Dynamic Compensator Design in Nonlinear Aerospace Systems

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Based on recently developed differential algebraic results, dynamic controllers are proposed for the feedback regulation of typical aerospace systems. Fliess' generalized observability canonical form (GOCF) is used for specifying a dynamic compensator that smoothly regulates the plant dynamics. The synthesis approach is also applicable to the design of nonlinear pulsewidth-modulation (PWM) controllers, as well as to sliding mode control strategies. The three underlying nonlinear control techniques, explored with the aid of illustrative examples, are commonly encountered in aerospace control system design problems. Simulations are also included.

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I. INTRODUCTION

In a series of recent articles [1–4], a new and general approach, based on *differential algebra*, has been introduced by M. Fliess for the study of controlled dynamic systems. This approach was shown to be remarkably suitable for the unified treatment of linear and nonlinear, lumped, or distributed, controlled plants. Feedback decoupling, invertibility, model matching, and realization, have also been conceptually clarified and generalized by Fliess via this powerful and most elegant approach. The fundamentals of the differential algebraic approach were also independently proposed by Professor J. F. Pommaret [5]. Crucially based on the extension to *differential fields* of the *theorem of the primitive element* (see Kolchin [6]), any controlled dynamic system, described by a set of forced ordinary differential equations, was shown to posses a *generalized controller canonical form* (GCCF) explicitly depending on the input and a finite number of its time derivatives [4]. For those cases in which the output of the system qualifies as a *differential primitive element*, the corresponding *generalized observability canonical form* (GOCF) is seen to entirely coincide with such GCCF. The GOCF is obtainable by means of output-based state coordinate transformations which are, in general, control dependent and, possibly, include a finite number of time derivatives of the control input components. As a direct consequence of these results, the problem of *feedback linearization* and of *input-output* linearization of a controlled dynamic system is always trivially solvable, in a local manner, using nonlinear, possibly time-varying, dynamic feedback. The linearizing dynamic compensators are clearly suggested by the canonical forms themselves. However, for systems with constant operating points, the asymptotic stability of the input-output linearized closed-loop dynamics, around such an equilibrium point, crucially depends on the *minimum phase* character of the nonlinear GOCF about such an equilibrium.

In this article, dynamic feedback controllers, based on Fliess' GOCF, are proposed for three typical examples of aerospace control problems. The first example is a variation of Zermelo's problem for an aircraft attempting a perfectly circular (tracking) maneuver in a region of strong air currents (Bryson and Ho [7]). In this example, the control actions related to the heading rate of the aircraft are assumed to be hard limited and piecewise continuous. As a second illustrative example, a dynamic variable structure controlled compensator is proposed for the smooth controlled descent of a thrusted vehicle towards a small hovering height in a planet exhibiting atmospheric resistance. A dynamic feedback controller, for a soft controlled landing on a planet with nonnegligible atmospheric resistance, has also been reported in Sira-Ramírez [8], from the perspective of

ON-OFF PWM control strategies. A third example deals with the problem of an orbital transfer for a multiinput jet-controlled space vehicle (Brockett [9]). In this case we use a dynamic compensator based on an ON-OFF-ON PWM control strategy. The dynamic feedback regulator design is based on a suitable *average* model for the PWM controlled system. The obtained average piecewise smooth dynamic controller solution is then translated into an actual discontinuous (PWM) control scheme which approximates, arbitrarily closely, the average designed behavior (see the Appendix).

Section II presents some general results and derivations about dynamic feedback linearizing controllers using Fliess's GOCF. Section III presents the three application examples. Simulations are included to illustrate the performance of the proposed controllers. The Appendix contains some background material on the ON-OFF, ON-OFF-ON, PWM control and their design-oriented average models. The Appendix also contains a brief summary about output stabilization of nonlinear systems based on dynamic variable structure control strategies.

II. FLIESS' DIFFERENTIAL ALGEBRAIC APPROACH TO CONTROLLED SYSTEMS DYNAMICS

In this section, Fliess' derivation of the GOCF for nonlinear dynamic systems is presented. The results are directly taken from [1], but they are presented here for the sake of self-containment. Background on *differential algebra* can be found in [6].

A. Fliess' GCCF and GOCF for Nonlinear Systems and Exact Dynamic Linearization [1]

Let u be a differential scalar indeterminate and let k be a *differential field*, with derivation denoted by d/dt. A *dynamics* is defined as a *finitely generated differentially algebraic extension* $K/k\langle u\rangle$ of the differential field $k\langle u\rangle$. The *input* u is said to be *independent* if u is a *differential transcendence basis* of K/k. Suppose $x = (x_1, x_2, \ldots, x_n)$ is a *nondifferential transcendence basis* of $K/k\langle u\rangle$. It follows that the derivatives dx_i/dt $(i = 1,\ldots,n)$ are $k\langle u\rangle$-algebraically dependent on the components of x. Thus, there exists exactly n polynomial differential equations of the form:

$$P_i(dx_i/dt, x, u, du/dt, \ldots, d^\nu u/dt^\nu) = 0; \quad i = 1,\ldots,n \tag{1}$$

implicitly describing the controlled dynamics. Under the assumption that such equations can be locally solved in *normal form*, i.e., as:

$$dx_i/dt = p_i(x, u, dt/dt, \ldots, d^\nu u/dt^\nu); \quad i = 1,\ldots,n \tag{2}$$

one obtains a nonredundant description of the dynamics. Such is not the case if one uses a generator system of $K/k\langle u\rangle$ which strictly contains a transcendence basis. Any other transcendence basis, say $z = (z_1, z_2, \ldots, z_n)$ also qualifies as a "state" and similar expressions can be obtained for the given dynamics. The components of x are $k\langle u\rangle$-algebraically dependent upon the components of z and vice versa. Such transformations from one state to another, involve equations dependent upon the control input u and its time derivatives.

According to the *theorem of the differential primitive element* [6], there exists an element $x \in K$ such that $K = k\langle u, x\rangle$. The (nondifferential) transcendence degree n of $K/k\langle u\rangle$ is the smallest integer such that $x^{(n)}$ is $k\langle u\rangle$-algebraically dependent on $x, dx/dt, \ldots, d^{(n-1)}x/dt^{(n-1)}$. Let $q_1 = x_1, q_2 = dx/dt, \ldots, q_n = d^{(n-1)}x/dt^{(n-1)}$. It follows that $q = (q_1, \ldots, q_n)$ is also a transcendence basis of $K/k\langle u\rangle$. One then obtains a generalization of the controller canonical form known as the *global generalized controller canonical form* [3]:

$$\begin{aligned} \frac{d}{dt}q_i &= q_{i+1}; \qquad i = 1,2,\ldots,n-1 \\ C(\dot{q}_n, q, u, \dot{u}, \ldots, u^{(\nu)}) &= 0. \end{aligned} \tag{3}$$

Suppose that the output $y = h(x)$ of the system also qualifies as a differential primitive element, i.e.:

$$\operatorname{rank}\frac{\partial(y, \dot{y}, \ldots y^{(n-1)})}{\partial x} = \operatorname{rank}\frac{\partial(y, \dot{y}, \ldots, y^{(n)})}{\partial x} = n. \tag{4}$$

(See Conte et al. [10].) Then, we may similarly have: $y = q_1, q_2 = dy/dt, \ldots, q_n = d^{(n-1)}y/dt^{(n-1)}$. One, hence, obtains a nonlinear generalization of the observability canonical form called the *global generalized observability canonical form* [1]:

$$\begin{aligned} \frac{d}{dt}q_i &= q_{i+1}; \qquad i = 1,2,\ldots,n-1 \\ C(\dot{q}_n, q, u, \dot{u}, \ldots, u^{(\nu)}) &= 0 \\ y &= q_1 \end{aligned} \tag{5}$$

where C is a polynomial with coefficients in k. If one can locally solve for the time derivative of q_n in the last equation, one obtains an explicit system of first-order differential equations, known as the *local generalized observability canonical form* (LGOCF):

$$\begin{aligned} \frac{d}{dt}q_i &= q_{i+1}; \qquad i = 1,2,\ldots,n-1 \\ \frac{d}{dt}q_n &= \mathrm{C}(q, u, \dot{u}, \ddot{u}, \ldots, u^{(\nu)}) \\ y &= q_1. \end{aligned} \tag{6}$$

Local exact dynamic input-output linearization may be simply achieved by equating the expression in the last differential equation in (6) to a (stable) linear equation in the components of the vector q, possibly including an external reference input signal ν, as follows:

$$C(q,u,\dot{u},\ddot{u},\ldots,u^{(\nu)}) = -\alpha_1 q_1 - \alpha_2 q_2 - \cdots - \alpha_n q_n + k\nu. \tag{7}$$

The last equation *implicitly* defines a dynamic nonlinear control law which locally accomplishes an exact linearization of the input-output dynamics. Smooth stabilization of the output signal $y = q_1$ is thus entirely possible via this feedback scheme. The obtained linear system has preestablished asymptotic stability properties chosen by means of the αs, provided the solutions for u in (7) are "well behaved" in a sense to be made more precise below.

