger" than the input $x(t)$, i.e.

$$\| \xi(t) \| = \sqrt{\xi_1^2(t) + \ldots + \xi_n^2(t)} \leq V_\xi < V_x \ \forall t \tag{44}$$

The time derivatives of the components of $\xi(t)$ are assumed to be also bounded

$$\| \dot{\xi}(t) \| = \sqrt{\dot{\xi}_1^2(t) + \ldots + \dot{\xi}_n^2(t)} \leq V_{\dot{\xi}} \ \forall t \tag{45}$$

We define the *augmented external perturbation input* vector as

$$\tilde{\xi}(t) = (\xi_1(t), \ldots, \xi_n(t), 0) \tag{46}$$

This means that it is implicitly assumed that the constant input B to the *bias* weight $\omega_{n+1}(t)$ is a fixed value which does not contain the influence of perturbation signals.

The *perturbed learning error* $\hat{e}(t) = y(t) - y_d(t)$ is now given by

$$\hat{e}(t) = [\tilde{x}(t) + \tilde{\xi}(t)]^T \tilde{\omega}(t) - y_d(t) \tag{47}$$

Note that, in spite of the fact that the perturbed input signal $\tilde{x}(t) + \tilde{\xi}(t)$ is actually *available for measurement*, its time derivative $\dot{\tilde{x}}(t) + \dot{\tilde{\xi}}(t)$ *is not*. This means that such time derivatives can not be used in the weight adaptation law. Hence, only an adaptation law of the type proposed in (10) can be actually devised for sliding mode creation on the zero learning error hyperplane.

By virtue of the above considerations, we shall center our attention of the perturbed adaptation law:

$$\dot{\tilde{\omega}}(t) = -\left(\frac{\tilde{x}(t) + \tilde{\xi}(t)}{[\tilde{x}(t) + \tilde{\xi}(t)]^T [\tilde{x}(t) + \tilde{\xi}(t)]} \right) k \ \text{sign}\ \hat{e}(t) \tag{48}$$

The weight adaptation law (48) results, as it easily verified, in the following discontinuous perturbed learning error dynamics

$$\dot{\hat{e}}(t) = -k \ \text{sign}\ \hat{e}(t) = \tilde{\omega}^T(t)[\dot{\tilde{x}}(t) + \dot{\tilde{\xi}}(t)] - \dot{y}_d(t) \tag{49}$$

The robustness result is summarized in the following theorem whose proof is rather similar to that of Theorem 2.1.

THEOREM 2.5. *Consider the sliding mode creation problem on the zero learning error hypersurface of an adaline including a perturbed input vector. If the adaptation law for the augmented weight vector $\tilde{\omega}(t)$ is chosen as in (48) with k being a positive constant satisfying*

$$k > \tilde{W}(V_{\dot{x}} + V_{\dot{\xi}}) + V_{\dot{y}} \tag{50}$$

then, given an arbitrary initial condition $\hat{e}(0)$, the perturbed learning error $\hat{e}(t)$ converges to zero in finite time $\hat{t}_h$, estimated by

$$\hat{t}_h \leq \frac{|\, e(0)\, |}{k - \tilde{W}(V_{\dot{x}} + V_{\dot{\xi}}) - V_{\dot{y}}} \tag{51}$$

in spite of all possible assumed (bounded) values of the perturbation inputs and its time derivatives. Moreover a sliding motion is sustained on $\hat{e}(t) = 0$ for all $t > \hat{t}_h$.

Proof. Immediate upon consideration of (49) and use of similar arguments as in the proof of theorem 2.1. □

The equivalent input $v_{eq}(t)$ is now defined, from (49) as

$$v_{eq}(t) = \tilde{\omega}^T(t)[\dot{\tilde{x}}(t) + \dot{\tilde{\xi}}(t)] - \dot{y}(t) \tag{52}$$

The equivalent feedback adaptation law is obtained by substituting the discontinuous term in the adaptation law (48) by $v_{eq}(t)$. The obtained average adaptation law is now a perturbation-dependent feedback adaptation law given by

$$\begin{aligned} \dot{\tilde{\omega}}_{eq}(t) &= -\left(\frac{\tilde{x}(t) + \tilde{\xi}(t)}{[\tilde{x}(t) + \tilde{\xi}(t)]^T[\tilde{x}(t) + \tilde{\xi}(t)]}\right)[\tilde{\omega}_{eq}^T(t)(\dot{\tilde{x}}(t) + \\ &\quad + \dot{\tilde{\xi}}(t)) - \dot{y}_d(t)] \\ &= -\left(\frac{[\tilde{x}(t) + \tilde{\xi}(t))][\dot{\tilde{x}}(t) + \dot{\tilde{\xi}}(t)]^T}{[\tilde{x}(t) + \tilde{\xi}(t)]^T[\tilde{x}(t) + \tilde{\xi}(t)]}\right)\tilde{\omega}_{eq}(t) + \\ &\quad + \left(\frac{\tilde{x}(t) + \tilde{\xi}(t)}{[\tilde{x}(t) + \tilde{\xi}(t)]^T[\tilde{x}(t) + \tilde{\xi}(t)]}\right)\dot{y}_d(t) \end{aligned} \tag{53}$$

Enforcing the invariance condition, corresponding to zero perturbed learning error $\dot{\hat{e}}(t) = 0$ on equation (54), one obtains, after some algebraic manipulations, the following expression

$$\left(I - \frac{[\tilde{x}(t) + \tilde{\xi}(t)][\tilde{x}(t) + \tilde{\xi}(t)]^T}{[\tilde{x}(t) + \tilde{\xi}(t)]^T[\tilde{x}(t) + \tilde{\xi}(t)]}\right)\dot{\tilde{\omega}}_{eq}(t) = 0 \tag{54}$$

According to the results of proposition 2.1, the matrix multiplying $\dot{\omega}_{eq}(t)$ in (54) is a time-varying projection operator. Hence, equation (54) implies that $\dot{\omega}(t)$ lies in the range space of the perturbed input vector, $\tilde{x}(t)+\tilde{\xi}(t)$ and has zero projection onto the zero perturbed learning error hyperplane. The adaptation mechanism for the perturbed input case has similar geometric features as the unperturbed case.

