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Adaptive dynamical feedback regulation strategies for linearizable uncertain systems

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In this paper, we address the design of adaptive dynamical feedback strategies of the continuous and discontinuous types, for the output stabilization of nonlinear systems. The class of systems considered corresponds to nonlinear controlled systems exhibiting linear parametric uncertainty. Dynamical feedback controllers, ideally achieving output stabilization via exact linearization, are obtained by means of repeated output differentiation and, either, pole placement, or, sliding mode control techniques. The adaptive versions of the dynamical stabilizing controllers are then obtainable through standard, direct, overparametrized adaptive control strategies available for linearizable systems. Illustrative examples which deal with the regulation of electro-mechanical systems are provided.

1. Introduction

Asymptotic output stabilization of parametrically uncertain nonlinear systems constitutes an important problem in control systems design. Contributions, from differential geometric viewpoints, have been given by Sastry and Isidori (1989), Kanellakopoulos *et al.* (1989, 1991), Taylor *et al.* (1989), Campion and Bastin (1990), Teel *et al.* (1991) and many others. For enlightening details, and general results, the reader is referred to the books by Sastry and Bodson (1989), and Narendra and Annaswamy (1989). On-going developments in this area are contained in the collection of lectures edited by Kokotovic (1991). For other contributions to the area, the reader is referred to the reprint of the book edited by Narendra *et al.* (1991).

In this article, using the results of Sastry and Isidori (1989), an adaptive asymptotic output stabilization scheme is proposed for dynamical pole placement, and sliding-mode based, exactly-linearizing controllers, obtained via repeated output differentiation. The schemes are restricted to the class of nonlinear systems with vector fields that exhibit linear parametric dependence. The availability of the dynamical controller state variables and overparametrization (Campion and Bastin, 1990) are the key issues that allow an extension of direct adaptive control techniques, available for static input–output linearizable systems, to dynamically controlled systems. Two design examples are presented. The first one involves the control a DC motor by means of adaptive dynamical pole placement. The second example deals with the stabilization of a magnetic suspension system via adaptive dynamical variable structure control strategies.

In § 2 of this paper, the adaptive dynamical pole placement stabilization

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scheme is presented along with the DC motor control design example. The adaptive dynamical variable structure control stabilization problem is presented in § 3, including applications to a magnetic suspension system. In both examples, computer simulations are provided to assess the performance of the proposed controllers. Concluding remarks, and proposals for further research, are collected in § 4.

2. Adaptive output stabilization of linearizable nonlinear systems via dynamical pole placement

2.1. *Input–output linearization by dynamical pole placement techniques*

Consider the following n-dimensional state-space realization of a single-input single-output nonlinear system

$$\left.\begin{aligned} \dot{x} &= f(x, \theta) + g(x, \theta)u \\ y &= h(x, \theta) \end{aligned}\right\} \tag{2.1}$$

where $f\colon R^{n+p} \to R^n$ and $g\colon R^{n+p} \to R^n$ are, for fixed θ in R^p, C^∞ vector fields globally defined on R^n, and $h\colon R^{n+p} \to R$ is a C^∞ function. It is assumed that the system has strong relative degree $r < n$ (Isidori 1989). The parameter vector θ is assumed to be constant and f, g and h are *linear* functions of θ.

The ith time derivative of the output function may be written in terms of the state vector x and the control input u as

$$\left.\begin{aligned} &y^{(i)} = b_i(x, \theta) \quad \text{for } i < r; \quad \text{with } b_0(x, \theta) = h(x, \theta) \\ &y^{(i)} = b_i(x, \theta, u, u^{(1)}, \ldots, u^{(i-r-1)}) + a(x, \theta)u^{(i-r)} \quad \text{for } r \leqslant i \leqslant n \end{aligned}\right\} \tag{2.2}$$

In particular, the nth time derivative of y may be obtained as

$$y^{(n)} = b_n(x, \theta, u, u^{(1)}, \ldots, u^{(n-r-1)}) + a(x, \theta)u^{(n-r)} \tag{2.3}$$

We assume that the 'observability' matrix, constituted by the (row vector) gradients, with respect to x, of $y^{(i)}$ $(i = 0, 1, \ldots, n-1)$ is full rank n, i.e.

$$\operatorname{rank} \frac{\partial(y, y^{(1)}, \ldots, y^{(n-1)})}{\partial x} = \operatorname{rank} \frac{\partial(y, y^{(1)}, \ldots, y^{(n)})}{\partial x} = n \tag{2.4}$$

This assumption implies that (2.1) can be described by an nth order input–output scalar differential equation (see Conte *et al.* 1988, Diop 1991). The implicit function theorem allows one to solve for x locally, from (2.2), in terms of u and its time derivatives, as well as in terms of the derivatives of y. In other words, there exists a set of n independent functions ϑ_i, implicitly defined by (2.2), such that

$$x_i = \vartheta_i(y, y^{(1)}, \ldots, y^{(n-1)}, u, u^{(1)}, \ldots, u^{(n-r-1)}); \quad i = 1, 2, \ldots, n \tag{2.5}$$

In general, one *locally* obtains a representation of (2.1) in the form

$$y^{(n)} = c(y, y^{(1)}, \ldots, y^{(n-1)}, \theta, u, u^{(1)}, \ldots, u^{(n-r)}) \tag{2.6}$$

Definition 2.1 (Fliess 1990 a): Let the output y be identically zero for an indefinite amount of time. The *zero dynamics*, associated with (2.1), are defined as

$$c(0, \theta, u, u^{(1)}, \ldots, u^{(n-r)}) = 0 \tag{2.7}$$

We assume that (2.7) is locally asymptotically stable to a constant operating point, $u = U$. In such a case we say (2.1) is locally *minimum phase* around the equilibrium point of interest. □

Proposition 2.2: *Let $u^{[i]}$ denote the following set $\{u, u^{(1)}, \ldots, u^{(i)}\}$ of control input derivatives. Then, the dynamical feedback controller*

$$u^{(n-r)} = -\frac{b_n(x, \theta, u^{[n-r-1]}) + \sum_{i=0}^{r-1} \alpha_i b_i(x, \theta) + \sum_{j=r}^{n-1} \alpha_j [b_j(x, \theta, u^{[j-r-1]}) + a(x, \theta) u^{(j-r)}]}{a(x, \theta)}$$

$$\alpha_n = 1$$

drives the output of system (2.1) to satisfy closed loop linearized dynamics of the form

$$y^{(n)} + \alpha_{n-1} y^{(n-1)} + \cdots + \alpha_1 y^{(1)} + \alpha_0 y = 0 \tag{2.9}$$

Proof: The proof is immediate upon direct substitution of (2.8) in (2.3) and the use of the definitions in (2.2). □

Provided that the system is minimum phase, then the scalar time-varying differential equation (2.8) defines a dynamical feedback controller which can accomplish exponential output stabilization to zero, in a manner entirely prescribed by the set of chosen design coefficients $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$. Typically, one chooses the α to obtain asymptotically stable dynamics for (2.9). The set of input derivatives $u^{[n-r-1]}$, in (2.8), naturally qualifies as a state vector for the dynamical controller, which is available for measurement. If the quantity $a(x, \theta)$ is bounded away from zero then no impasse points need be considered for the dynamical system representing the linearizing controller (see Fliess and Hasler 1990). This assumption is equivalent to the strong relative degree assumption (Sastry and Isidori 1989).