It is evident that the nonlinear dynamic linearization scheme presented above is based on *exact cancellation* of the nonlinear plant dynamics by means of the proposed controller. One intuitively expects that this cancellation may lead to internal instabilities in some special cases. For instance, by means of a straightforward linearization around a constant equilibrium point (if any) of the combined GOCF and the proposed dynamic feedback controller, one can easily demonstrate, in the single-input single-output case, that the dynamic linearizing controller is locally asymptotically stable if and only if the linearized transfer function of the given plant is *minimum phase*, i.e., if the linearized *zero dynamics* is asymptotically stable. The corresponding statement for multiinput systems is also valid. However, in this case, it has been shown that there exists three possible, nonequivalent, extensions of the concept of *zero dynamics* (Isidori and Moog [11]). The zero dynamics has also been defined by Fliess [12] in a more direct fashion, from (7), as:

$$C(0,u,\dot{u},\ddot{u},\ldots,u^{(\nu)}) = 0. \tag{8}$$

A second possible source of difficulty in the use of the implicit dynamical controller (7) lies in the existence of *impasse points*. These points are, generally speaking, singularity points at which values of q produce a singular Jacobian matrix $\partial C/\partial u^{(\nu)}$.

III. SOME AEROSPACE APPLICATIONS

A. Dynamically Controlled Circular Path Maneuver for Aircraft Flying Through Region of Strong Winds with Limited Heading Rate

An aircraft must fly through a region of strong air currents whose magnitude and direction are only nominally known as functions of position: $h = h(x,y)$, $z = z(x,y)$, where (x,y) are rectangular coordinates and (h,z) are the velocity components for the air current in the x and y directions, respectively. The magnitude of the velocity of the aircraft, relative to the air, is assumed to be a constant V. The problem consists on maneuvering the aircraft along a circular path of given radius R with limited heading rate. The nonlinear model describing the controlled motions is (see [7, pp. 96]):

$$\begin{aligned}\dot{x} &= V\cos\phi + h(x,y);\\ \dot{y} &= V\sin\phi + z(x,y);\\ \dot{\phi} &= u\end{aligned} \tag{9}$$

where the scalar parameter u, acting as the control input, is the heading rate. ϕ is the heading angle of the axis of the aircraft, relative to the fixed Cartesian coordinates. The components (x,y) represent the position of the aircraft in such Cartesian coordinates. The heading angle ϕ takes values in the real line, while the control action u is assumed to be limited to the interval $[-1,1]$. This limited control action is to regulate, in a smooth manner, the motions of the aircraft toward the required circular path and to sustain a trajectory in an indefinite manner.

REMARK Fliess' differential algebraic approach applies, primarily, to systems described by algebraic differential equations. System (9), without the perturbation terms h and z, contains terms which are not algebraic. However, such terms are not *transcendentally differentially algebraic*, i.e., they can be expressed as solutions of purely algebraic differential equations with coefficients in the differential field $k\langle u\rangle$. Conceivably, the system can, therefore, be transformed into an *algebraic* differential system. Indeed, if we let $\xi = \cos\phi$ and $\zeta = \sin\phi$ then, two additional algebraic differential equations would be obtained describing ξ and ζ, i.e., $d\xi/dt = -u\zeta$ and $d\zeta/dt = u\xi$. Notice, however, that an algebraic constraint, of the type $\xi^2 + \zeta^2 = 1$, exists now for these new "state" variables ξ and ζ. The developments below would carry, much in the same manner, for the resulting, constrained, fifth-dimensional system. At the end, we would revert to the original expressions and use the algebraic restriction on the defined variables. Fliess has also shown [3] that the theory extends to systems with analytic coefficients. In this case, we thus indirectly use the *local generalized analytic observability canonical form* for system (9), without need for resorting to a higher dimensional algebraic system.

Defining a tracking error as $s(x,y) = x^2 + y^2 - R^2$, then, the problem of sustaining a circular trajectory in the (x,y) plane is translated into the problem of zeroing the nonlinear output function $s(x,y)$ associated to the nonlinear dynamic system (9).

It is easy to verify that $q := s(x,y)$ is a *differential primitive element* that allows one to write the model (9) in a GOCF of the form (6) with $\nu = 1$. For simplicity in the derivations ahead we assume that the distributed velocity components, $h(x,y)$ and $z(x,y)$, of the air currents in the x and y directions are nominally constant, i.e., of fixed direction and magnitude. Hence, in the computations below, the first- and second-order partial derivatives of h and z, with respect to x and y, are set to zero. However, the performance of our controller will be judged in the presence of distributed unmodelled wind gusts. Under these assumptions, one computes the required time derivatives of the primitive element q as

$$\begin{aligned} q &= s(x,y) = x^2 + y^2 - R^2 \\ \dot{q} &= 2V(x\cos\phi + y\sin\phi) + 2(xh + yz) \\ \ddot{q} &= 2[V^2 + 2V(h\cos\phi + z\sin\phi) \\ &\quad + h^2 + z^2 + uV(y\cos\phi - x\sin\phi)] \qquad (10) \\ \dddot{q} &= 6uV(-h\sin\phi + z\cos\phi) \\ &\quad + 2\dot{u}V(-x\sin\phi + y\cos\phi) \\ &\quad - 2u^2V(x\cos\phi + y\sin\phi). \end{aligned}$$

The controller design is based on imposing, for the exactly linearized GOCF, asymptotically stable dynamics, with suitably chosen eigenvalues (whether real or complex conjugate) a_1, a_2, and a_3, with negative real parts, and at least one real eigenvalue, as follows:

$$\dddot{q} - (a_1 + a_2 + a_3)\ddot{q} + (a_1a_2 + a_1a_3 + a_2a_3)\dot{q} - a_1a_2a_3 = 0. \qquad (11)$$

The nonlinear dynamic feedback controller will impose any desirable asymptotically stable behavior on the tracking error q, as well as on its time derivatives dq/dt and d^2q/dt^2. Such a nonlinear dynamic feedback controller, is immediately obtained from (10) and (11) as

$$\begin{aligned} \frac{du}{dt} &= \frac{1}{y\cos\phi - x\sin\phi} \\ &\quad \times \Big\{ -3u(-h\sin\phi + z\cos\phi) + u^2(x\cos\phi + y\sin\phi) \\ &\quad - \alpha_2\Big[V + 2(h\cos\phi + z\sin\phi) \\ &\quad + u(-x\sin\phi + y\cos\phi) + \frac{h^2 + z^2}{V}\Big] \\ &\quad - \alpha_1\Big[x\cos\phi + y\sin\phi + \frac{xh + yz}{V}\Big] \\ &\quad - \frac{\alpha_0}{2V}(x^2 + y^2 - R^2)\Big\} \end{aligned} \qquad (12)$$

with

$$\begin{aligned} \alpha_2 &= -(a_1 + a_2 + a_3); \\ \alpha_1 &= (a_1a_2 + a_1a_3 + a_2a_3); \\ \alpha_0 &= -a_1a_2a_3. \end{aligned}$$

It is easy to see that the factor $y\cos\phi - x\sin\phi$, in the denominator of the controller expression, does not yield any singularity, or *impasse points*, along the prescribed circular path. Indeed, since along the circular trajectory, the heading angle ϕ differs $\pi/2$ rad with respect to the aircraft position angle $\tan^{-1}(y/x)$, the singularity condition $y\cos\phi = x\sin\phi$, is never satisfied. A singularity is obtained, however, if initial conditions are such that the aircraft is initially heading away, or towards, the target circle along a radial direction. Quite understandably, this singularity is seen to precisely coincide with the violation of the rank condition (4). It is also easy to see that any maneuver initiated with nonradial headings will produce evolutions of the heading angle that stray away from the above singularity condition.

Computer simulations were carried out for the controlled system (9), (12). The radius of the circular path was chosen as $R = 800$ mts, while the aircraft velocity was set to $V = 110$ mts/s. The poles for the exactly linearized system were all located at $-1.1\ \mathrm{s}^{-1}$. The functions $h(x,y)$ and $z(x,y)$ were nominally chosen as $h_0 = 10$ mts/s and $z_0 = 40$ mts/s (i.e., a wind current with fixed magnitude of 41.23 mts/s and direction of 75.96° with respect to the x axis). However, a large unmodelled wind gust perturbation, distributed around the point $(x,y) = (0,-R)$ of the circular path, was included in the simulation of the system model (9), but the corresponding expression for the perturbation function was never substituted on the designed controller (12). Such perturbation was assumed to be of the form:

$$z(x,y) = z_0\left\{1 + \exp\left(-\left[\left(\frac{x}{100}\right)^2 + \left(\frac{y+R}{100}\right)^2\right]\right)\right\}. \qquad (13)$$

Fig. 1 shows a computer simulated trajectory in the plane x, y, depicting the response of the dynamic feedback controlled maneuver under nominal and perturbed conditions for $z(x,y)$. In Fig. 2 the perturbed and unperturbed dynamic controller output signals are compared. Fig. 3 compares the time responses of the distance error to the required circular trajectory, for the perturbed and unperturbed controlled systems. This error was defined to be: $e(x,y) = (x^2 + y^2)^{1/2} - R$, just for numerical convenience as compared with $s(x,y)$.