2.3.2 Noisy measurements of the unperturbed inputs and neuron output

Consider now the case in which the *measurements* of the unperturbed input vector $x(t)$ are corrupted by some unknown but bounded noise signal, which we still denote by $\xi(t)$, satisfying the assumptions of the previous subsection. The measured vector is denoted by $x_m(t) = x(t) = \xi(t)$. We also assume that the bounded measurement noise component $\xi_{n+1}(t)$ distorts the measured value of the constant bias input to the threshold element, assumed to be nominally equals to B. However, the noise component $\xi_{n+1}(t)$ should satisfy the restriction $| \omega_{n+1}(t) | \leq V_{\xi_{n+1}} < B$. The *measured* input vector function $\tilde{x}_m(t) = \tilde{x}(t) + \tilde{\xi}(t)$ has the obvious meaning. Its constitutive parts are defined as before and they satisfy the same assumptions. In particular the assumption $V_\xi < V_x$ clearly implies that the vector function $\tilde{x}(t)$ is never orthogonal to the exterded measured input vector function $\tilde{x}_m(t)$, i.e.

$$\tilde{x}_m^T(t)\tilde{x}(t) = [\tilde{x}(t) + \tilde{\xi}(t)]^T\tilde{x}(t) \neq 0$$

Additionally, the measured perceptron output $y_m(t) = y(t) = \zeta(t)$ is assumed to be corrupted by some additive noise signal, $\zeta(t)$, which has a bounded time derivative, i.e. $| \dot{\zeta}(t) | \leq V_{\dot{\zeta}} \forall t$

It is assumed that the time derivative of the perturbed measured input vector $\dot{\tilde{x}}_m(t) = \tilde{x}(t) + \xi(t)$ is not synthesizable in practise. The perturbed adaptation law, proposed for this case, is of the same form as that in (48). The perturbed learning error dynamics is now obtained as

$$\begin{aligned}
(55) \qquad \dot{\hat{e}}(t) &= \dot{\tilde{\omega}}^T(t)\tilde{x}(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t) = \dot{\zeta}(t) - \dot{y}_d(t) \\
&= -\left(\frac{[\tilde{x}(t) + \tilde{\xi}(t)]^T\tilde{x}(t)}{[\tilde{x}(t) + \tilde{\xi}(t)]^T[\tilde{x}(t) + \tilde{\xi}(t)]}\right) k \text{ sign } \hat{e}(t) + \\
&\quad +\tilde{\omega}^T(t)\dot{\tilde{x}}(t) + \dot{\zeta}(t) - \dot{y}_d(t)
\end{aligned}$$

To guarantee the existence of a sliding regime on the hyperplane $\hat{e}(t) = 0$, the smallest possible value of the product of the switch gain factor k, and the time varying scalar quantity modulating its value, has to be sufficiently large as to overcome the unknown but bounded values of the term $\tilde{\omega}^T(t)\dot{\tilde{x}}(t) + \dot{\zeta}(t) - \dot{y}_d(t)$ in the error dynamics (56).

Note that

$$\left|\frac{[\tilde{x}(t)+\tilde{\xi}(t)]^T\tilde{x}(t)}{[\tilde{x}(t)+\tilde{\xi}(t)]^T[\tilde{x}(t)+\tilde{\xi}(t)]}\right| = \left|\frac{[x(t)+\xi(t)]^T x(t) + B(B+\xi_{n+1}(t))}{[x(t)+\xi(t)]^T[x(t)+\xi(t)] + (B+\xi_{n+1}(t))^2}\right|$$

$$\geq \frac{B^2 - (V_x + V_\xi)V_x - BV_{\xi_{n+1}}}{(V_x+V_\xi)^2 + (B+V_{x_{n+1}})^2} =: \eta \tag{56}$$

We assume also that $\eta > 0$. The preceeding developments establish the basis for the proof of the following theorem which summarizes the robustness result for this case.

THEOREM 2.6. *Consider the problem of a sliding mode creation in a neuron with noisy measurements of the unperturbed input vector $\tilde{x}(t)$ and output signal $y(t)$. If the adaptation law for the augmented weight vector $\tilde{\omega}(t)$ is chosen as in (48) with k being a positive constant satisfying*

$$k > \frac{\tilde{W}V_{\dot{x}} + V_{\dot{\zeta}} + V_{\dot{y}}}{B^2 - (V_x + V_\xi)V_x - BV_{\xi_{n+1}}}\left[(V_x+V_\xi)^2 + (B+V_{\xi_{n+1}})^2\right] \tag{57}$$

then, given an arbitrary initial condition $\hat{e}(0)$, the perturbed learning error $\hat{e}(t)$ converges to zero in finite time $\hat{t}_h$ estimated by

$$\hat{t}_h \leq \frac{|\,e(0)\,|}{k\eta - \tilde{W}(V_{\dot{x}} + V_{\dot{\xi}}) - V_{\dot{\zeta}} - V_{\dot{y}}} \tag{58}$$

in spite of all possible assumed (bounded) values of the input measurement noise. Moreover a sliding motion is sustained on $\hat{e}(t) = 0$ for all $t > \hat{t}_h$.

Proof. Immediate from the preceeding considerations and arguments similar to those already used in the proof of theorem 2.1. □

We use again the method of the equivalent control on the basis of the error dynamics equation (56).

$$-\frac{\tilde{x}_m^T(t)\tilde{x}(t)}{\tilde{x}_m^T(t)\tilde{x}_m(t)}v_{eq}(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t) + \dot{\zeta}(t) - \dot{y}_d(t) = 0$$

Note that no singularity is present in the definition of the equivalent input $v_{eq}(t)$ since the product $\tilde{x}_m^T(t)\tilde{x}(t)$ is never zero, as remarked before.