2.2. *An adaptive regulation scheme for dynamically linearizable systems*

The effectiveness of the dynamical feedback controller (2.8) is highly dependent upon perfect knowledge of the involved system parameters θ. It is clear that exact cancellation of nonlinearities would not generally be possible if the dynamical controller (2.8) was computed using estimated values of such parameters, which are known to be in error with respect to their true values. In this section we assume that the components of θ are constant, but otherwise unknown, and present an adaptive approach to dynamical feedback linearization. We denote the estimated values of the parameter vector as $\hat{\theta}$.

Remark 2.3: It may be verified that the linearity of f, g and h with respect to θ implies that the quantities b_i $(i = 0, 1, \ldots, n-1)$ and a in (2.2) are multilinear functions of the components θ_i of θ. Hence, if one defines a large dimensional vector Θ containing, as individual components, all possible ordered homogeneous multinomial expressions in the θ_i of degree smaller or equal than n, then b_i $(i = 0, 1, \ldots, n-1)$ and a are indeed linear functions of Θ. This observation and the involved process, known as 'overparametrization' (Campion and Bastin 1990), allows us to extend recently proposed adaptive control techniques (Sastry and Isidori 1989), developed for *statically* linearizable systems, to systems linearizable by *dynamical* feedback (see Fliess 1990 b, and also Sira-Ramirez 1992 a). □

Define

$$u^{(n-r)} = \frac{b_n(x, \hat{\theta}, u^{[n-r-1]}) + \sum_{i=0}^{r-1} \alpha_i b_i(x, \hat{\theta}) + \sum_{j=r}^{n-1} \alpha_j [b_j(x, \hat{\theta}, u^{[j-r-1]}) + a(x, \hat{\theta}) u^{(j-r)}]}{a(x, \hat{\theta})} \tag{2.10}$$

Then, if a dynamical controller of the form (2.10), based on parameter estimates, is used to regulate the evolution of $y^{(n)}$, the expression (2.3) is found to be, after some manipulation

$$\begin{aligned} & y^{(n)} + \alpha_{n-1} y^{(n-1)} + \cdots + \alpha_1 y^{(1)} + \alpha_0 y = \\ & b_n(x, \theta, u^{[n-r-1]}) - b_n(x, \hat{\theta}, u^{[n-r-1]}) + \sum_{i=0}^{r-1} \alpha_i [b_i(x, \theta) - b_i(x, \hat{\theta})] \\ & + \sum_{j=r}^{n-1} \alpha_j \{ b_j(x, \theta, u^{[j-r-1]}) - b_j(x, \hat{\theta}, u^{[j-r-1]}) + [a(x, \theta) - a(x, \hat{\theta})] u^{(j-r)} \} \\ & + [a(x, \theta) - a(x, \hat{\theta})] u^{(n-r)} \end{aligned} \tag{2.11}$$

where $u^{(n-r)}$ represents the expression of the estimated controller given in (2.10).

By virtue of Remark 2.3 one may conclude that expression (2.11) can be written as a linear function of the parameter estimation error $\Theta - \hat{\Theta} := \phi$.

$$\begin{aligned} & y^{(n)} + \alpha_{n-1} y^{(n-1)} + \cdots + \alpha_1 y^{(1)} + \alpha_0 y \\ & \qquad = (\Theta - \hat{\Theta})^{\mathrm{T}} W(x, \hat{\theta}, u^{[n-r-1]}) = \phi^{\mathrm{T}} W(x, \hat{\theta}, u^{[n-r-1]}) \end{aligned} \tag{2.12}$$

where W is the nonlinear state-dependent regressor vector, depending also on the vector of parameter estimates, $\hat{\theta}$, and the measurable 'state' of the dynamical controller, represented here by u and the derivatives of u up to order $n - r - 1$, i.e. by $u^{[n-r-1]}$. By slightly abusing notation we shall write W as a function of $\hat{\Theta}$ rather than as a function of $\hat{\theta}$.

In order to find an appropriate adaptation law, the developments given in Sastry and Isidori (1989), or in Sastry and Bodson (1989), can be followed very closely in a rather straightforward fashion. We summarize the developments in Sastry and Isidori (1989) as follows.

Let $L(s)$ be defined as the characteristic polynomial of the linear differential equation (2.9) and let $L^{-1}(s)$ stand for the linear time-invariant operator

$$L^{-1}(s) = \frac{1}{s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0} \tag{2.13}$$

The output variable y may then be written as the convolution of the linear operator (2.13) with the nonlinear time-varying function obtained in the right hand side of (2.12). One has

$$y = L^{-1}(s) * [\phi^{\mathrm{T}} W(x, \hat{\Theta}, u^{[n-r-1]})] \tag{2.14}$$

where the '*' denotes the convolution operation in the hybrid notation of (2.14).

Let e_1 denote the augmented output error, defined as

$$e_1 = y + \hat{\Theta}^{\mathrm{T}} L^{-1}(s)^*[W(x, \hat{\Theta}, u^{[n-r-1]})] - L^{-1}(s)^*[\hat{\Theta}^{\mathrm{T}} W(x, \hat{\Theta}, u^{[n-r-1]})] \tag{2.15}$$

Notice that e_1 can be calculated from measurable signals. It is now easy to see, using (2.14) and the commutativity between the operator $L^{-1}(s)$ and the (constant) value of the actual parameter, that

$$e_1 = \phi^{\mathrm{T}}(L^{-1}(s)^*[W(x, \hat{\Theta}, u^{[n-r-1]})]) =: \phi^{\mathrm{T}}\xi \tag{2.16}$$

Where ξ is the vector of filtered regressor components. From the fact that e_1 is a linear error equation (Sastry and Bodson 1989) in ϕ, several update laws may be proposed. One such possibility is represented by the following gradient type of update law (see Sastry and Bodson 1989, p. 57)

$$\dot{\hat{\Theta}} = -g e_1 W(x, \hat{\Theta}, u^{[n-r-1]}) \tag{2.17 a}$$

where g is a positive constant called the adaptation gain.

A second possibility is represented by the normalized gradient update law (see Isidori and Sastry 1989, and Sastry and Bodson 1989, p. 58)

$$-\dot{\hat{\Theta}} = \dot{\phi} = -\frac{e_1\xi}{1 + \xi^{\mathrm{T}}\xi} \tag{2.17 b}$$

The parameter estimation error ϕ can converge to zero, provided persistency of excitation conditions are satisfied during the stabilization transient (see Isidori and Sastry 1989, Sastry and Bodson 1989 chapter 2, and Narendra and Annaswamy 1989 chapter 6). In such a case the output signal y is asymptotically stable.

2.3. *A DC motor example*

2.3.1. *Non-adaptive dynamical linearizing control for angular velocity regulation in a DC motor.* Consider a field controlled DC-motor model (see Rugh 1981, p. 98) given by

$$\left.\begin{aligned} \dot{x}_1 &= -\frac{R_{\mathrm{a}}}{L_{\mathrm{a}}} x_1 - \frac{K}{L_{\mathrm{a}}} x_2 u + \frac{V_{\mathrm{a}}}{L_{\mathrm{a}}} \\ \dot{x}_2 &= -\frac{B}{J} x_2 + \frac{K}{J} x_1 u \\ y &= x_2 - \Omega \end{aligned}\right\} \tag{2.18}$$

where x_1 is the armature circuit current, x_2 is the angular velocity of the rotating axis. The armature circuit voltage, V_{a}, is assumed to be constant while the field winding input voltage, u, acts as a control variable. The quantity Ω represents a desired constant angular velocity.