B. Smooth Variable Structure Controlled Landing on Surface of Nonatmosphere-free Planet

The following example has been previously treated in Sira-Ramírez [13], using a *memoryless* sliding mode feedback controller, and it was also treated in [8] in the context of a dynamic feedback ON-OFF PWM controller scheme. General background on the use of Fliess' GOCF to "smoothed" sliding mode control, and

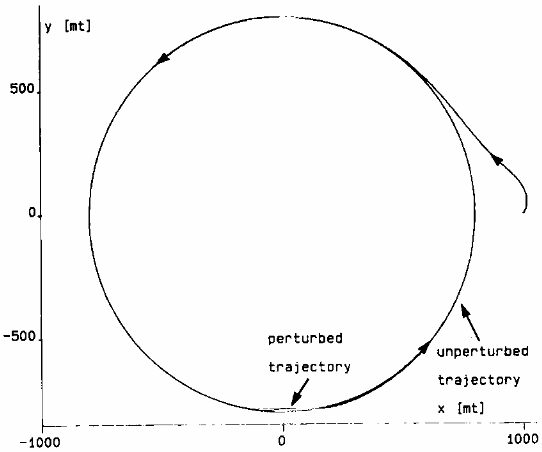

Fig. 1. Dynamic feedback controlled airplane tracking of circular path under nominal and unmodelled wind gust perturbation.

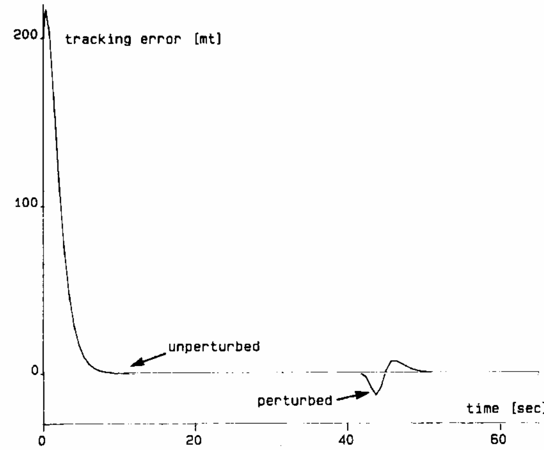

Fig. 3. Tracking error of dynamic feedback controlled airplane following a circular path under nominal and strong wind perturbations.

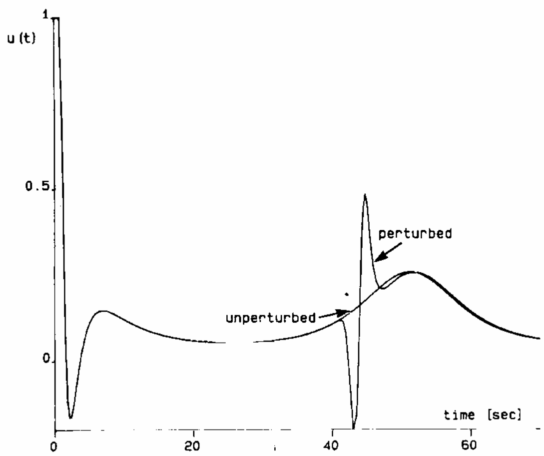

Fig. 2. Time responses of limited dynamic controller output regulating a nominal and wind perturbed tracking of the aircraft circular path.

asymptotic output stabilization, is summarized in the Appendix.

Consider the nonlinear dynamic model describing the vertical descent, including the spacecraft mass behavior, of a thrust controlled vehicle attempting a regulated landing on the surface of a planet of gravity acceleration g and nonnegligible atmospheric resistance force opposing the vertical downwards motion.

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = g - \left(\frac{\gamma}{x_3}\right)x_2^2 - \left(\frac{\sigma\alpha}{x_3}\right)u \qquad (14)$$

$$\frac{dx_3}{dt} = -\alpha u$$

where x_1 is the position (height) on the vertical axis, chosen here to be positively oriented downwards (i.e., $x_1 < 0$, for actual positive height), x_2 is the downwards velocity and x_3 represents the combined mass of the vehicle and the residual fuel. The function u is a control function taking values in the closed interval [0,1] regulating, in an *amplitude modulated* manner, the constant rate of ejection per unit time α and effectively acting as a control parameter. The constant σ represents the relative ejection velocity of the gases in the thruster. Thus, $\sigma\alpha$ is the maximum thrust of the braking engine, while γ is a positive quantity representing the atmospheric resistance coefficient.

A soft landing on the surface $x_1 = 0$ may be seen as a particular case of a controlled descent toward a sustained hovering about certain preestablished height $x_1 = K$. Usually, the landing maneuver entitles a regulated descent toward a small height (typically 1 mt, or so, i.e., $K = -1$) on which a short hovering takes place before the main thruster is safely shut off. The final touchdown stage is actually a free fall toward the surface from the small hovering height. Taking the output function of the system as $y = h(x) = x_1 - K$, the problem of sustained hovering is translated into the problem of zeroing the output y that one can associate to the nonlinear system (14).

REMARK It is evident from the dynamic system equations (14) that the maximum value of the downwards velocity x_2 takes place only under free fall (i.e., uncontrolled) conditions ($u = 0$, $x_3 =$ constant $= M$). This maximum velocity value is precisely given by $(gM/\gamma)^{1/2}$. In such a case the downwards spacecraft acceleration is zero. A braking maneuver toward a sustained hovering, starting from free fall conditions, entitles a negative controlled acceleration until reaching zero downwards velocity at the preestablished hovering height $x_1 = K$. At this point the controlled acceleration should also become zero. It follows that, during the controlled descent, the downwards acceleration is always bounded above by zero.

We proceed to specify the GOCF of system (14) which allows us to derive a nonlinear dynamic variable structure feedback controller for achieving a slow descent maneuver.

It is easy to verify that $q_1 = x_1 - K$ is *differential primitive element* that allows one to write the model (14) in a GOCF. Thus, the control-dependent state coordinate transformation of the controlled system (14) given by

$$q_1 = x_1 - K, \qquad q_2 = x_2, \qquad q_3 = g - \frac{\gamma x_2^2 + \sigma\alpha u}{x_3}$$
$$x_1 = q_1 + K, \qquad x_2 = q_2, \qquad x_3 = \frac{\gamma q_2^2 + \sigma\alpha u}{g - q_3} \tag{15}$$

yields the following transformed system in GOCF:

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= q_3 \\ \dot{q}_3 &= -(g - q_3)\left[\frac{2\gamma q_2 g + \sigma\alpha\dot{u}}{\gamma q_2^2 + \sigma\alpha u}\right] \\ &\quad + (2\gamma q_2 - \alpha u)\left[\frac{(g - q_3)^2}{\gamma q_2^2 + \sigma\alpha u}\right] \\ y &= q_1. \end{aligned} \tag{16}$$

Consider the auxiliary output function ω given by (see the Appendix):

$$\omega = q_3 + m_1 q_2 + m_0 q_1 \tag{17}$$

with m_1 and m_0 representing design constants. A dynamic variable structure feedback controller may be synthesized by enforcing on ω the discontinuous dynamic behavior described by

$$\dot{\omega} = -\lambda(\omega + \Omega\,\mathrm{sign}\,\omega) \tag{18}$$

with λ and Ω being positive quantities. It is easily seen that (18) autonomously undergoes a sliding motion on $\omega = 0$. If this sliding mode is appropriately sustained, it can be seen, from the results in the Appendix, that by virtue of (16) and (17), the canonical coordinates q_1, q_2, q_3 exhibit an asymptotically stable motion towards the origin. Thus, the equality:

$$\begin{aligned} &-(g - q_3)\left[\frac{2\gamma q_2 g + \sigma\alpha\dot{u}}{\gamma q_2^2 + \sigma\alpha u}\right] \\ &\quad + (2\gamma q_2 - \alpha u)\left[\frac{(g - q_3)^2}{\gamma q_2^2 + \sigma\alpha u}\right] + m_1 q_3 + m_0 q_2 \\ &= -\lambda[q_3 + m_1 q_2 + m_0 q_1 \\ &\qquad + \Omega\,\mathrm{sgn}(q_3 + m_1 q_2 + m_0 q_1)] \end{aligned} \tag{19}$$

leads to synthesizing the *computed control function* (henceforth denoted by μ) as

$$\begin{aligned} \frac{d}{dt}\mu = \frac{\gamma q_2^2 + \sigma\alpha\mu}{\sigma\alpha(g - q_3)}\Big\{ &(m_1 + \lambda)q_3 + (m_0 + \lambda m_1)q_2 \\ &+ \lambda m_0 q_1 + \lambda\Omega\,\mathrm{sgn}[q_3 + m_1 q_2 + m_0 q_1] \\ &+ (2\gamma q_2 - \alpha\mu)\left[\frac{(g - q_3)^2}{\gamma q_2^2 + \sigma\alpha\mu}\right]\Big\} - \frac{2\gamma q_2 g}{\sigma\alpha}. \end{aligned} \tag{20}$$

The *actual control input function* u is obtained by properly limiting between 0 and 1 the values of the computed control function μ, obtained as a solution of the nonlinear time-varying differential equation (20), i.e., the actual control input u is given by

$$u = \begin{cases} 1 & \text{if } \mu \geq 1 \\ \mu & \text{if } 0 < \mu < 1. \\ 0 & \text{if } \mu \leq 0 \end{cases} \tag{21}$$