The ideal sliding dynamics, obtained from the invariance condition $\dot{\hat{e}}(t) = 0$, yields, in this case, the following description of the equivalent adaptation law for the vector of variable weights

$$\dot{\tilde{\omega}}_{eq}(t) = -\left(\frac{\tilde{x}_m(t)\dot{\tilde{x}}^T(t)}{\tilde{x}_m^T(t)\tilde{x}(t)}\right)\tilde{\omega}_{eq}(t) + \left(\frac{\tilde{x}_m(t)}{\tilde{x}_m^T(t)\tilde{x}(t)}\right)[\dot{y}_d(t) - \dot{\zeta}(t)] \tag{59}$$

In accordance with the invariance condition $\dot{\hat{e}}(t) = 0$ one substitutes $\dot{y}_d(t)$ by $\dot{y}(t)$ in the preceeding equation. After some simple algebraic manipulations the following expression is obtained

$$\left(I - \frac{\tilde{x}_m(t)\tilde{x}^T(t)}{\tilde{x}_m^T(t)\tilde{x}(t)}\right)\dot{\tilde{\omega}}_{eq}(t) = 0 \tag{60}$$

which clearly indicates that the velocity vector for the weight evolution belongs to the range space of the vector $\tilde{x}_m(t)$ and has zero projection onto its normal hyperplane. Once sliding occurs, the vector of variable weights is attached to a fixed point of a hyperplane normal to $\tilde{x}_m(t)$. However, the zero error learning hyperplane is *skewed* with respect to this hyperplane and the weight vector evolution is no longer attached to a fixed, but a variable, point on the zero learning error hyperplane. We say that an *unmatched* evolution is obtained for the vector of adaptive weights.

The projection onto the normal hyperplane to $\tilde{x}(t)$, of the velocity vector of the weight adaptation trajectory, $\dot{\tilde{\omega}}_{eq}(t)$, now exhibits a nonzero component. This means that the projection of the vector of weights *does* move relative to the error hyperplane $\hat{e}(t) = 0$ in a sliding fashion. Moreover, since the velocity vector of $\tilde{\omega}_{eq}(t)$ is no longer orthogonal to the zero learning error hyperplane, the adopted weight evolution law does not guarantee the fastest approach to the zero learning error condition and boundedness of $\tilde{\omega}_{eq}(t)$ becomes highly dependent upon the nature of the noise signal $\dot{\xi}(t)$. The following result establishes structural condition which guarantees the fastest rate of approach of the vector of adaptive weights to satisfy the zero learning error condition.

THEOREM 2.7. *Let $\tilde{\mathcal{X}}(t)$ denote the one-dimensional range space, at time t, of the vector $\tilde{x}(t)$, then the equivalent feedback adaptation laws, given by (59), satisfies*

$$\dot{\tilde{\omega}}_{eq}(t) \in \tilde{\mathcal{X}}(t) \quad \forall\ t \tag{61}$$

if, and only if,

$$\xi(t) \in \tilde{\mathcal{X}}(t)\,. \tag{62}$$

Proof. Evidently if $\xi(t) \in \tilde{\mathcal{X}}(t)$, then, according to the assumption lating the bounds of $\tilde{x}(t)$ and $\tilde{\xi}(t)$, by which $V_x < V_\xi$, there exists a ti varying scalar function $\mu(t)$ taking values in the open interval $(-1,+1)$, su that $\xi(t) = \mu(t)\tilde{x}(t)\forall t$. It is then easy to see that the projection along $\tilde{\mathcal{X}}$ onto its normal hyperplane, of the generated average vector velocity of t weight adaptation trajectories satisfy

$$
\begin{aligned}
\left(I - \frac{\tilde{x}_m(t)\tilde{x}^T(t)}{\tilde{x}_m^T(t)\tilde{x}(t)}\right)\dot{\omega}_{eq}(t) &= \left(I - \frac{[\tilde{x}(t) + \mu(t)\tilde{x}(t)]\tilde{x}^T(t)}{[\tilde{x}(t) + \mu(t)\tilde{\xi}(t)]^T\tilde{x}(t)}\right)\dot{\omega}_{eq}(t) \\
&= \left(I - \frac{\tilde{x}(t)\tilde{x}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\dot{\omega}_{eq}(t) = 0 \qquad (63)
\end{aligned}
$$

and, therefore $\dot{\omega}_{eq}(t) \in \tilde{\mathcal{X}}(t)$.

Assume now that $\dot{\omega}_{eq}(t) \in \tilde{\mathcal{X}}(t)\forall t$. From (60) it follows that $\dot{\omega}_{eq}(t)$ m be expressed, as

$$\dot{\omega}_{eq}(t) = \alpha(t)\tilde{x}_m(t) \qquad (64)$$

where the scalar function $\alpha(t)$ is given by

$$\alpha(t) = \frac{\tilde{x}^T(t)\dot{\omega}_{eq}(t)}{\tilde{x}_m^T(t)\tilde{x}(t)}$$

in other words $\tilde{x}_m(t) = \tilde{x}(t) + \tilde{\xi}(t) \in \tilde{\mathcal{X}}(t)$. But this is only possible, and only if, $\tilde{\xi}(t) \in \tilde{\mathcal{X}}(t)\forall t$

Condition (62) is the well known *matching condition* which is to be s isfied by the structure of the input measurement perturbation noise. T matching condition means that all effects of the measurement perturbatic will be confined to the time varying subspace $\tilde{\mathcal{X}}(t)$ where the discontinuc feedback actions will overcome them. The boundedness of the perturl tion signals implies furthermore that the regulated motions of the adapt weights vector will be robustly brought to the zero learning error hyperpla

3. Application to inverse and direct dynamics indentification

In discrete time feedforward neural networks, the basic building block unit connecting physically available input variables to the neurons, or Adalines, is constituted by a *transversal filter* consisting of an ideal sampler and a string of pure delay elements in a "ladder" array (see [4]). This unit is usually addressed as the *IS/F-module* (for Ideal Sampler-Filter). The output of each pure delay unit constitutes a component of the discrete-time state vector of the ladder filter. These states are provided, as input signals, to the neuron module.