System (2.18) is of the form

$$\left.\begin{aligned} \dot{x} &= \theta_1 f_1(x) + \theta_2 f_2(x) + \theta_3 f_3(x) + [\theta_4 g_1(x) + \theta_5 g_2(x)]u \\ y &= h(x) \end{aligned}\right\} \tag{2.19}$$

with

$$f_1(x) = \begin{bmatrix} -x_1 \\ 0 \end{bmatrix}, \quad f_2(x) = \begin{bmatrix} 0 \\ -x_2 \end{bmatrix}, \quad f_3(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$g_1(x) = \begin{bmatrix} -x_2 \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$$

and

$$\theta_1 = \frac{R_a}{L_a}, \quad \theta_2 = \frac{B}{J}, \quad \theta_3 = \frac{V_a}{L_a}, \quad \theta_4 = \frac{K}{L_a}, \quad \theta_5 = \frac{K}{J}$$

It is easy to verify that for the given system (2.18), the rank of the following 2 by 2 matrix

$$\left[\frac{\partial y}{\partial x}, \frac{\partial y^{(1)}}{\partial x}\right] = \begin{bmatrix} 0 & 1 \\ \theta_5 u & -\theta_2 \end{bmatrix}$$

is everywhere equal to 2, except when $u = 0$. Angular velocity stabilization tasks which require polarity reversals in the field winding input voltage u have to be treated separately by different techniques.

A constant equilibrium point, parametrized in terms of the desired angular velocity Ω, for this system is given by

$$\left.\begin{aligned} x_1(\Omega) &= \frac{\theta_3}{2\theta_1}\left(1 + \left[1 - \frac{4\theta_1\theta_2\theta_4\Omega^2}{\theta_3^2\theta_5}\right]^{1/2}\right) \\ x_2(\Omega) &= \Omega \\ u(\Omega) &= \frac{2\theta_1\theta_2\Omega}{\theta_3\theta_5\left(1 + \left[1 - \dfrac{4\theta_1\theta_2\theta_4\Omega^2}{\theta_3^2\theta_5}\right]^{1/2}\right)} \end{aligned}\right\} \tag{2.20}$$

An input–output representation of system (2.18) readily follows by elimination of the state vector x from the expressions of y and $\mathrm{d}y/\mathrm{d}t$

$$y^{(2)} - \theta_2^2(y + \Omega) + (\theta_2 + \theta_1)[y^{(1)} + \theta_2(y + \Omega)] + \theta_4\theta_5(y + \Omega)u^2 - \theta_3\theta_5 u - \frac{u^{(1)}}{u}[y^{(1)} + \theta_2(y + \Omega)] = 0 \tag{2.21}$$

The zero dynamics associated with system (2.18) are obtained from (2.21) by letting $y = y^{(1)} = y^{(2)} = 0$, as

$$u^{(1)} = u\left(\frac{\theta_4\theta_5}{\theta_2}u^2 - \frac{\theta_3\theta_5}{\theta_2\Omega}u + \theta_1\right) \tag{2.22}$$

The three constant equilibrium points for the zero dynamics are: $u = 0$, (which was discarded as a singularity), and

$$u = \frac{\theta_3}{2\theta_4\Omega}\left[1 \pm \left[1 - 4\frac{\theta_1\theta_2\theta_4}{\theta_5}\left(\frac{\Omega}{\theta_3}\right)^2\right]^{1/2}\right] \tag{2.23}$$

Under the condition that $\theta_5\theta_3^2 > 4\theta_1\theta_2\theta_4\Omega^2$ (i.e. $V_a^2 - 4R_aB\Omega^2 > 0$), one finds quite straightforwardly, by plotting $u^{(1)}$ against u, from (2.22), that the larger solution in (2.23) is unstable while the solution with smaller u is asymptotically stable. The system is thus locally minimum phase.

Let $\omega_n > 0$ and $\zeta > 0$. Imposing on the output y of (2.18) the following linear asymptotically stable dynamics

$$y^{(2)} + 2\zeta\omega_n y^{(1)} + \omega_n^2 y = 0 \tag{2.24}$$

one readily obtains, using the result of proposition 2.2 above, the following stabilizing dynamical feedback controller

$$\begin{aligned}\dot{u} = \frac{1}{\theta_5 x_1}\big[&(2\zeta\omega_n\theta_2 - \omega_n^2 - \theta_2^2)x_2 - \theta_3\theta_5 u \\ &+ (\theta_2 + \theta_1 - 2\zeta\omega_n)\theta_5 x_1 u + \theta_4\theta_5 x_2 u^2 + \omega_n^2\Omega\big]\end{aligned} \tag{2.25}$$

This dynamical controller achieves asymptotic output stabilization around the stable equilibrium point (see Sira-Ramirez 1992 a for the non-adaptive tracking version of this controller).

2.3.2. *Adaptive dynamical linearizing control for angular velocity regulation in a DC motor.* Owing to a lack of parameter knowledge, instead of the exactly linearizing controller (2.25), one uses a dynamical controller, based on the estimates of the parameters, their products, and powers, as

$$\dot{u} = \frac{[(2\zeta\omega_n\hat{\theta}_2 - \omega_n^2 - \hat{\theta}_2^2)x_2 - \widehat{\theta_3\theta_5}u + (\widehat{\theta_2\theta_5} + \widehat{\theta_1\theta_5} - 2\zeta\omega_n\hat{\theta}_5)x_1 u + \widehat{\theta_4\theta_5}x_2 u^2 + \omega_n^2\Omega]}{\hat{\theta}_5 x_1} \tag{2.26}$$

or, equivalently, in terms of the components of an overparametrization vector $\hat{\Theta}$ defined as

$$\left.\begin{aligned}\hat{\Theta} &= (\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_{20}) = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_5, \hat{\theta}_1^2, \widehat{\theta_1\theta_2}, \ldots, \widehat{\theta_1\theta_5}, \hat{\theta}_2^2, \widehat{\theta_2\theta_3}, \ldots, \hat{\theta}_5^2) \\ \dot{u} &= \frac{[(2\zeta\omega_n\hat{\Theta}_2 - \omega_n^2 - \hat{\Theta}_{11})x_2 - \hat{\Theta}_{17}u + (\hat{\Theta}_{14} + \hat{\Theta}_{10} - 2\zeta\omega_n\hat{\Theta}_5)x_1 u + \hat{\Theta}_{19}x_2 u^2 + \omega_n^2\Omega]}{\hat{\Theta}_5 x_1}\end{aligned}\right\} \tag{2.27}$$

Let ϕ_i denote the parameter estimation error $\Theta_i - \hat{\Theta}_i$ ($i = 1, \ldots, 20$), then, using the results of the previous section we obtain the following expression for the closed-loop behaviour of the output variable

$$y^{(2)} + 2\zeta\omega_n y^{(1)} + \omega_n^2 y = [\phi_1 \ldots \phi_{20}]\begin{bmatrix} w_1(x, \hat{\Theta}, u) \\ \vdots \\ w_{20}(x, \hat{\Theta}, u)\end{bmatrix} \tag{2.28}$$

The elements constituting the parameter estimation error update law are summarized below.