Notice that no singularity is implied by the presence of the factor $(g - q_3)^{-1}$ in (20) due to the established negativity of the vertical acceleration q_3 during the descent maneuver. In original state coordinates, the dynamic variable structure feedback controller is given by

$$\begin{aligned} \frac{d}{dt}\mu = \frac{x_3}{\sigma\alpha}\Bigg[&\lambda m_0(x_1 - K) + (m_0 + \lambda m_1)x_2 \\ &+ (m_1 + \lambda)\left(g - \frac{\gamma x_2^2 + \sigma\alpha\mu}{x_3}\right) \\ &+ \lambda\Omega\,\mathrm{sgn}\left[\left(g - \frac{\gamma x_2^2 + \sigma\alpha\mu}{x_3}\right) + m_1 x_2 + m_0(x_1 - K)\right] \\ &+ (2\gamma x_2 - \alpha\mu)\left(\frac{\gamma x_2^2 + \sigma\alpha\mu}{x_3^2}\right)\Bigg] - \frac{2\gamma x_2 g}{\sigma\alpha}. \end{aligned} \tag{22}$$

Under sliding mode behavior on the switching manifold, the invariance conditions $\omega = 0$ and $d\omega/dt = 0$, result in the following ideal sliding dynamics:

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= -m_1 q_2 - m_0 q_1 \\ y &= q_1 \end{aligned} \tag{23}$$

which can be made exponentially asymptotically stable to $y = 0$ by appropriately choosing of the (positive) coefficients m_0 and m_1. In other words, the dynamic variable structure feedback controller accomplished, within nonsaturating conditions for the limiting values of the actuator, any desirable exponential rate of decay on the height error q_1 and vertical velocity q_2. The *equivalent control function* (see Utkin [14]) is obtained as the solution of

$$\frac{d}{dt}u_{EQ} = \frac{\gamma q_2^2 + \sigma\alpha u_{EQ}}{\sigma\alpha(g - q_3)} \times \left\{ -m_1 q_3 - m_0 q_2 + (2\gamma q_2 - \alpha u_{EQ}) \times \left[-\frac{(g - q_3)^2}{\gamma q_2^2 + \sigma\alpha u_{EQ}} \right] \right\} - \frac{2\gamma q_2 g}{\sigma\alpha} \tag{24}$$

with the restriction:

$$q_3 = -m_1 q_2 - m_0 q_1. \tag{25}$$

A sliding regime exists in those regions of the sliding manifold where $0 < u_{EQ} < 1$ (see Sira-Ramírez [15]).

Notice that under ideal sliding conditions, both q_1 and q_2 asymptotically approach zero. Hence, by virtue of (17), q_3 also decreases to zero. The hovering condition is then characterized by $q_1 = q_2 = q_3 = 0$. It follows from (23) that the equivalent control is governed by

$$\dot{u}_{EQ} = -\frac{g}{\sigma} u_{EQ} \tag{26}$$

and by (15) the total mass is governed by

$$x_3 = \frac{\sigma\alpha}{g} u_{EQ}. \tag{27}$$

It is easy to see from (26), (27), that the hovering conditions cannot be indefinitely sustained since this would imply that the total mass of the spacecraft asymptotically converges to zero. This cannot physically happen, due to earlier total depletion of the fuel mass. The mathematical model, thus, becomes unrealistic after the fuel mass has been totally consumed. Under these circumstances, the reached equilibrium point would not be physically meaningful. This fact coincides with the *zero dynamics* analysis developed in [8] for a related, but different control strategy, based on dynamic ON-OFF PWM control of the descending spacecraft.

Simulations were performed with the following constant parameters: $\sigma = 200$ [mt/s], $\alpha = 50$ [Kg/s], $g = 3.72$ [mt/s^2], $\gamma = 1$ [Kg/mt], and $K = -1$ [mt]. The poles of the ideal sliding dynamics were both located at $-1.0\ s^{-1}$, while Ω and λ were set to 1 mt/s and 5 s^{-1}, respectively. On a planet with the given physical constants, the free fall limit velocity is 51.03 [mt/s]. Fig. 4 shows the evolution of the controlled state variables x_1, x_2 (height and vertical velocity) and the total mass x_3. Fig. 5(a) represents the time evolution of the actual (i.e., limited) control input u during the controlled descent maneuver. Fig. 5(b) contrasts the auxiliary output function ω and the input signal u in relation to sliding mode existence for the auxiliary output function. Initial states were chosen, from an ongoing free fall condition, at $x_1(0) = -500$ [mt], $x_2 = 51.03$ [mt/s], and $x_3(0) = 700$ [Kg].

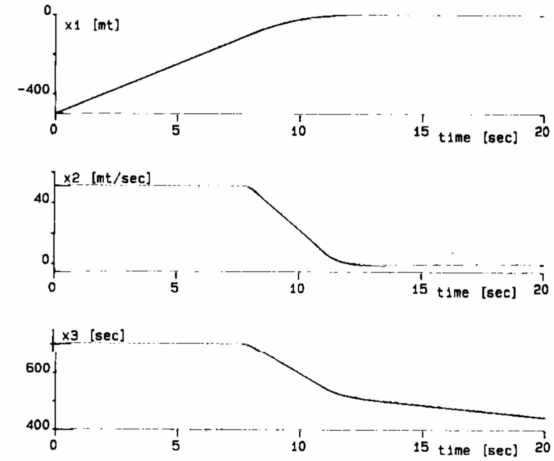

Fig. 4. Dynamic variable structure controlled height, vertical velocity, and total spacecraft mass evolution for soft landing maneuver.

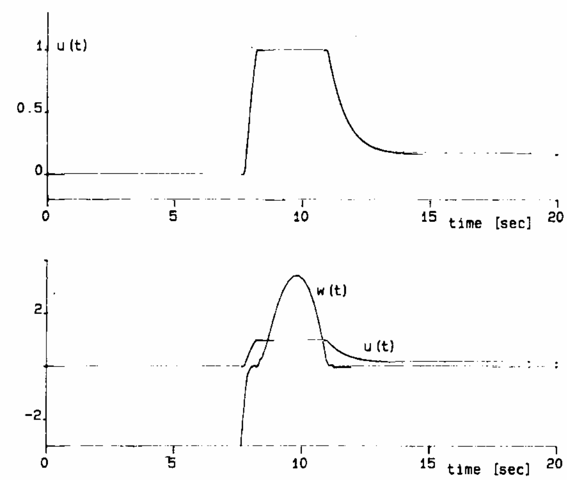

Fig. 5. (a) Actual amplitude modulated control input for sliding mode soft controlled landing. (b) Auxiliary output function exhibiting sliding regime in regions of nonsaturated controller output.

C. Dynamic PWM Feedback Controlled Orbital Transfer Maneuver

A well-known model for a normalized (unit) mass spacecraft performing a controlled orbital transfer is given by (see [9, pp. 14], and, also [7, pp. 66]):

$$\begin{aligned} \dot{r} &= v; \\ \dot{v} &= r\omega^2 - \frac{k}{r^2} + u_1; \\ \dot{\omega} &= -\frac{2v\omega}{r} + \frac{1}{r}u_2 \end{aligned} \tag{28}$$

with r being the radial distance of the spacecraft from the center of the Earth, v is the radial component of the velocity, ω is the angular velocity, k is the Earth gravitational constant, and u_1 and u_2 represent, respectively, thrust in the radial and tangential directions. The orbital parameters are constrained to satisfy, in steady state equilibrium conditions: $r = R$, $\omega = \Omega$, $u_1 = u_2 = 0$, the relation $R^3\Omega^2 = k$. A PWM ON-OFF-ON control policy, such as that described by (43) and (44) (in the Appendix), is assumed for each thruster. The control inputs u_1 and u_2 take values, respectively, in the discrete sets $\{+U_1, 0, -U_1\}$ and $\{+U_2, 0, -U_2\}$. We summarize below the steps leading to a PWM dynamic feedback controller design based on the *average PWM* model of the given multivariable system. The developments in the Appendix justify the use of such an average model for design purposes.

Average PWM Model: According to the results of the Appendix, the average PWM model is simply obtained by formally replacing the discontinuous control inputs u_1 and u_2 by their corresponding piecewise smooth *duty ratio* functions μ_1 and μ_2.

$$\begin{aligned} \dot{r} &= v; \\ \dot{v} &= r\omega^2 - \frac{k}{r^2} + U_1\mu_1; \\ \dot{\omega} &= -\frac{2v\omega}{r} + \frac{1}{r}U_2\mu_2. \end{aligned} \tag{29}$$

REMARK Although this system is trivially exactly linearizable by means of a *static* multivariable controller, we proceed to synthesize a dynamic controller just to illustrate the design process.