In *continuous time* (i.e. analog) neuron units, the IS/F-module must be replaced by a string of integrators, which is the dynamical continuous-time "equivalent" of the pure delay element. However, such an arrangement is inherently unstable and some internal feedback must be devised so that the resulting unit provides stable, i.e. bounded, signals as inputs to the neuron unit. We thus propose the use of a ***stable filter***, i.e. a stable time-invariant linear dynamical system whose (available) states will be used as inputs to the neuron unit. The scalar input function $u(t)$ is the input to the filter and represents the physically available signal to be processed by the neuron (usually a *plant* input or output). Figure 3 depicts a schematic representation of the Stable Filter (SF) module connected to the Adaline module.

Let A denote the constant matrix representing the internal, time invariant, feedback connections of the SF-module. Let b be the vector representing the input channel structure to the stable filter. The pair (A, b), with state vector $x(t)$ represents the SF-module.

Consider the augmented version of the input pair (A, b), as follows

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \; ; \quad \tilde{b} = \begin{bmatrix} b \\ 0 \end{bmatrix} \tag{65}$$

The SF-module state equations are therefore given by

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{b}\, u(t) \tag{66}$$

where $\tilde{x}(t)$ is the *state vector* of the SF-module, constituting also the vector of augmented inputs, considered in the previous section, to the neuron unit.

Note that since the vector function $x(t)$ is implicitly assumed to be available for measurement, the vector $\tilde{x}(t)$, and the vector $\dot{\tilde{x}}$, are indeed, available for measurement from the particular topology of the SF module (note that each state variable component describing the filter is physically measurable, and so are all their first order time derivatives, which are just the inputs to the several integrators present in the constructed filter). Any sliding mode control strategy to be used on the basis of the SF-Adaline combination can, therefore, assume that these two signals are actually available for measurement. Note also that any noise affecting the signal $u(t)$ influences the filter state vector $x(t)$ in such a manner that the filtered noise components present on each input $x_i(t)$ to the neuron is already of the "matched" type.

The composite discontinuous weight adaptation dynamics takes then the form

$$
\begin{aligned}
\dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{b}u(t) \\
\dot{\tilde{\omega}}(t) &= -\left(\frac{\tilde{x}(t)\tilde{x}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\tilde{A}^T + u(t)\frac{\tilde{x}(t)\tilde{b}^T}{\tilde{x}^T(t)\tilde{x}(t)}\right)\tilde{w}(t) - \\
&\quad -\left(\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)k \text{ sign } e(t) \\
y(t) &= \tilde{x}^T(t)\tilde{\omega}(t)
\end{aligned} \tag{67}
$$

The corresponding (average) equivalent adaptation law is simply obtained now as

$$
\begin{aligned}
\dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{b}u(t) \\
\dot{\tilde{\omega}}_{eq}(t) &= -\left(\frac{\tilde{x}(t)\tilde{x}^T}{\tilde{x}^T(t)\tilde{x}(t)}\tilde{A}^T + u(t)\frac{\tilde{x}(t)\tilde{b}^T}{\tilde{x}(t)\tilde{x}(t)}\right)\tilde{\omega}_{eq}(t) + \\
&\quad +\left(\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\dot{y}_d(t) \\
y(t) &= \tilde{x}^T(t)\tilde{\omega}_{eq}(t) = y_d(t)
\end{aligned} \tag{68}
$$

The particular form adopted for $\dot{\tilde{x}}(t)$ does not have any bearing on the geometric features associated to the sliding mode adaptation algorithm. As it can be easily verified, condition (34), is independent of $\dot{\tilde{x}}(t)$ and hence, it is independent of the particular values of the pair $(\tilde{A}, \tilde{b})$. The stability features of $\tilde{\omega}(t)$, or of its average value $\tilde{\omega}_{eq}(t)$ do depend, however, on the values adopted by the pair $(\tilde{A}, \tilde{b})$, and the value given to the input function $u(t)$.

PROPOSITION 3.1. *The actual and the average sliding mode controlled dynamics for $y(t)$ (67), (68) are independent of the matrices $\tilde{A}$ and $\tilde{b}$ i.e. the convergence of the output function $y(t)$ to the desired output $y_d(t)$ by means* *of the sliding mode adaptation algorithm is insensitive with respect to the SF-module parameters*

Proof. We only prove the proposition for the case of (67), since the proof corresponding to (68) proceeds along similar lines.

Indeed, taking the time derivative of $y(t)$ in (67) and using the expressions in the first two equations in (67), one finds, after some straightforward algebraic manipulations, that:

$$(69) \qquad \begin{aligned} \dot{y}(t) &= \dot{\tilde{\omega}}^T(t)\tilde{x}(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t) \\ &= -k \text{ sign } e(t) = -k \text{ sign } [y(t) - y_d(t)] \end{aligned}$$

□

For the forward and inverse dynamics identification tasks we use the following definitions, which may also be found in [4].

The *forward dynamics* identification problem consists in making the output signal of the adaline $y(t)$, follow the output of a given (unknown) dynamical system $y_p(t)$, acting as the desired output signal $y_d(t)$, when the input to the system $u(t)$ also acts as an input to the SF-Adaline combination. Upon convergence of the learning error $e(t)$ to zero, the adaline "emulates", in a certain sense, the input-output behaviour of the given system. The structure of the forward dynamics identification scheme using a SFadaline combination is shown in Figure 4.

The *inverse dynamics* identification task consists in having the output of the adaline $y(t)$ follow the input $u(t)$ of a given dynamical system (i.e. $y_d(t) = u(t)$) when the output of such a dynamical system $y_p(t)$ acts as an *input* to the SF-adaline arrangement. The inverse dynamics identification scheme is shown also in Figure 4. When the output of the adaline $y(t)$ converges to the input to the system $u(t)$, we say that the *inverse dynamics* of the plant is being "emulated" by the neuron-filter combination.

3.1. Identification of forward and inverse dynamics for the Kapitsa pendulum

Here we consider a truly nonlinear system of the *non-flat* type, studied by Fliess and coworkers in [3], consisting of a unit mass rod with a suspension point which freely moves only on a vertical direction. The Kapitsa pendulum is, thus, an inverted pendulum where the control actions are constrained to move the suspension point only along a vertical axis (see Figure 5).