Parameter estimation error update law

$$\dot{\phi}_i = -\dot{\hat{\Theta}}_i = -e_1 \frac{\xi_i}{1 + \xi^{\mathrm{T}}\xi}; \quad i = 1, \ldots, 20$$

Regressor vector components

$$w_2(x, \hat{\Theta}, u) = -2\zeta\omega_n x_2$$

$$w_5(x, \Theta, u)$$

$$= \frac{[(2\zeta\omega_n\hat{\Theta}_2 - \omega_n^2 - \hat{\Theta}_{11})x_2 - \hat{\Theta}_{17}u + (\hat{\Theta}_{14} + \hat{\Theta}_{10} - 2\zeta\omega_n\hat{\Theta}_5)x_1 u + \hat{\Theta}_{19}x_2u^2 + \omega_n^2\Omega}{\hat{\Theta}_5 x_1}$$

$$w_{10}(x, \hat{\Theta}, u) = -x_1 u, \quad w_{11}(x, \hat{\Theta}, u) = x_2, \quad w_{14}(x, \hat{\Theta}, u) = -x_1 u$$

$$w_{17}(x, \hat{\Theta}, u) = u, \qquad w_{19}(x, \hat{\Theta}, u) = -2x_2u^2$$

where those regressor vector entries not listed above have value equal to zero.

Augmented output stabilization error

$$e_1 = \sum_{i=1}^{i=20} \phi_i \xi_i$$

Filtered regressor components (with zero initial conditions)

$$\ddot{\xi}_i = -2\zeta\omega_n\dot{\xi}_i - \omega_n^2\xi_i + w_i(x, \hat{\Theta}, u^{[n-r-1]})$$
$$i = 2, 5, 10, 11, 14, 17, 19$$

$$(\zeta_i = 0, \text{ for } i = 1, 3, 4, \ldots, 18, 20)$$

Parameter estimation error update law

$$\dot{\phi}_i = -\dot{\hat{\Theta}}_i = -e_1\frac{\xi_i}{1 + \xi^{\mathrm{T}}\xi}; \quad i = 2, 5, \ldots, 19$$

2.3.3. *Simulation results.* Computer simulations were run to assess the performance of the adaptive dynamical controller for a DC motor with the following nominal values for the system parameters

$$R_a = 7\ \Omega, \qquad L_a = 120\ \mathrm{mH}$$
$$K = 1{\cdot}41 \times 10^{-2}\ \mathrm{N\,m\,A^{-1}} \qquad B = 6{\cdot}04 \times 10^{-6}\ \mathrm{N\,m\,s\,rad^{-1}}$$
$$B = 6{\cdot}04 \times 10^{-6}\ \mathrm{N\,m\,s\,rad^{-1}} \quad J = 1{\cdot}06 \pm 10^{-6}\ \mathrm{N\,m\,s^2\,rad^{-1}}, \quad V_a = 5\ V$$

The dynamically controlled state variable trajectories $x_1(t)$ and $x_2(t)$ are depicted, respectively, in Figs 1 and 2, while the adaptive control input trajectory $u(t)$ is shown in Fig. 3. The state components slowly converge to $x_1 = 0{\cdot}661$ A, and $x_2 = 202{\cdot}3\ \mathrm{rad\,s^{-1}}$. These values are within 4% of their ideal equilibrium values given by: $x_1 = 0{\cdot}702$ A, $x_2 = \Omega = 200\ \mathrm{rad\,s^{-1}}$. In Fig. 4, the value of the estimated parameter $\hat{\theta}_2 = \hat{\Theta}_2$ is shown to converge slowly to a constant value of 5·555 which does not coincide with its nominal value of 5·698. The rest of the parameters have small variations and they are not shown here. The dynamical controller parameters were set as: $\zeta = 0{\cdot}7$, $\omega_n = 30$.

3. Adaptive output stabilization of linearizable nonlinear systems via dynamical sliding mode control

3.1. *Linearization by discontinuous dynamical feedback control*

In this section we present an adaptive dynamical variable structure linearization scheme for asymptotic output stabilization problems in systems described by (2.1). In spite of the fact that sliding mode control is, *per se*, a control technique

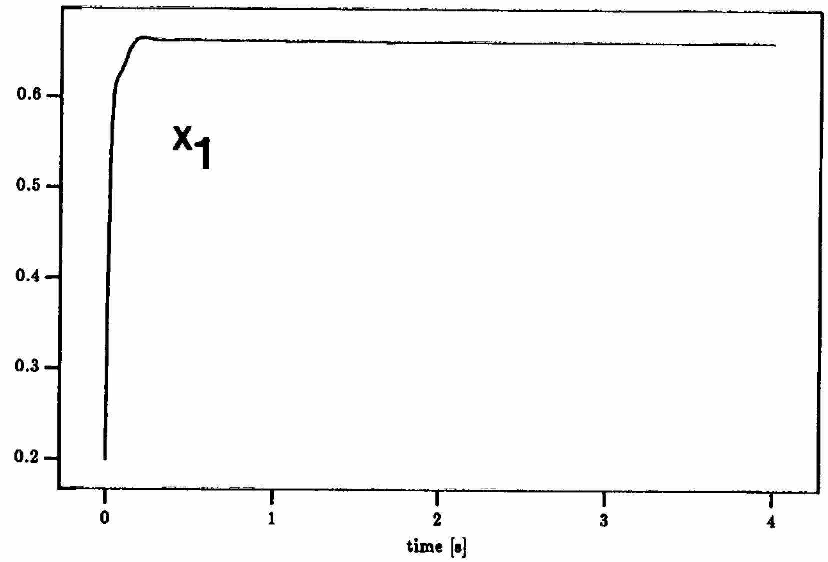

Figure 1. Time response of armature current for adaptive dynamically controlled DC motor example.

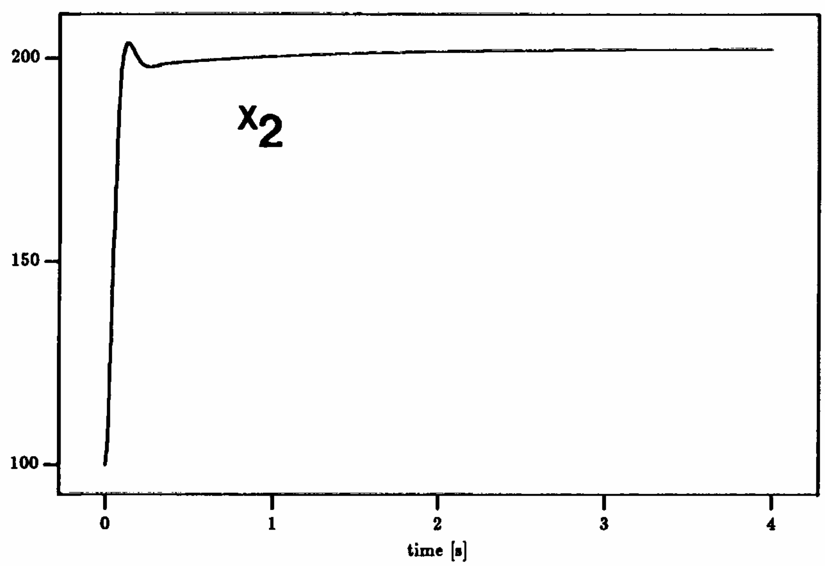

Figure 2. Time response of angular velocity for adaptive dynamically controlled DC motor example.