GOCF for the Average PWM Model: Taking as the output variable $y = r - R$, it is easy to see, by application of condition (4), that $q_1 = y$ qualifies as a *differential primitive element* for (29), on the basis of which we can generate a GOCF:

$$\begin{aligned} \dot{q}_i &= q_{i+1}; \qquad i = 1,2 \\ \dot{q}_3 &= -3q_2\left[\frac{q_3 - U_1\mu_1}{q_1 + R} + \frac{k}{(q_1 + R)^3}\right] \\ &\quad + 2U_2\mu_2\sqrt{\frac{q_3 - U_1\mu_1}{q_1 + R} + \frac{k}{(q_1 + R)^3}} \\ &\quad + \frac{2kq_2}{(q_1 + R)^3} + U_1\dot{\mu}_1 \\ y &= q_1. \end{aligned} \tag{30}$$

Desirable Linearized PWM Average Dynamics for the Output Variable:

$$\begin{aligned} \dot{q}_1 &= q_2; \\ \dot{q}_2 &= q_3; \\ \dot{q}_3 &= -\alpha_1 q_1 - \alpha_2 q_2 - \alpha_3 q_3; \\ y &= q_1. \end{aligned} \tag{31}$$

Dynamic Average Feedback Controller Synthesizing the Unrestricted Computed Duty Ratios: Equating the last state equation in (30) and (31) leads to a time-varying differential equation for μ_1 with indeterminate quantity represented by the duty ratio function μ_2. This duty ratio function can thus be independently chosen. We propose to use a μ_2 which exactly linearizes, to an asymptotically exponentially stable dynamics, the last equation in (29).

$$\begin{aligned} \frac{d}{dt}\hat{\mu}_1 &= -\frac{\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3}{U_1} \\ &\quad + \frac{3q_2}{U_1}\left[\frac{q_3 - U_1\hat{\mu}_1}{q_1 + R} + \frac{k}{(q_1 + R)^3}\right] \\ &\quad - \frac{2U_2\hat{\mu}_2}{U_1}\sqrt{\frac{q_3 - U_1\hat{\mu}_1}{q_1 + R} + \frac{k}{(q_1 + R)^3}} \\ &\quad - \frac{2kq_2}{U_1(q_1 + R)^3} \\ \hat{\mu}_2 &= \frac{\alpha_4}{U_2}\sqrt{\frac{k}{R^3}}(q_1 + R) + \frac{[2q_2 - \alpha_4(q_1 + R)]}{U_2} \\ &\quad \times \sqrt{\frac{q_3 - U_1\hat{\mu}_1}{q_1 + R} + \frac{k}{(q_1 + R)^3}}. \end{aligned} \tag{32}$$

The dynamic controller specifying the computed duty ratio μ_1 is synthesized on the basis of achieving an average asymptotically stable behavior of the orbital radial error $q_1 = r - R$. The nondynamical, time-varying, feedback controller specifying the computed duty ratio μ_2 yields an asymptotically exponentially stable motion toward the corresponding constant angular velocity: $\Omega = (k/R^3)^{1/2}$, (i.e., the desired dynamics for ω is imposed as $d\omega/dt = -\alpha_r(\omega - \Omega)$). The exponential rate of decay of ω is thus specified by α_4. The *actual duty ratio* functions μ_1 and μ_2 are obtained constraining the solutions μ_1, μ_2 of (32) to the closed interval $[-1, 1]$. The discontinuous ON-OFF-ON PWM multiinput controller is now easily synthesized on the basis of these actual feedback duty ratio functions μ_1 and μ_2, according to the switching policy described in (43) and (44).

A computer-simulated experiment was also carried out to illustrate the quality of the response obtained with the designed discontinuous dynamic feedback controller. The radius of the Earth was taken as 6371 Kmts and the gravity constant k, for such magnitude of heights, was set to $k = 389258.1$ Km^3/s^2. Initial conditions were taken for a circular orbit located some 150 Km high above the surface of the Earth (orbit data: $r = 6521$ Km, $v = 0$ Km/s, $\omega = 1.1832 \times 10^{-3}$ rad/s). A controlled maneuver was performed which brought the spacecraft to a second orbit of 175 Km of height (orbit data: $r = 6546$ Km, $v = 0$ Km/s, $\mu = 1.1780 \times 10^{-3}$ rad/s). The three poles of the linearized system were located at -0.1 s^{-1}. Fig. 6 shows the time response of the radial position,

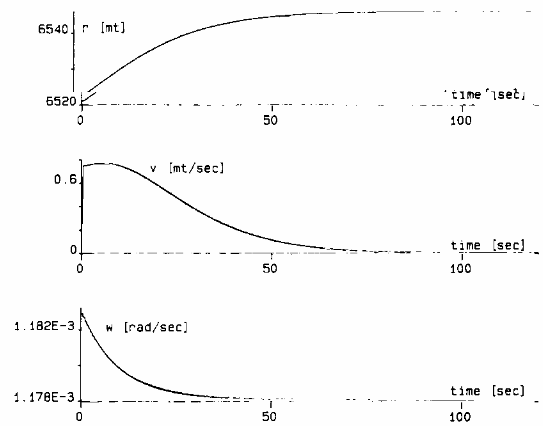

Fig. 6. Radial position, radial velocity and angular velocity responses for dynamic ON-OFF-ON PWM feedback controlled spacecraft orbital transfer.

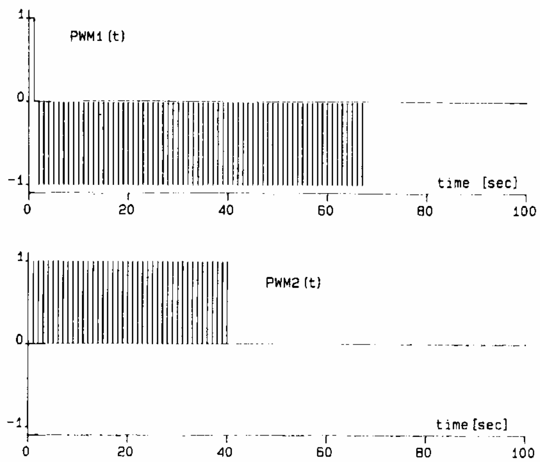

Fig. 7. Discontinuous ON-OFF-ON dynamically generated PWM control inputs.

the radial velocity response and the time evolution of the corresponding angular velocity of the PWM controlled spacecraft. Fig. 7 shows the discontinuous control signals commanding the orbitral transfer of the spacecraft.

IV. CONCLUSIONS

Synthesis of nonlinear dynamic feedback regulators, based on the differential algebraic approach to controlled systems dynamics, has been carried for some typical aerospace problems. In spite of the exact linearizations involved in the determination of the dynamic feedback policies, the simulation results demonstrate a certain degree of robustness of the obtained control schemes with respect to unmodeled disturbances. Moreover, the synthesis approach was seen to be suitable for either continuously or discontinuously controlled plants (PWM and sliding modes). In the first example, a local exact linearization is accomplished by means of a smooth dynamic controller with bounded input amplitude. In the second example a dynamic variable structure controller was designed which exhibits drastically reduced chattering both at the commanded input and at the regulated output. In the third example, an infinite frequency average model transforms the discontinuous multivariable ON-OFF-ON PWM controller design problem into a problem of continuous nature. In this example, the average solution was shown to be approximated, in an arbitrarily close manner, by the discontinuous control policy provided a sufficiently high sampling frequency is used.

APPENDIX

A. Generalities about ON-OFF PWM Control of Nonlinear Systems

Consider a multiinput nonlinear dynamic system defined on an open set of R^n described by

$$\dot{x} = f(x) + G(x)\mathrm{v} \tag{33}$$

with $\mathrm{v} \in R^m$, being a discontinuous feedback control strategy of the PWM type, given, componentwise, by

$$\mathrm{v}_i = \begin{cases} \mathrm{v}_i^+(x) & \text{for} \quad t_k < t \le t_k + \mu_i[x(t_k)]T \\ \mathrm{v}_i^-(x) & \text{for} \quad t_k + \mu_i[x(t_k)]T < t \le t_k + T; \end{cases}$$

$$i = 1,2,\ldots,m; \quad k = 0,1,2,\ldots \tag{34}$$

where T is a fixed sampling period, also known as the *duty cycle*, t_k is the kth sampling instant and $\mu_i(x(t))$ is a continuous piecewise smooth feedback function for the ith controller, known as the *duty ratio* function. This function determines the control pulsewidth of the ith control vector component during the ongoing inter-sampling interval $[t_k, t_k + T]$. The pulsewidths $\mu_i(x(t_k))T$ are determined at the beginning of each sampling interval t_k, uniformly for all controllers, on the basis of the value of the sampled state vector at such instants (schemes based on output functions, or output errors are also possible). The continuous piecewise smooth duty ratio functions are assumed to be bounded by $0 < \mu_i(x(t)) < 1$ for all t. Then $n \times m$ matrix $G(x)$ is assumed to be locally full rank with smooth vector fields $g_i(x)$ $(i = 1,\ldots,m)$ constituting its columns.

The effect of such a discontinuous feedback strategy on the controlled state trajectories is to produce a zig-zag motion, very much reminiscent of actual sliding mode controlled trajectories ([14]). The analysis and design of such class of hybrid systems (33), (34) is extremely difficult and can only be carried out in an approximate manner.

However, the inconveniences of the nonlinear discrete-time approximations can be eliminated if some smooth continuous average model is adopted as an approximation for the actual PWM controlled system. Such a smooth average behavior may be considered on the basis of a high sampling frequency for systems which are relatively slow as compared with such fast control changes. In the following paragraphs we justify the use of an average continuous model based on an *infinite sampling frequency* assumption for (33), (34). The advantages of such an averaging procedure, aside from some intimate connections with sliding modc control (see Sira-Ramírez [16–18]), lay in the possibility of using modern nonlinear feedback control design techniques for the synthesis of the duty ratio as a truly feedback function. Furthermore, the smooth average designed behavior can be arbitrarily closely approximated by the actual discontinuous feedback controlled trajectories as the sampling frequency of the PWM actuator is suitably increased within a finite bound.