We considered a nonstabilizing open loop control $u(t)$, applied to the plant, and obtained the corresponding output $y_p(t)$ of the nonlinear system, represented by the angular position of the rod with respect to the vertical axis. In the forward dynamics identification problem $y_p(t)$ is regarded as the desired signal, $y_d(t)$, to be followed by the neuron output $y(t)$. In that case, the input function $u(t)$ to the system, is also the input to the SF unit. For the inverse dynamics identification the roles of $u(t)$ and $y_p(t)$ were reversed, with respect to the neuron system.

The open loop control function $u(t)$ was chosen, according to ([3]), of the form

$$u(t) = A_1 + A_2 \cos(\frac{t}{2\epsilon}) + A_3 \sin(\frac{t}{\epsilon}) \tag{70}$$

where A_1, A_2 and A_3 are constant parameters. The nonlinear system is assumed to be unknown and only its input and output signals assumed to be measurable for the adaptation process. For simulation purposes, however, the following model was used

$$\begin{aligned} \dot{\alpha} &= p(t) + \frac{u(t)}{l} \sin \alpha(t) \\ \dot{p}(t) &= \left(\frac{g}{l} - \frac{u^2(t)}{l^2} \cos \alpha(t)\right) \sin \alpha(t) - \frac{u(t)}{l} p(t) \cos \alpha(t) \\ \dot{z}(t) &= u(t) \\ y_p(t) &= \alpha(t) \end{aligned} \tag{71}$$

where $\alpha(t)$ is the angle of the rod with the vertical axes, $p(t)$ is proportional to the generalized impulsion. The constants g and l represents, respectively, the gravity acceleration and the length of the rod. The velocity of the suspension point acts as the control variable $u(t)$. The variable $z(t)$ is then the vertical position of the suspension point.

Numerical values for the parameters of the Kapitsa pendulum model were set to be $g = 9.81[\frac{m}{\sec^2}]$ and $l = 0.7$[mts]. An open loop control input signal $u(t)$ of the form given in (70) with the following constant parameters

$$A_1 = 0.4\,; \quad A_2 = 2\,; \quad A_3 = 3\,; \quad \epsilon = 0.05$$

was used, for both tasks.

The SF module was designed as a stable low pass filter with the following state representation

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \dot{x}_3(t) &= -x_1(t) - 3x_2(t) - 3x_3(t) + u(t) \\ \dot{x}_4(t) &= 0 \end{aligned} \tag{72}$$

where the state variable $x_4(t)$ represents the *bias* component with initial condition equals to B. Such a constant parameter is taken, for this example, as $B = 1$.

The results of a simulated forward dynamics identification taks, for an Adaline with a total of 4 weights (including one bias variable weight) are shown in Figure 6. In this figure, the desired output trajectory $y_d(t)$ is constituted by the nonlinear pendulum system output i.e. $y_p(t) = \alpha(t)+y_d(t)$ and the input $u(t)$ to the SF-module is the same input given to the nonlinear system. The learning (tracking) error response $e(t)$ is shown to converge to zero in approximately 0.02 sec. To alleviate the "chattering" phenomena, present in the neuron output and learning error responses, as well as to speed up the simulation time for the SIMNON package, the following standard substitution was adopted for the ideal switch function

$$k \text{ sign } e(t) \approx k \frac{e(t)}{\mid e(t) \mid +\delta}$$

with $\delta = 0.05$.

Highly accurate following is seen to be achieved without chattering around the desired output signal. The open loop unperturbed input signal trajectory $u(t)$, affecting both the pendulum and the SF-neuron arrangement, is also shown in this figure. In the simulation no additive noise affecting the input signal $u(t)$ was assumed. The value used for the variable structure gain k was set to $k = 5$.

For the comparison and qualitative neuron performance evaluation, simulations were also carried out for the same forward dynamics identification task with an input signal $u(t)$ subject now to a computer generated additive bounded noise $\xi(t)$. The generated noise signal is a discrete time stochastic process normally distributed at each instant of time with zero mean and standard deviation equals to 1. The value of k was set to be the same as for the previous simulation and the same switch substitution was carried out. The perturbed input $u(t) + \xi(t)$ affects, as before, both the input signal to the pendulum and to the SF-adaline system. The measure filter state is now a perturbed vector function which is measurable. The noisy states are used to conform the sliding mode adaptive strategy in accordance with equation (48). The perturbed filter state constituting the input to the adaline is, thus, of the "matched" type. The corresponding simulation results are shown in Figure 7.

The inverse dynamics identification task was also implemented using the same SF-module described above. The variable structure control gain used in this case was $k = 90$. The simulation results, without additive noise for the measured output signal $y_p(t)$ of the nonlinear system, are presented, for a 4 weights Adaline, in Figure 8. Figure 9 presents the corresponding results for an additive noise input signal, of the same characteristics as before, affecting the measured signal $y_p(t)$ given as an input to the filter-neuron combination. In this case the value of k was substantially increased to $k = 1000$ due to the large values of the first order time derivative of the desired output signal $y_d(t)$, represented now by the noisy signal $u(t) + \xi(t)$, with $u(t)$ as given in (70).

4. Conclusions

In this article a new dynamical discontinuous feedback adaptive learning algorithm has been proposed, for linear adaptive combiners, which robustly drives the learning error to zero in finite time. The components of the vector of variable weights are assumed to be provided with continuous time adaptation possibilities. The dynamical adaptive learning scheme is based on sliding mode control ideas and represents a simple, yet robust, mechanism for guaranteeing finite time reachability of a zero learning error condition. The approach is also highly insensitive to bounded external perturbation inputs, measurement noises and designed input filter parameters. The assumptions made about the bounded nature of external input signals and desired outputs, as well as of their time derivatives, are quite standard in the literature about adaptive neuron elements, but one which prevents the consideration of discontinuous memoryless activation functions, of the non-differentiable type.