devised to deal efficiently with parametric and external uncertainty, the class of systems where the switching surface does not depend on system parameters may be very limited indeed. Some of the advantages of dynamical sliding mode control for nonlinear systems lie in the possibility of chattering-free control inputs and state responses (for more details, and an application example, from the chemical process control area, the reader is referred to Sira-Ramirez

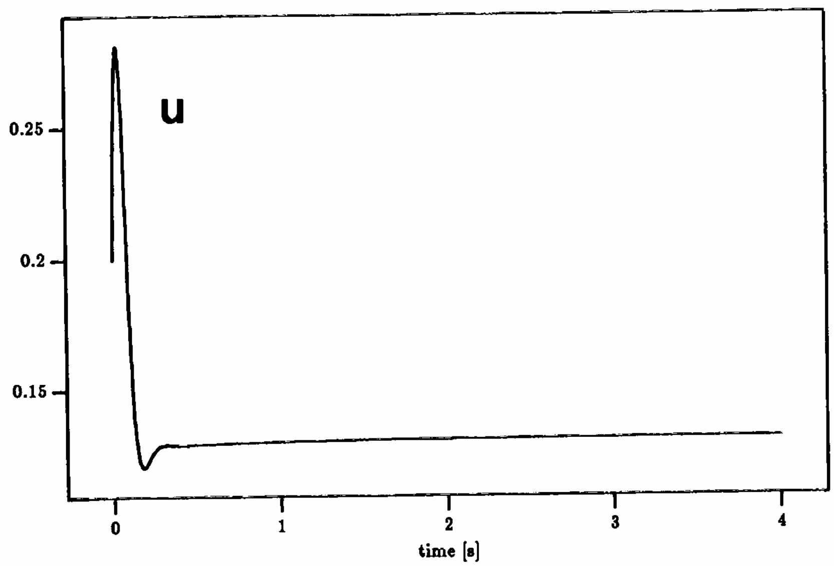

Figure 3. Time response of field winding input current for adaptive dynamically controlled DC motor example.

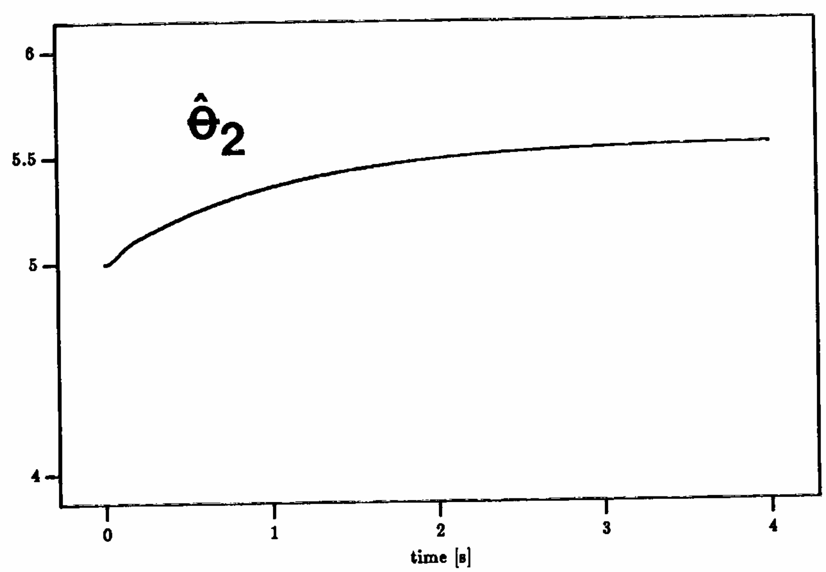

Figure 4. Parameter estimate trajectory for adaptive dynamically controlled DC motor example.

1992 b). However, dynamical sliding modes are naturally created on suitable input-dependent sliding surfaces, which crucially depend upon system parameters. These, in turn, may be completely unknown making the sliding surface definition somewhat contradictory. In this section we shall address such a class of problems from an adaptive control viewpoint.

Proposition 3.1: *Let μ be a strictly positive scalar quantity. Then, the following dynamical discontinuous feedback controller*

$$
\begin{aligned}
a(x,\theta)u^{(n-r)} = & -b_n(x,\theta,u^{[n-r-1]}) - \sum_{i=1}^{r-1} \alpha_i b_i(x,\theta) \\
& - \sum_{j=r}^{n-1} \alpha_j [b_j(x,\theta,u^{[j-r-1]}) + a(x,\theta)u^{(j-r)}] \\
& - \mu \operatorname{sgn} \Big\{ \sum_{i=1}^{r} \alpha_i b_{i-1}(x,\theta) \\
& + \sum_{j=r+1}^{n} \alpha_j [b_{j-1}(x,\theta,u^{[j-r-2]}) + a(x,\theta)u^{(j-r-1)}] \Big\}; \quad \alpha_n = 1
\end{aligned}
\tag{3.1}
$$

drives the output of system (2.1) *to satisfy, in finite time, linearized dynamics of the form*

$$
y^{(n-1)} + \alpha_{n-1} y^{(n-2)} + \cdots + \alpha_1 y = 0 \tag{3.2}
$$

Proof: Define the quantity: $s = y^{(n-1)} + \alpha_{n-1}y^{(n-2)} + \cdots + \alpha_1 y$, and let $s(0)$ stand for the value of s at time $t = 0$. One easily verifies that $\mathrm{d}s/\mathrm{d}t = -\mu \operatorname{sgn}(s)$. Hence the condition $s = 0$ is reached in finite time T, given by: $T = \mu^{-1}|s(0)|$, and the condition $s = 0$ is indefinitely sustained in a sliding mode fashion (Utkin 1978). □

Provided that the system is minimum phase, the scalar time-varying differential equation (3.2) defines a dynamical discontinuous feedback controller which can accomplish exponential output stabilization to zero. As before, one typically chooses the gains α_i $(i = 1, 2, \ldots, n-1)$, to obtain an asymptotically stable dynamics for (3.2).

3.2. *An adaptive regulation scheme for linearizable systems using dynamical sliding-mode control*

Consider the time derivative of the quantity s, defined in the proof of proposition 3.1.:

$$
\dot{s} = \sum_{i=1}^{r-1} \alpha_i b_i(x,\theta) + \sum_{j=r}^{n} \alpha_j [b_j(x,\theta,u^{[j-r-1]}) + a(x,\theta)u^{(j-r)}] \tag{3.3}
$$

Let $\hat{s}$, the estimate of the sliding surface coordinate function, be defined as

$$
\hat{s} = \sum_{i=1}^{r} \alpha_i b_{i-1}(x,\hat{\theta}) + \sum_{j=r+1}^{n} \alpha_j [b_{j-1}(x,\hat{\theta},u^{[j-r-2]}) + a(x,\hat{\theta})u^{(j-r-1)}] \tag{3.4}
$$

Define also

$$
\begin{aligned}
a(x,\hat{\theta})u^{(n-r)} = & -b_n(x,\hat{\theta},u^{[n-r-1]}) - \sum_{i=1}^{r-1} \alpha_i b_i(x,\hat{\theta}) \\
& - \sum_{j=r}^{n-1} \alpha_j [b_j(x,\hat{\theta},u^{[j-r-1]}) + a(x,\hat{\theta})u^{(j-r)}]
\end{aligned}
$$

$$-\mu \operatorname{sgn}\left\{\sum_{i=1}^{r} \alpha_i b_{i-1}(x, \hat{\theta}) + \sum_{j=r+1}^{n} \alpha_j [b_{j-1}(x, \hat{\theta}, u^{[j-r-2]}) + a(x, \hat{\theta}) u^{(j-r-1)}]\right\} \tag{3.5}$$