It is easily seen that the discontinuously controlled model (33), (34) is equivalent to the following switch-controlled model:

$$\frac{dx}{dt} = f(x) + \sum_{i=1}^{m} g_i(x)[u_i v_i^+(x) + (1-u_i)v_i^-(x)] \tag{35}$$

with

$$u_i = \begin{cases} 1 & \text{for} \quad t_k < t \le t_k + \mu_i[x(t_k)]T \\ 0 & \text{for} \quad t_k + \mu_i[x(t_k)]T < t \le t_k + T; \end{cases}$$
$$i = 1,\dots,m; \quad k = 0,1,2,\dots. \tag{36}$$

DEFINITION 1 An *average PWM model* for the discontinuously controlled system (33), (34) (or equivalently (35), (36) is defined by the dynamic system formally obtained by letting the sampling frequency $1/T$ of the PWM actuator grow to infinity, i.e., letting the duty $T \to 0$. We denote the state of the averaged system by $z(t)$, just to differentiate it from the state vector $x(t)$ of the discontinuously controlled system.

PROPOSITION 1 *The average PWM model, obtained by formally imposing an infinitely large sampling frequency, $1/T$, for the controlled system* (35), (36) *is given by*

$$\frac{dz}{dt} = f(z) + \sum_{i=1}^{m} g_i(z)[\mu_i v_i^+(z) + (1-\mu_i)v_i^-(z)] \tag{37}$$

PROOF. Sira-Ramírez et al. [19]) We can rewrite the differential equation for the controlled plant in (35), (36) as an equivalent integral-difference equation exactly relating the values of the state vector $x(t)$ at subsequent sampling instants:

$$x(t_k+T) = x(t_k) + \int_{t_k}^{t_k+T} f(x(\sigma))\,d\sigma + \sum_{i=1}^{m}\int_{t_k}^{t_k+\mu_i[x(t_k)]T} g_i(x(\sigma))v_i^+(x(\sigma))\,d\sigma + \sum_{i=1}^{m}\int_{t_k+\mu_i[x(t_k)]T}^{t_k+T} g_i(x(\sigma))v_i^-(x(\sigma))\,d\sigma.$$

Substracting $x(t_k)$ from $x(t_k+T)$, dividing by T and taking the limit as the sampling period goes to zero, while letting the sampling instant t_k adopt an arbitrary value t of the time coordinate, one obtains

$$\lim_{\substack{T\to 0\\ t_k\to t}}\left[\frac{x(t_k+T)-x(t_k)}{T}\right] = \lim_{\substack{T\to 0\\ t_k\to t}}\left\{\int_{t_k}^{t_k+T} f(x(\sigma))d\sigma + \sum_{i=1}^{m}\left[\int_{t_k}^{t_k+\mu_i[x(t_k)]T} g_i(x(\sigma))v_i^+(x(\sigma))\,d\sigma + \int_{t_k+\mu_i[x(t_k)]T}^{t_k+T} g_i(x(\sigma))v_i^-(x(\sigma))\,d\sigma\right]\right\}.$$

Using now the *fundamental lemma of calculus*, and denoting the average state coordinates by the vector z, instead of x, one immediately obtains that, at each instant of time, the time derivative of the average state vector is precisely given by the expression in (37).

REMARK The average PWM model (37) is related to the *Filippov average* dynamics (see Filippov [20]) of the underlying discontinuous system, componentwise controlled by $v_i^+(x)$ and $v_i^-(x)$, when an infinitely fast switching strategy takes place around a certain discontinuity surface on which the resulting controlled trajectory can be locally sustained. Notice, furthermore, that (37) is a linear-in-the-control vector differential equation formally obtained from the original discontinuous model (35), (36) just by replacing the binary-valued control vector parameter components u_i by the continuous piecewise smooth duty ratio function vector components u_i.

The following result states that under identical initial conditions, the controlled trajectories of the actual discontinuous feedback controlled system (35), (36) continuously tend toward the average PWM controlled trajectories generated by (37) as the sampling frequency associated to the PWM actuator (36) is increased without limit. Hence, to arbitrarily closely retain the qualitative and quantitative stability characteristics of the average PWM designed trajectories, a sufficiently high sampling frequency is required for the PWM actuator of the actual discontinuously controlled system. This is the key feature that allows an efficient design scheme based on the continuous average PWM model. We prove the result for single input systems ($m = 1$) and refer for the details of the extension for multiinput systems to [19].

THEOREM 1 *Let $\mu(t)$ be a given continuous piecewise smooth duty ratio functioned bounded by $0 < \mu(t) < 1$. Under identical initial conditions for the actual and average PWM controlled models, the corresponding controlled state trajectories of the discontinuous PWM system* (35), (36) *continuously and globally converge toward those of the corresponding average PWM system* (37) *as the sampling frequency $1/T$ grows without bound.*

PROOF. Notice that if in (35) and (37) the smooth vector $g(x) = 0$, the theorem is trivially true and, as a matter of fact, both trajectories $x(t)$ and $z(t)$ coincide for all t. The same statement holds true for identical initial conditions and $\mu(t) \equiv 1$, or $\mu(t) \equiv 0$, on open intervals of time. Thus assume $g(x) \neq 0$. By virtue of the *theorem of rectifiability of vector fields* (see Arnol'd [21, pp. 85]) there exists a diffeomorphic state coordinate transformation $\Phi(\cdot) : R^n \to R^n$, yielding $\xi = \Phi(x)$ and $\zeta = \Phi(z)$ for the actual and average state coordinates, such that in the transformed coordinates the vector field $g(x)$ is expressed as a constant vector of value, say, b, i.e.,

$$\left[\frac{\partial \Phi(x)}{\partial x}\right]_{x=\Phi^{-1}(\xi)} g(\Phi^{-1}(\xi)) = b;$$

$$\left[\frac{\partial \Phi(z)}{\partial z}\right]_{z=\Phi^{-1}(\zeta)} g(\Phi^{-1}(\zeta)) = b.$$

Evidently, such a diffeomorphic state coordinate transformation is not expected to produce any particularly special structure on the *drift* vector fields $f(x)$ and $f(z)$ in (35) and (37). We denote the transformed drift vector fields respectively, by $\phi(\xi)$ and $\phi(\zeta)$, i.e.,

$$\left[\frac{\partial \Phi(x)}{\partial x}\right]_{x=\Phi^{-1}(\xi)} f(\Phi^{-1}(\xi)) = \phi(\xi);$$

$$\left[\frac{\partial \Phi(z)}{\partial z}\right]_{z=\Phi^{-1}(\zeta)} f(\Phi^{-1}(\zeta)) = \Phi(\zeta).$$

Then the controlled dynamic systems (35) and (37) are expressed, in new coordinates, as

$$\frac{d\xi}{dt} = \phi(\xi) + bu; \qquad \frac{d\zeta}{dt} = \phi(\zeta) + b\mu \tag{38}$$

with identical initial conditions being assumed ($\xi(t_0) = \zeta(t_0) = \psi_0$).

Let T be any finite time interval containing an integer number N of sampling periods T, i.e., $\mathrm{T} = NT$, with N being of order $[K/T]$, i.e., the order of T is independent of T. The differential equations (38) can be equivalently expressed as integral equations of the form:

$$\xi(\mathrm{T}) = \psi_0 + \int_{t_0}^{\mathrm{T}} \phi(\xi(\sigma))\, d\sigma + b \sum_{k=0}^{N} \int_{t_k}^{t_k+\mu(t_k)T} d\sigma \tag{39}$$

$$\zeta(\mathrm{T}) = \psi_0 + \int_{t_0}^{\mathrm{T}} \phi(\zeta(\sigma))\, d\sigma + b \sum_{k=0}^{N} \int_{t_k}^{t_k+T} \mu(\sigma)\, d\sigma. \tag{40}$$

Evidently, the sum of integral terms in (39) represents a second-order approximation to the sum of integral terms in (40). Hence, the integral equation in (40) is a *regular second-order perturbation*, in terms of the sampling parameter T, of the integral equation (39). Indeed, using a Taylor series expansion of $\mu(\sigma)$ around t_k on each summand in (40), one may rewrite equations (39), (40) as

$$\zeta(\mathrm{T}) = \psi_0 + \int_{t_0}^{\mathrm{T}} \phi(\zeta(\sigma))\, d\sigma + b \sum_{k=0}^{N} \mu(t_k) T \tag{41}$$

$$\xi(\mathrm{T}) = \psi_0 + \int_{t_0}^{\mathrm{T}} \phi(\xi(\sigma))\, d\sigma + b \sum_{k=0}^{N} \left[\mu(t_k)T + \frac{1}{2} \left.\frac{d\mu(\sigma)}{d\sigma}\right|_{\sigma=t_k} T^2 + O(T^3) \right] \tag{42}$$

If follows from well-known results in the theory of perturbations of integral equations (see Miller [22, 273–285]), that as the regular perturbation decreases to zero (i.e., as sampling period T decreases to zero), the solution of the first integral equation, representing the actual discontinuously PWM controlled system, *continuously converges*, in a global manner, toward the solution of the second integral equation representing the average PWM system (see also: Tikhonov et al. [23, 180–185] for the same basic result in the context of ordinary differential equations).

The above result easily extends to multiinput systems, but in the proof of such a case one cannot resort to the *theorem of rectifiability of vector fields* (see [19] for details).