Bounded average weight evolution is guaranteed under several conditions relative to the underlying linear time-varying system describing the average evolution of the vector of adaptive weights. Some of these conditions are closely related to those of persistency of excitation and thus links our approach with standard adaptive control results. The type of imposed requirements for stability also represent quite natural and prevailing assumptions in most available adaptive linear and nonlinear control schemes.

The matching condition, with respect to bounded input signal and neuron output measurment noises, guarantees a minimum norm solution for the velocity of weight adaptation and fastest convergence to the zero error hyperplane. Measurable inputs, already containing external perturbation components, result in a "matched" input channel structure which always guarantees orthogonal velocity of convergence to the sliding hyperplane. The matched structure appears to be trivially satisfied in most automatic control oriented applications.

Chattering-free dynamical sliding mode controllers for nonlinear systems have been recently proposed by Sira-Ramírez ([14]) using input-dependent sliding surfaces. The adaline case study presented here represents an instance in which the sliding surface (zero learning error condition) is actually an "input" dependent manifold. The obtained sliding "controller" is thus continuous rather than bang-bang.

Extensions of the results to more general classes of multilayer neuron arrangements is being pursued at the present time, with encouraging results.

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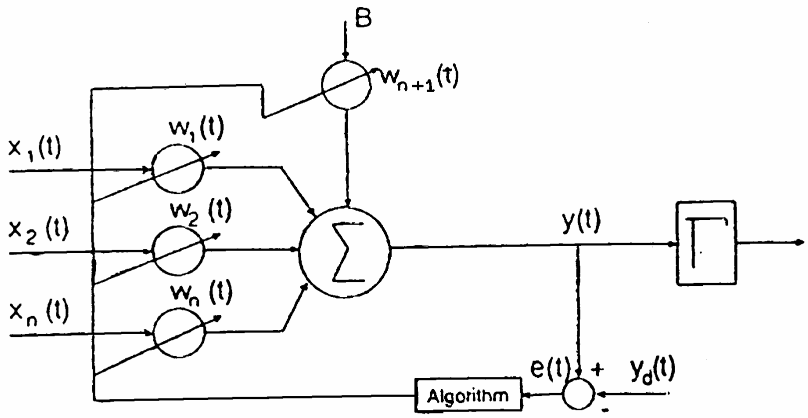

Figure 1. Adaptive Linear Element.

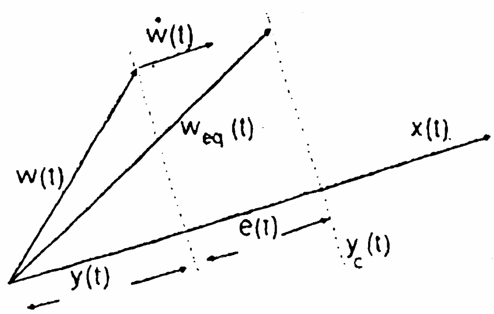

Figure 2. Geometric Interpretation of Sliding Mode Learning Algorithm.

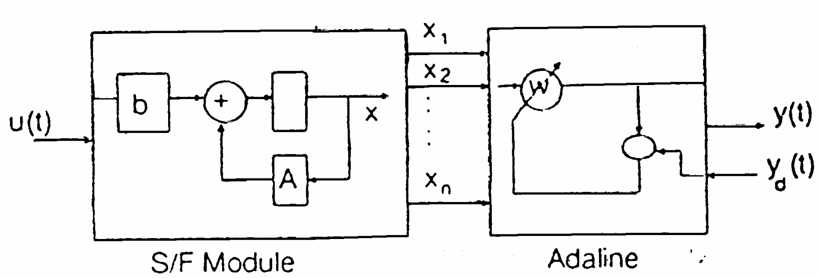

Figure 3. Schematic Representation of the Stable Filter Module and Adaline.

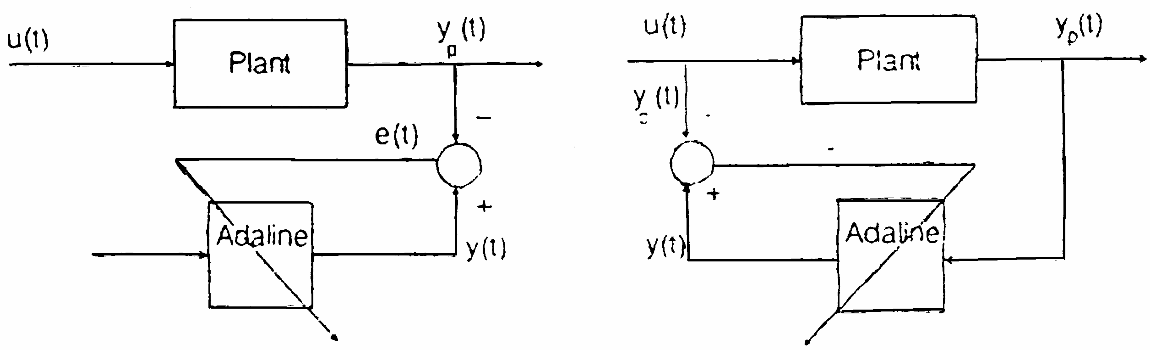

Figure 4. Forward and Inverse Dynamics Identification Schemes using *SF*-Adalines.

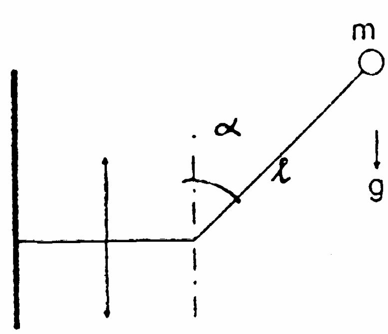

Figure 5. Kapitsa Pendulum.