Then, if a dynamical controller of the form (3.5), based on parameter estimates, is used to regulate the evolution of ds/dt, the expression (3.3) is found to be, after some manipulation

$$\begin{aligned}\dot{s} = & -\mu \operatorname{sgn} \hat{s} + \sum_{i=1}^{r-1} \alpha_i [b_i(x, \theta) - b_i(x, \hat{\theta})] \\ & + \sum_{j=r}^{n-1} \alpha_j \{b_j(x, \theta, u^{[j-r-1]}) - b_j(x, \hat{\theta}, u^{[j-r-1]}) + [a(x, \theta) - a(x, \hat{\theta})] u^{(j-r)}\} \\ & + b_n(x, \theta, u^{[n-r-1]}) - b_n(x, \hat{\theta}, u^{[n-r-1]}) + [a(x, \theta) - a(x, \hat{\theta})] u^{(n-r)}\end{aligned} \tag{3.6}$$

where $u^{(n-r)}$ above represents the expression of the estimated controller, given by (3.5).

By virtue of Remark 2.3 one may conclude that expression (3.6) can be written as a linear function of the parameter estimation error $\Theta - \hat{\Theta} := \phi$

$$\dot{s} = -\mu \operatorname{sgn} \hat{s} + (\Theta - \hat{\Theta})^{\mathrm{T}} W(x, \hat{\theta}, u^{[n-r-1]}) = -\mu \operatorname{sgn} \hat{s} + \phi^{\mathrm{T}} W(x, \hat{\theta}, u^{[n-r-1]}) \tag{3.7}$$

where W is the nonlinear state-dependent regressor vector depending also on the vector of parameter estimates, $\hat{\theta}$, and the 'state' of the dynamical controller, represented here by u and the derivatives of u up to order $n - r - 1$, i.e. by $u^{[n-r-1]}$. By slightly abusing notation we shall write W as a function of $\hat{\Theta}$ rather than as a function of $\hat{\theta}$.

It is easy to see that the switching surface coordinate estimation error $s - \hat{s}$ is given by

$$\begin{aligned} s - \hat{s} &= \sum_{i=1}^{r} \alpha_i [b_{i-1}(x, \theta) - b_{i-1}(x, \hat{\theta})] \\ &\quad + \sum_{j=r+1}^{n} \alpha_j \{b_{j-1}(x, \theta, u^{[j-r-2]}) - b_{j-1}(x, \hat{\theta}, u^{[j-r-2]}) \\ &\quad + [a(x, \theta) - a(x, \hat{\theta})] u^{(j-r-1)}\} \\ &= (\Theta - \hat{\Theta})^{\mathrm{T}} W_s(x, u^{[n-r-1]}) = \phi^{\mathrm{T}} W_s(x, u^{[n-r-1]}) \end{aligned} \tag{3.8}$$

where $W_s(x, u^{[n-r-1]})$ is a switching surface regressor vector which does not depend on the parameter estimates.

Let K be a known positive definite (diagonal) matrix of entires K_{ii}. Consider the Lyapunov function given by

$$V(s, \phi) = \tfrac{1}{2}s^2 + \tfrac{1}{2}\phi^{\mathrm{T}} K \phi \tag{3.9}$$

The time derivative of such a Lyapunov function is obtained as

$$\dot{V}(s, \phi) = s\dot{s} + \phi^{\mathrm{T}} K \dot{\phi} = -\mu s \operatorname{sgn} \hat{s} + \phi^{\mathrm{T}}[sW(x, \hat{\Theta}, u^{[n-r-1]}) + K\dot{\phi}]$$

Choosing the variations of the parameter adaptation error according to the law

$$\dot{\phi} = -\dot{\hat{\Theta}} = -sK^{-1}W(x, \hat{\Theta}, u^{[n-r-1]})$$
$$= -[\hat{s} + \phi^{\mathrm{T}} W_s(x, u^{[n-r-1]})]K^{-1}W(x, \hat{\Theta}, u^{[n-r-1]}) \tag{3.10}$$

one obtains

$$\dot{V}(s, \phi) = = -\mu s \operatorname{sgn} \hat{s} = \begin{cases} -\mu|s| & \text{for } \operatorname{sgn} s = \operatorname{sgn} \hat{s} \\ \mu|s| & \text{for } \operatorname{sgn} s = -\operatorname{sgn} \hat{s} \end{cases} \tag{3.11}$$

It follows from (3.11) that the values of s will converge towards the manifold $s = 0$ as long as s and $\hat{s}$ exhibit the same sign. However, in the region bounded by the manifolds $s = 0$ and $\hat{s} = 0$, both quantities have different signs and the trajectories of s are actually 'repelled' from $s = 0$. It is easy to see from (3.7) that if μ is large enough to overcome the supremum of the absolute value of $\phi^{\mathrm{T}} W$, then a sliding motion exists, for the trajectory of s, on the switching manifold $\hat{s} = 0$. Hence, the values of s will not converge to zero, but, rather, they will be 'trapped' on the estimated surface $\hat{s} = 0$ in a sliding motion and (3.2) will only be approximately satisfied.

Remark 3.2: It follows from (3.8) that, if the parameter estimation error ϕ converges to zero then the actual value of the surface coordinate function s will indeed converge to zero, while sliding on $\hat{s} = 0$. However, convergence of the estimation error ϕ to zero is very much attached to the condition of persistency of excitation (see Sastry and Bodson 1989, Narendra and Annaswamy 1989). This condition may not be fulfilled while the output is being driven to zero in a stable fashion. □

We have thus proven the following result.

Theorem 3.3: *Let μ be such that*

$$\mu > \sup|\phi^{\mathrm{T}} W(x, \hat{\Theta}, u^{[n-r-1]})| \tag{3.12}$$

Then, the adaptive dynamical discontinuous control law (3.4), (3.5), (3.10) renders a sliding mode trajectory on the switching manifold $\hat{s} = 0$ which asymptotically stabilizes the output of the system (2.1) to the equilibrium value of the approximately linear dynamics given by

$$y^{(n-1)} + \alpha_{n-1} y^{(n-2)} + \cdots + \alpha_1 y = \phi^{\mathrm{T}} W_s(x, u^{[n-r-1]}) \tag{3.13}$$

Remark 3.4: Condition (3.12) cannot be verified *a priori* due to is dependance on the state of the system (2.1) and on the state of the dynamical controller (2.12). If a 'modulated' gain μ is allowed for the discontinuous controller then one may choose $\mu = k|\phi^{\mathrm{T}} W(x, \hat{\Theta}, u^{[n-r-1]})|$, with $k > 1$. This guarantees existence of a sliding regime on $\hat{s} = 0$. □

3.2. *A magnetic suspension system example*

3.2.1. *Non adaptive dynamical linearizing sliding mode controller for a magnetic suspension system.* Consider the magnetic ball suspension system described by (see also Kuo 1991):

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= g - \frac{c}{M}\frac{u}{x_1} = g - \theta_1 \frac{u}{x_1} \\ y &= x_1 - X \end{aligned}\right\} \tag{3.14}$$

Where x_1 represents the position of the ball measured from the magnet. The state variable x_2 represents the ball's downwards velocity and u is the non-negative control variable (actually representing the square of the current flowing through the electromagnet coils). M is the mass of the ball and c is a constant. The ratio c/M is assumed to be unknown.