The final step in completing a design procedure based on the average PWM model consists in translating the average continuous stabilizing feedback controller design into a suitable ON-OFF (i.e., discontinuous) feedback controller of PWM nature. Such ON-OFF controller must retain the stabilizing features of the continuous average designed controller and, at the same time, it should yield actual discontinuous responses that remain arbitrarily close to the smooth designed responses. This is primarily accomplished by specifying a sufficiently high sampling frequency for the actual PWM actuator and, secondly, by suitable smoothing of the state variables before using them in the synthesis of the average stabilizing designed controller. The smoothing action may be accomplished by introducing low-pass filtering effects on the state variables measurements. One then simply relies on the high-frequency rejection characteristics of most sensing devices.

B. Generalities about ON-OFF-ON PWM Control of Nonlinear Systems

In a typical ON-OFF-ON PWM control strategy for the control of the nonlinear system (33), the switching actions are specified according to the sign of the duty ratio function components as

For $\mu_i[x(t_k)] > 0$:

$$\mathrm{v}_i = \begin{cases} \mathrm{v}_i^+(x) & \text{for} \quad t_k < t \le t_k + \mu_i[x(t_k)]T \\ 0 & \text{for} \quad t_k + \mu_i[x(t_k)]T < t \le t_k + T \end{cases} \qquad i = 1,\dots,m; \quad k = 0,1,2,\dots . \tag{43}$$

For $\mu_i[x(t_k)] < 0$:

$$\mathrm{v}_i = \begin{cases} \mathrm{v}_i^-(x) & \text{for} \quad t_k < t \le t_k + |\mu_i[x(t_k)]|T \\ 0 & \text{for} \quad t_k + |\mu_i[x(t_k)]|T < t \le t_k + T \end{cases} \qquad i = 1,\dots,m; \quad k = 0,1,2,\dots \tag{44}$$

where $\mu_i(x(t))$ is the ith *duty ratio* function constrained now within the bounds $-1 < \mu_i(x(t)) < 1$.

It is easily seen that the discontinuously controlled model (33), (43), (44) is equivalent to the following switch-controlled model:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\left[\frac{u_i}{2}(\mathrm{v}_i^+ - \mathrm{v}_i^-) + \frac{|u_i|}{2}(\mathrm{v}_i^+ + \mathrm{v}_i^-)\right] \tag{45}$$

with

$$u_i = \begin{cases} \operatorname{sign}\mu_i[x(t_k)] & \text{for} \quad t_k < t \le t_k + |\mu_i[x(t_k)]|T \\ 0 & \text{for} \quad t_k + |\mu_i[x(t_k)]|T < t \le t_k + T \end{cases} \qquad i = 1,\dots,m; \quad k = 0,1,\dots . \tag{46}$$

To obtain the average model of (45), (46) when the switching frequency tends to infinity, i.e., when $T \to 0$, we express the exact solution of (45), at any time $t_k + T$, as a function of the state at time t_k, and then investigate the description of the state in the limit.

$$\begin{aligned} x(t_k + T) = x(t_k) &+ \int_{t_k}^{t_k+T} f(x(\sigma))\,d\sigma \\ &+ \sum_{i=1}^{m} \int_{t_k}^{t_k+|\mu_i[x(t_k)]|T} g_i(x(\sigma)) \\ &\times \left\{ \frac{\operatorname{sign}\mu_i[x(t_k)]}{2}[\mathrm{v}_i^+(x(\sigma)) - \mathrm{v}_i^-(x(\sigma))] \right. \\ &\left. + \tfrac{1}{2}[\mathrm{v}_i^+(x(\sigma)) + \mathrm{v}_i^-(x(\sigma))] \right\} d\sigma. \end{aligned} \tag{47}$$

Forming the quotient $[x(t_k + T) - x(t_k)]/T$ and taking the limit as $T \to 0$, and as t_k tends to any generic instant t, one obtains

$$\begin{aligned} \lim_{\substack{T\to 0 \\ t_k \to t}} \left[\frac{x(t_k + T) - x(t_k)}{T}\right] &= \lim_{\substack{T\to 0 \\ t_k \to t}} \frac{1}{T}\left[\int_{t_k}^{t_k+T} f(x(\sigma))\,d\sigma \right. \\ &+ \sum_{i=1}^{m} \int_{t_k}^{t_k+|\mu_i[x(t_k)]|T} g_i(x(\sigma)) \\ &\times \left\{ \frac{\operatorname{sign}\mu_i[x(t_k)]}{2}[\mathrm{v}_i^+(x(\sigma)) - \mathrm{v}_i^-(x(\sigma))] \right. \\ &\left.\left. + \tfrac{1}{2}[\mathrm{v}_i^+(x(\sigma)) + \mathrm{v}_i^-(x(\sigma))] \right\} d\sigma \right]. \end{aligned}$$

After a straightforward application of the *fundamental lemma of calculus*, the derivative of the (average) state at each instant of time t is given by

$$\begin{aligned} \dot{z} = f(z) + \sum_{i=1}^{m} g_i(z)&\left[\frac{\mu_i(z)}{2}(\mathrm{v}_i^+(z) - \mathrm{v}_i^-(z)) \right. \\ &\left. + \frac{|\mu_i(z)|}{2}(\mathrm{v}_i^+(z) + \mathrm{v}_i^-(z))\right] \end{aligned} \tag{48}$$

i.e., the average PWM system (48) is obtained by formally substituting, in the discontinuous model (45), each switch position function u_i by the corresponding duty ratio function μ_i.

As corollaries to the above result, if the control inputs v_i take values in the discrete sets $\{-U_i, 0, +U_i\}$, i.e., if $\mathrm{v}_i^+ = U_i$ and $\mathrm{v}_i^- = -U_i$, the average model is simply given by $dx/dt = f(x) + \sum \mu_i U_i g_i(x)$, and finally, if each of the U_is happen to be equal to 1, the average model is simply $dx/dt = f(x) + \sum \mu_i g_i(x)$.

It can be demonstrated, in a similar fashion to the ON-OFF PWM case that the average PWM responses obtained from (48) remain arbitrarily close to those of (45), (46), provided the same initial state is used for the two models and a sufficiently high sampling frequency is specified to the PWM actuator.

The reader is referred to [19] for a concrete application and further details about multivariable PWM feedback control design of the ON-OFF type.

C. Dynamic Variable Structure Control Approach to Output Stabilization Problems

Consider the n-dimensional single input-single output nonlinear system in state space form:

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{49}$$

where f is an analytic nonlinear algebraic function of the state and input variables. It may be argued that this represents no great loss of generality since the function f might have arisen from an elimination procedure carried out on a lower dimensional nonlinear state space description of the controlled system, which included differential equations whose coefficients satisfied, in turn, algebraic differential equations (see [3]). We refer to (49) as the pair (f,h).

The system is assumed to be transformable to the GOCF as

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= q_3 \\ &\vdots \\ \dot{q}_{n-1} &= q_n \\ \dot{q}_n &= \mathbf{c}(q,u,u,\ldots,u^{(\alpha)}) \\ y &= q_1 \end{aligned} \tag{50}$$

where α is strictly smaller than n and coincides with $n-r$, with r being the *relative degree* or *relative order* of the pair (f,h) (see Isidori [24]).

If we let $b^T = [0,\ldots,0,1]$, $c = [1,0,\ldots,0]^T$ and A an $n \times n$ matrix in companion form, with zeros in the last row, then (50) is briefly expressed as

$$dq/dt = Aq + b\mathbf{c}(q,u,\ldots u^{(\alpha)}); \qquad y = c^T q. \tag{51}$$

The following proposition is quite basic to our development below.

PROPOSITION 2 *The one-dimensional discontinuous system:*

$$\dot{\omega} = -\lambda(\omega + \Omega \operatorname{sgn} \omega) \tag{52}$$

globally exhibits a sliding regime on $\omega = 0$. *Here,* λ *and* Ω *are strictly positive quantities and "*sign*" stands for the signum function, defined as*

$$\operatorname{sign} \omega = \begin{cases} +1 & \text{if } \omega > 0 \\ 0 & \text{if } \omega = 0 \\ -1 & \text{if } \omega < 0 \end{cases}. \tag{53}$$

Furthermore, any trajectory starting on the value $\omega = \omega(0)$, *at time* $t = 0$, *reaches the condition* $\omega = 0$ *in finite time* T, *given by* $T = \lambda^{-1}\ln[1 + |\omega(0)|/\Omega]$.

PROOF. Immediately upon checking that globally, $\omega d\omega/dt < 0$ for $\omega \neq 0$, which is a well-known condition for sliding mode existence [14]. The second part follows from the linearity of the two intervening system "structures".