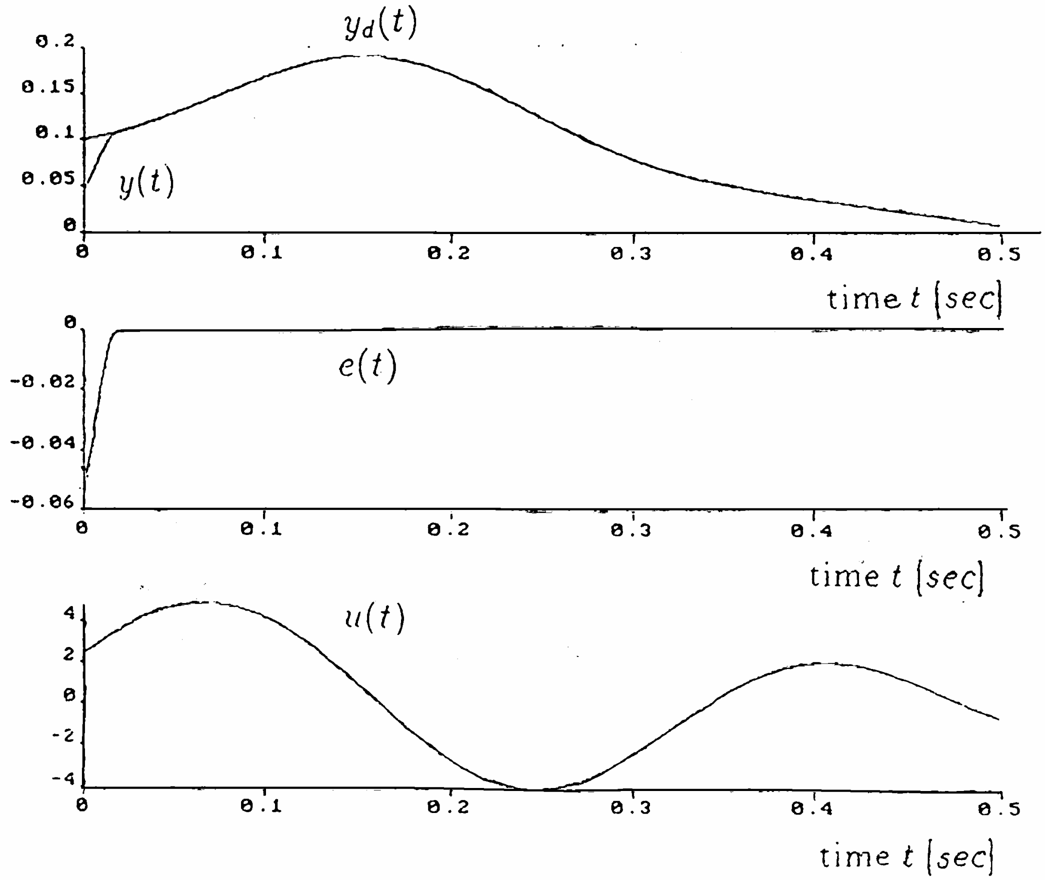

Figure 6. Noise-free Forward Dynamics Identification of the Kapitsa pendulum.

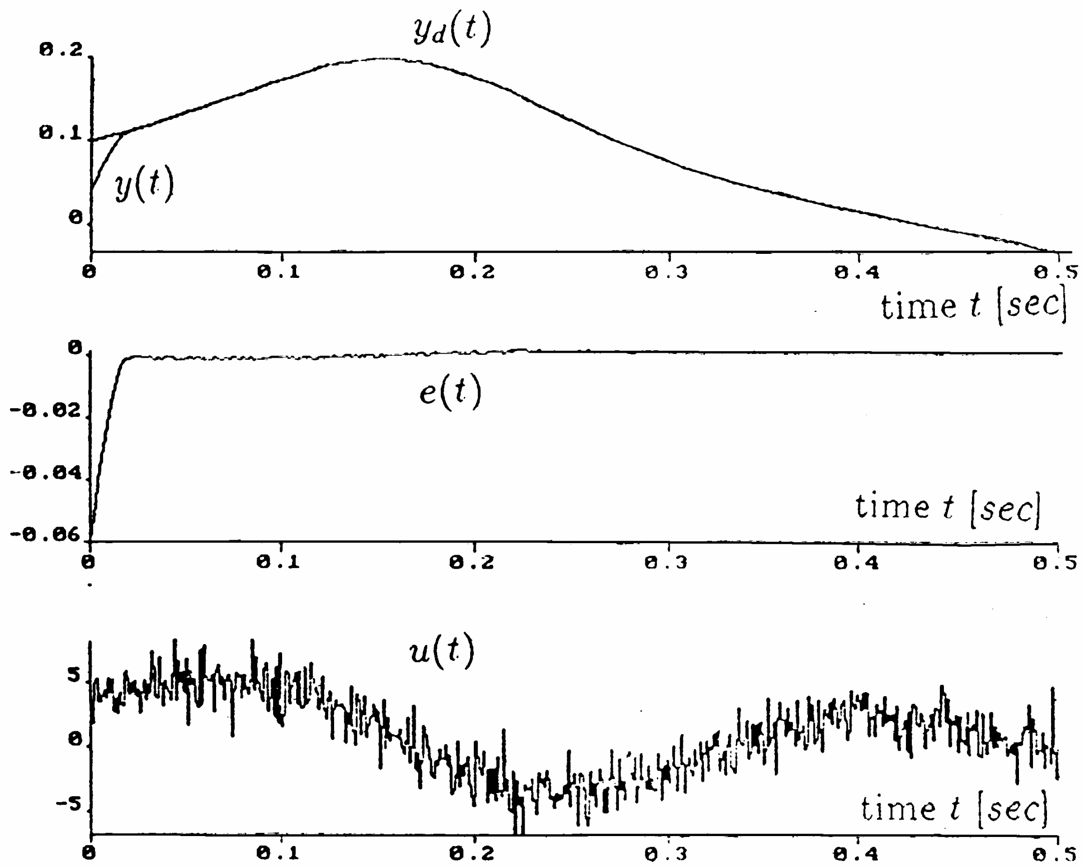

Figure 7. Robust Forward Dynamics Identification with Bounded Noise input for the Kapitsa pendulum.