It is desired to regulate the position of the ball to a prescribed set-point value specified by the constant X. It is assumed that the control variable u is naturally bounded in the closed interval $[0, U_{\max}]$.

System (3.14) is exactly linearizable by static-state feedback. A sliding mode controller design would entitle large chattering of the input variable. However, a dynamical sliding mode controller can still be designed for (3.14) by considering the extended system model (see Nijmeijer and van der Schaft 1990) of (3.14).

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= g - \theta_1 \frac{u}{x_1} \\ \dot{u} &= v \\ y &= x_1 - X \end{aligned}\right\} \tag{3.15}$$

Consider the following input-dependent sliding surface for (3.15):

$$s = g - \theta_1 \frac{u}{x_1} + 2\zeta\omega_n x_2 + \omega_n^2(x_1 - X) \tag{3.16}$$

If s can be brought to zero in finite time, the ideal sliding dynamics are seen to satisfy

$$y^{(2)} + 2\zeta\omega_n y^{(1)} + \omega_n^2 y = 0 \tag{3.17}$$

Using the results of proposition 3.1, one finds that the dynamical variable structure controller is represented by

$$\dot{u} = \frac{x_1}{\theta_1}\left[2\zeta\omega_n\left(g - \theta_1 \frac{u}{x_1}\right) + \omega_n^2 x_2 + \mu \operatorname{sgn} s\right] + \frac{x_2 u}{x_1} \tag{3.18}$$

3.2. *Adaptive dynamical sliding mode linearizing control for magnetic suspension system*

Owing to a lack of exact parameter knowledge, instead of the controller (3.18), one uses a dynamical variable controller, based on estimates of the parameter and the sliding surface coordinate function

$$\dot{u} = \frac{x_1}{\hat{\theta}_1}\left[2\zeta\omega_n\left(g - \hat{\theta}_1 \frac{u}{x_1}\right) + \omega_n^2 x_2 + \mu \operatorname{sgn} \hat{s}\right] + \frac{x_2 u}{x_1} \tag{3.19}$$

where

$$\hat{s} = g - \hat{\theta}_1 \frac{u}{x_1} + 2\zeta\omega_n x_2 + \omega_n^2(x_1 - X) \tag{3.20}$$

Let ϕ_1 denote the parameter estimation error, $\theta_1 - \hat{\theta}_1$. Then, the evolution of the sliding surface coordinate function s obeys

$$\dot{s} = -\mu \operatorname{sgn} \hat{s} - \frac{\phi_1}{\hat{\theta}_1}\left[2\zeta\omega_n\left(g - \hat{\theta}_1 \frac{u}{x_1}\right) + \omega_n^2 x_2 + \mu \operatorname{sgn} \hat{s}\right] \tag{3.21}$$

A sliding motion is induced on the estimate of the switching surface $\hat{s} = 0$. Notice that, from (3.16) and (3.20) one obtains

$$s = \hat{s} - \phi_1 \frac{x_3}{x_1} \tag{3.22}$$

Using the result in (3.10), we obtain a parameter estimation error update law of the form

$$\dot{\phi}_1 = -\dot{\hat{\theta}}_1 = \left(\hat{s} - \phi_1 \frac{x_3}{x_1}\right) \frac{2\zeta\omega_n g + \omega_n^2 x_2}{\hat{\theta}_1} \tag{3.23}$$

Simulations were run to assess the performance of the adaptive dynamical sliding mode controller (3.19), (3.20), (3.23) on a magnetic ball suspension system with the following parameters

$$\theta_1 = \frac{c}{M} = 100\,\mathrm{Nm\,A^{-2}}, \quad g = 9{\cdot}81\,\mathrm{m\,s^{-2}}$$

The state variable trajectories $x_1(t)$, $x_2(t)$ are shown in Fig. 5. The non-chattering control input trajectory is depicted in Fig. 6. The state trajectories

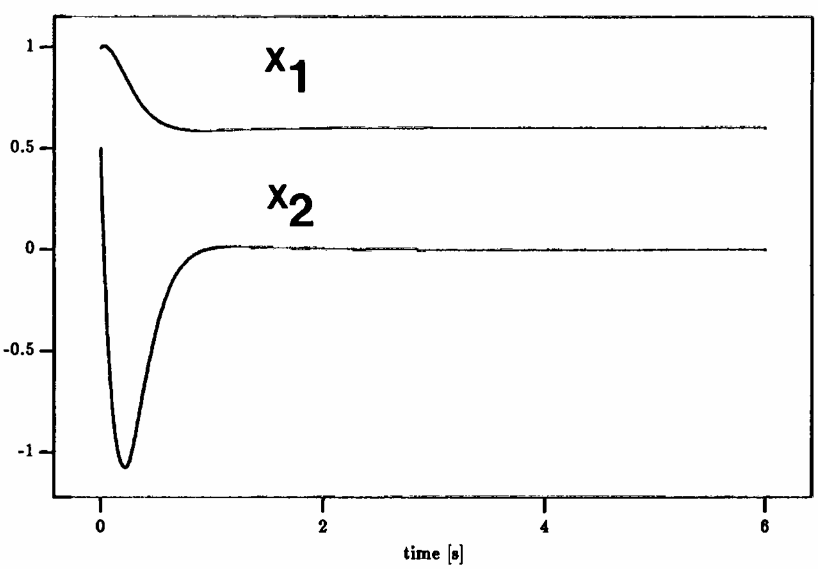

Figure 5. Time response of states variables for adaptive dynamical sliding mode controlled magnetic suspension system example.

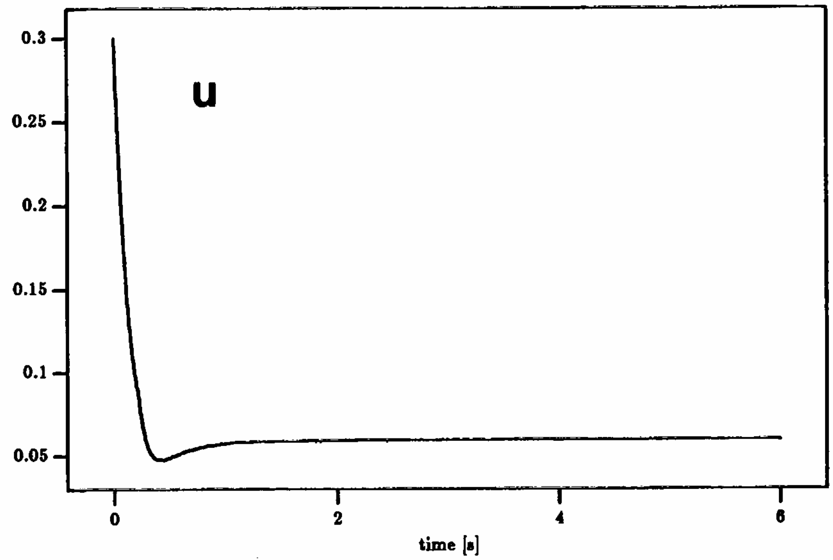

Figure 6. Time response of control input variable for adaptive dynamical sliding mode controlled magnetic suspension system example.