Let the set of real coefficients $\{m_0,\ldots,m_{n-2}\}$ be such that the following polynomial is Hurwitz:

$$s^{n-1} + m_{n-2}s^{n-2} + \cdots + m_1 s + m_0. \tag{54}$$

Consider the auxiliary output variable:

$$\omega = \sum_{i=1}^{n} m_{i-1} y^{(i-1)}; \qquad \text{with} \quad m_{n-1} = 1 \tag{55}$$

which, in terms of the generalized phase coordinates vector η, is also simply expressed as

$$\omega = \sum_{i=1}^{n} m_{i-1} q_i = m^T q. \tag{56}$$

From (56), (51), and the fact that $m^T b = 1$, it follows easily that:

$$\dot{\omega} = m^T A q + \mathbf{c}(q,u,u^{(1)},\ldots,u^{(\alpha)}). \tag{57}$$

A dynamic variable structure feedback controller is obtained if we impose on the evolution of the auxiliary output variable ω, the discontinuous dynamics considered in (52). From (52), (56), and (57) one obtains

$$\mathbf{c}(q,u,u^{(1)},\ldots,u^{(\alpha)}) = -m^T[\lambda I + A]q - \lambda\Omega \operatorname{sgn} m^T q \tag{58}$$

which is to be viewed as an implicit differential equation with discontinuous right-hand side. On each one of the regions $\omega = m^T q > 0$, and $\omega = m^T q < 0$, a different "structure" is valid and the implicit differential equation is to be solved for the controller u, on the basis of knowledge of q. Since ω was shown to exhibit a sliding regime on the discontinuity surface $\omega = 0$, *Filippov's continuation method* (see [20]), or, alternatively, the *method of the equivalent control* [14], is to be used for defining the idealized solutions of (58) on the switching manifold $\omega = 0$.

According to the *method of the equivalent control*, the discontinuous motions on the sliding surface $\omega = 0$ can be described, in an idealized fashion, by the *invariance conditions* $\omega = 0$ and $d\omega/dt = 0$. These conditions allow, in turn, the definition of a *virtual* control action, known as the *equivalent control*, which would be responsible for locally smoothly maintaining the evolution of the state variables on the manifold $\omega = 0$, should the motions start on this manifold. The resulting autonomous dynamics, ideally constrained to the switching manifold and "controlled" by the equivalent control, is known as the *ideal sliding dynamics*. It follows from (55) and (51) that such an ideal sliding dynamics is given by

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= q_3 \\ &\vdots \\ \dot{q}_{n-1} &= -m_{n-2}q_{n-1} - m_{n-3}q_{n-2} - \cdots - m_0 q_1 \end{aligned} \tag{59}$$

which represents an asymptotically stable motion toward the origin of generalized phase coordinates with eigenvalues uniquely specified by the prescribed set of coefficients $\{m_0, \ldots, m_{n-2}\}$. In particular the output function $y = h(x) = q_1$ asymptotically converges to zero. From the invariance conditions the equivalent control, denoted by u_{EQ} is defined as the solution of the implicit differential equation:

$$c(q, u, u_{\text{EQ}}^{(1)}, \ldots, u_{\text{EQ}}^{(\alpha)}) = -m^T A q; \qquad \text{with} \quad m^T q = 0. \tag{60}$$

REMARK Of course one does not really need to prescribe dynamic variable structure controllers to asymptotically stabilize the output function to zero. As a matter of fact, a static sliding mode controller can always achieve such a stabilization with the aid of an auxiliary output function which involves only r output time derivatives. These derivatives, in turn, can always be asymptotically synthesized with the aid of "postprocessors" (see [24] and also Sire-Ramírez [25]). Furthermore, since the generalized phase variables in (58) are locally obtained from input dependent state coordinate transformations (involving also a finite number of input time derivatives) it is clear that full original state feedback is required, without the benefit of being able to use postprocessors in the synthesis of the controller equations and of the auxiliary output functions. However, two important advantages can be readily established about the dynamic variable structure controller represented by (58). The first one is the fact that the output function $y = h(x)$ asymptotically approaches zero with substantially reduced, or smoothed out, "chattering". Notice that at least n integrators stand between the output variable y and the regulated chattering behavior of the auxiliary output variable ω. Therefore with respect to the static variable structure controller alternative, $n - r$ additional integrations contribute to further smooth out the controlled output signal. Secondly, and this is possibly the most important advantage, a canonical phase variable representation for the dynamic controller (58) indicates that the control input u is the outcome of at least α $(= n - r)$ integrations performed on a nonlinear function of the discontinuous actions that lead the auxiliary output ω to zero. This means substantially smoothed control actions which do not result in a "bang-bang" behavior for the actuator, something that cannot be avoided in the static controller alternative.

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REFERENCES

[1] Fliess, M. (1989)
Généralisation non Linéaire de la Forme Canonique de Commande et Linearisation par Bouclage.
C. R. Academie de Sciences, Paris, **I-308** (1989), 377–379.

[2] Fliess, M. (1989)
Automatique et Corps Differentiels.
Forum Mathematicum, **1** (1989), 227–238.

[3] Fliess, M. (1990)
Generalized controller canonical form for linear and nonlinear systems.
IEEE Transactions on Automatic Control, **AC-35**, 9 (Sept. 1990), 994–1001.

[4] Fliess, M (1988)
Nonlinear control theory and differential algebra.
In Ch. I. Byrnes and A. Khurzhansky (Eds.), *Modeling and Adaptive Control*, Lecture Notes in Control and Information Sciences, *Vol. 105*.
New York: Springer, 1988, 134–145.

[5] Pommaret, J. F. (1987)
Lie Pseudogroups and Mechanics.
New York: Gordon and Breach, 1987.

[6] Kolchin, E. R. (1973)
Differential Algebra and Algebraic Groups.
New York: Academic Press, 1973.

[7] Bryson, A., and C. Ho, Y. (1969)
Applied Optimal Control.
Waltham, MA: Ginn, 1969.

[8] Sira-Ramírez, H.
Nonlinear dynamically feedback controlled descent on a non atmosphere-free planet: A differential algebraic approach.
Control-Theory and Applications. Submitted for publicationn.

[9] Brockett, R. W. (1970)
Finite Dimensional Linear Systems.
New York: Wiley, 1970.

[10] Conte, G., Moog, C., and Perdon, A. (1988)
Un théoreme sur la représéntation entréc-sortie d'un système non linéaire.
Compte Rendu Academie Science, Paris, Série I, **t. 307** (1988), 363–366.

[11] Isidori, A., and Moog, C. H. (1988)
On the nonlinear equivalent of the notion of transmission zeros.
In C. Byrnes and A. Kurszanski (Eds.), *Modeling and Adaptive Control*, Lecture Notes in Information and Control, *Vol. 105*.
New York: Springer, 1988.

[12] Fliess, M. (1990)
What the Kalman state variable representation is good for.
In *Proceedings of the 29th IEEE Conference on Decision and Control*, **3**, Honolulu, Hawaii, Dec. 1990, 1282–1287.

[13] Sira-Ramírez, H. (1990)
A variable structure control approach to the problem of soft landing on a planet.
Control-Theory and Advanced Technology, **6**, 1 (Mar. 1990), 53–73.

[14] Utkin, V. I. (1978)
Sliding Modes and Their Applications to Variable Structure Control.
Moscow: Mir, 1978.

[15] Sira-Ramírez, H. (1990)
Structure at infinity, zero dynamics and normal forms of systems undergoing sliding motions.
International Journal of Systems Science, **21**, 4 (Apr. 1990), 665–674.

[16] Sira-Ramírez, H. (1989)
A geometric approach to pulse-width modulated control in nonlinear dynamical systems.
IEEE Transactions on Automatic Control, **34**, 2 (1989), 184–187.

[17] Sira-Ramírez, H. (1993)
Nonlinear pulse-width-modulation controller design.
In K. K. D. Young (Ed.), *Variable Structure Control for Robotics and Aerospace Applications*.
New York: Elsevier Science.

[18] Sira-Ramírez, H. (1989)
Invariance conditions in nonlinear PWM controlled systems.
International Journal of Systems Science, **20**, 9 (1989), 1679–1690.

[19] Sira-Ramírez, H., Zribi, M., and Ahmad, S. (1990)
Pulse-width-modulated control of robotic manipulators.
Technical report TR-EE 90-16, Purdue University, Lafayette, IN, Feb. 1990.

[20] Filippov, A. F. (1988)
Differential Equations With Discontinuous Right-Hand Sides.
Boston, MA: Kluwer Academic, 1988.

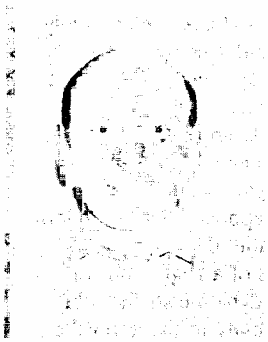
[21] Arnol'd, V. I. (1988)
Dynamical Systems III, Encyclopedia of Mathematical Sciences, *Vol. 3.*
New York: Springer, 1988.

[22] Miller, R. K. (1971)
Nonlinear Volterra Integral Equations.
New York: W. A. Benjamin, Inc., 1971.

[23] Tikhonov, A. N., Vasil'eva, A. B., and Sveshnikov, A. G. (1988)
Differential Equations.
New York: Springer, 1988.

[24] Isidori, A. (1989)
Nonlinear Control Systems, (2nd ed.)
New York: Springer, 1989.

[25] Sira-Ramírez, H. (1990)
Nonlinear variable structure systems in sliding mode: The general case.
IEEE Transactions on Automatic Control, **34**, 11 (Nov. 1990), 1186–1188.



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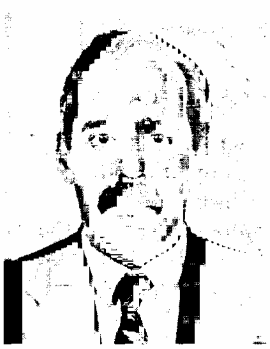

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