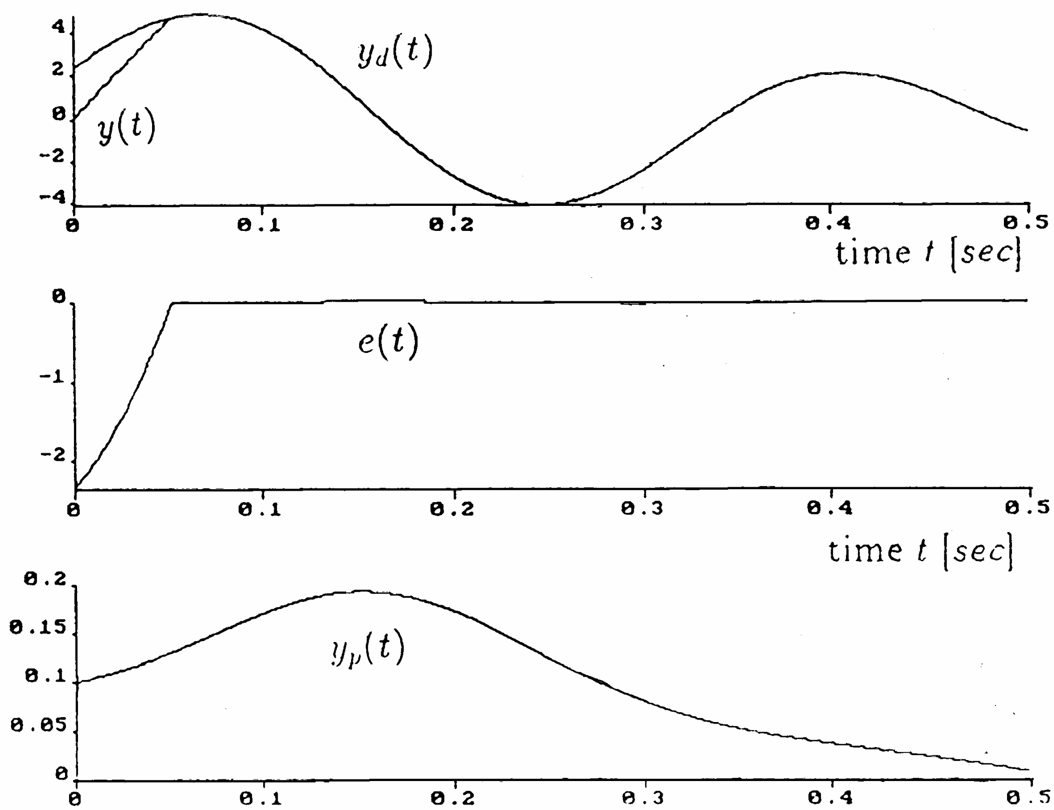

Figure 8. Noise-free Inverse Dynamics Identification of the Kapitsa pendulum.

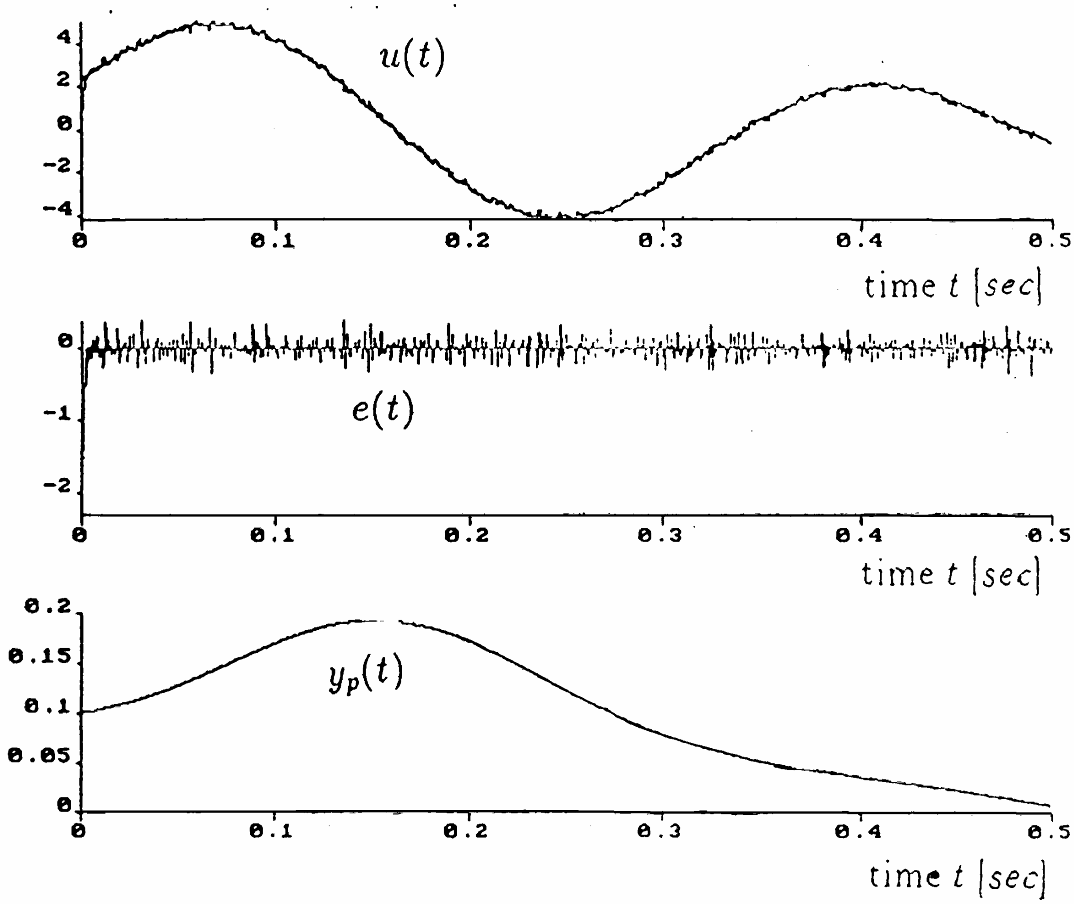

Figure 9. Robust Inverse Dynamics Identification with Bounded Noise input for the Kapitsa pendulum.