converge to the values $x_1 = 0{\cdot}605$ m, $x_2 = 6{\cdot}1 \times 10^{-5}$ m s^{-1}. These values are reasonably close to their ideal equilibrium values given by $x_1 = 0{\cdot}6$ m and $x_2 = 0$. In Fig. 7, the estimated parameter is shown. This parameter slowly converges to a constant value of $102{\cdot}4$ Nm A^{-2} which does not coincide with the 'true' value of 100 Nm A^{-2}. Figures 8 and 9 show, respectively, the time

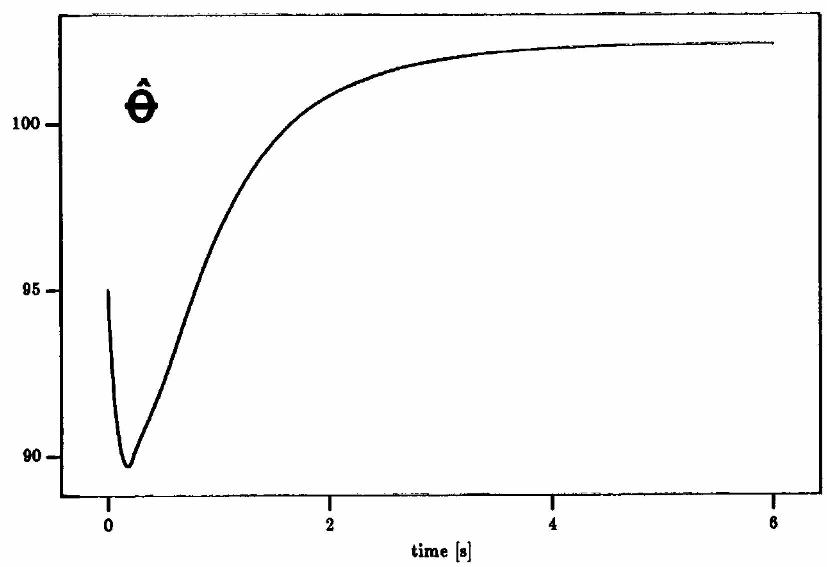

Figure 7. Parameter estimate trajectory for adaptive dynamical sliding mode controlled magnetic suspension system example.

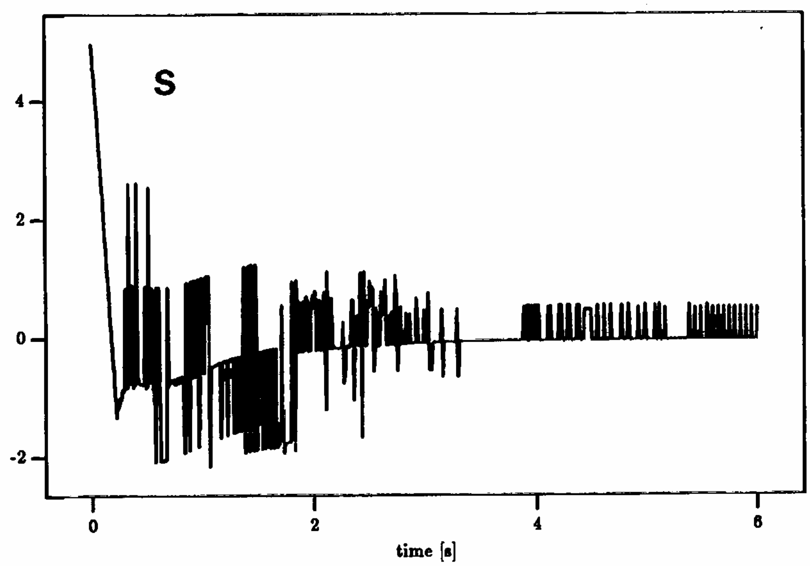

Figure 8. Evolution of sliding surface coordinate function for adaptive dynamical sliding mode controller in the magnetic suspension system example.

evolution of the sliding surface coordinate function s and its estimated value $\hat{s}$. It is clearly seen that sliding motions take place on $\hat{s} = 0$, while the value of s slowly converges towards $\hat{s} = 0$ yielding a steady-state error. The variable structure controller parameters and the constants for the adaptation laws were set as: $\mu = 20$, $\zeta = 0{\cdot}9$, $\omega_n = 7$, $K_{11} = K_{22} = 0{\cdot}1$.

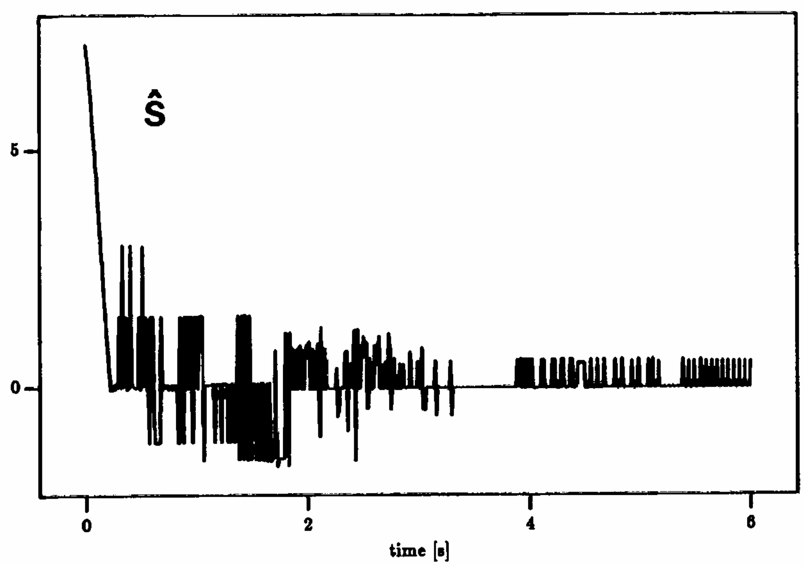

Figure 9. Evolution of estimate values of sliding surface coordinate function for adaptive dynamical sliding mode controller in the magnetic suspension system example.

4. Conclusions

In this paper, adaptive dynamical continuous and discontinuous feedback compensators, which approximately accomplish asymptotic output stabilization, were examined for a class of parametric uncertain systems linearizable by dynamical feedback strategies. Adaptive dynamical feedback linearization may be accomplished by extending the available results for adaptive statically linearizable systems. This simply entitles the incorporation of the states of the dynamical controller as part of the adaptation mechanism. A controller design example was presented for the asymptotic stabilization of the shaft's angular velocity in a nonlinear DC motor. The performance of the controller was evaluated through computer simulations which were encouraging.

An extension of the dynamical variable structure control techniques developed in Sira-Ramirez (1992 b, c) were presented for the adaptive case. The results show that whenever the input-dependent sliding surface exhibits an explicit dependance on the uncertain parameters, a sliding motion can only be generated on an estimate of the switching surface, which is known to be in error with respect to the exactly linearizing manifold. Thus, a small constant stabilization error, directly dependent on the steady-state parameter estimation error, may always be present in the proposed adaptive scheme, if the condition of persistency of excitation is not verified during the transient. However, if the persistency of excitation conditions are satisfied, these will, surely, induce more accurate results on the stabilization task. This condition, as is well known, is more naturally verified in adaptive output tracking tasks. An illustrative example was presented dealing with the adaptive dynamical variable structure stabilization of a magnetic suspension system. The proposed adaptive control approach inherits, from the underlying *dynamical* sliding mode control scheme, the chattering-free trajectories for the inputs and the associated state and output responses.

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