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Pulse-Width-Modulation Control of Some Mechanical Systems

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Abstract. Pulse-Width-Modulation (PWM) control strategies have been of limited interest, and use, in the regulation of a large class of mechanical systems, such as robotic manipulators and rigid spacecrafts. The chattering associated to discontinuous bang-bang feedback control input signals has traditionally posed serious concerns when considering the selection of this robust and efficient control technique over some other smooth control alternatives. This article proposes a means of circumventing the inconveniences caused by the PWM control signal discontinuities in such a class of systems, while retaining the robust features of the PWM control approach.

1. Introduction

The design of PWM control policies for the regulation of dynamical systems is generally based on approximating an average (i.e., smooth) desired behavior by means of high frequency discontinuous control signals which, somehow, emulate the effects of the designed average feedback control inputs. In spite of its well known robustness features, the PWM control approach induces undesirable bang-bang control inputs causing noticeable high frequency discontinuities, or *chattering*, in the controlled responses. This kind of discontinuous behavior severely limits, for a large class of mechanical and electromechanical systems, the applicability of PWM stabilizing feedback control policies, as well as that of some other closely related discontinuous feedback control techniques (such as Sliding Mode control and Pulse-Frequency-Modulation control). For mechanical systems, in general, and robotic manipulators, in particular, low limits to possible wear and tear effects and high accuracy of the regulated variables are important design concerns when choosing the regulating feedback control scheme.

PWM control constitutes a robust feedback control policy due to its insensitivity to external disturbance inputs, certain immunity to model parameter variations, within known bounds, and enhanced performance in spite of the effects of everpresent modeling errors. For basic work related to PWM control schemes, in linear and nonlinear dynamical systems, the reader is referred to Polak [1], Skoog [2], Skoog and Blankenship [3], Tsypkin [4], La Cava et al. [5], Csaki [6], and also Sira-Ramirez [7–12]. The parallels existing between a closely related discontinuous control technique, known as *Sliding Mode Control* (sec Utkin [13]), and PWM were established in [8] from a purely geometric viewpoint. From the viewpoint emphasized in this article, however, *sampled* sliding mode controllers turn out to be just a particularization of PWM control strategies in which a fixed, fully saturated, *duty ratio function* is adopted throughout the entire controller operation (see Sira-Ramirez [14] for further details and applications). The robustness associated to PWM control schemes for nonlinear systems were explored in [9], where it was shown that under certain, traditional *matching conditions*, PWM regulation achieves the same insensitivity to parametric variations and external (bounded) perturbations, as that exhibited by Variable Structure Control under sliding model conditions.

A large collection of mechanical nonlinear systems belong to the class of *systems exactly linearizable by state coordinate transformations and static state feedback*. The *generalized state models* (see Fliess [15, 16]) of such systems do not include input signal derivatives. Thus, static, instead of dynamical, controllers naturally arise from the exact linearization procedure introduced in [15]. In this article, we present a general design method for synthesizing *static*, as well *dynamical*, PWM feedback control laws stabilizing to a constant equilibrium point the state of any nonlinear dynamical system belonging to the above class. PWM controllers are proposed which effectively stabilize to zero a suitably designed auxiliary scalar output function of the system. The obtained restricted dynamics results, in turn, in a linearization and local asymptotic stabilization of the original system state to its equilibrium value. However, the obtained static PWM controller is shown to include undesirable chattering effects on the generated input signals. A dynamical PWM controller alternative is then proposed, and synthesized, on the basis of *Fliess's Generalized Controller Canonical Form* (FGCCF) (see [15]) of the associated *Extended System* (see Nijmeijer and Van der Schaft [17]). Continuous, instead of bang-bang, feedback control input signals are, thus, obtained which robustly stabilize to a constant operating point the closed loop system, while effectively smoothing out the chattering effects on the control signals and regulated state variables.

Section two presents a fundamental result on the PWM control of an elementary scalar dynamical system. It is shown that, based on this scalar result, a general PWM controller design procedure can be proposed for higher order nonlinear controlled plants. Section two also reviews FGCCF on which, both, the static, and dynamical, PWM controller design scheme for linearizable systems, proposed in this section, utilizes the concept of the *Extended System*. Section three presents an illustrative example, constituted by a purely mechanical system, representing a single link rigid robotic manipulator on which the performances of a static and a dynamical PWM feedback controller are tested by means of computer simulations. The conclusion of this work is collected in Section four.

2. PWM control of nonlinear systems

2.1. PWM control of a scalar system

Consider the scalar PWM controlled dynamical system with state variable s, in which the constants a, W and ρ are all strictly positive quantities.

$$\dot{s} = -a\, s - Wv$$

$$v = \mathrm{PWM}_\tau(s) = \begin{cases} sgn\,[s(t_k)] & \text{for } t_k \leq t < t_k + \tau[s(t_k)]T \\ 0 & \text{for } t_k + \tau[s(t_k)]T \leq t < t_k + T \end{cases} \tag{1}$$

$$\tau[s(t)] = \begin{cases} 1 & \text{for } |\, s(t)\,| > \dfrac{1}{\rho} \\ \rho\, |s(t)| & \text{for } |\, s(t)\,| \leq \dfrac{1}{\rho} \end{cases}$$

$$k = 0, 1, 2, \ldots;\ t_{k+1} = t_k + T.$$

The t_k's represent *regularly* spaced sampling instants. The sampling interval $[t_k, t_{k+1}]$ is known as the *duty cycle* and its width is here represented by the constant T, i.e., $T = t_{k+1} - t_k$. At each sampling instant, t_k, the value of the width of the next sign-modulated, fixed amplitude, control pulse v is determined by the sampled value of the *duty ratio function*, represented by $\tau[s(t_k)]$. The function *sgn* stands for the *signum function*:

$$\mathrm{sgn}(s) = \begin{cases} +1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0 \end{cases}$$

The pulse width, $\tau[s(t_k)]T$, saturates to the value of the duty cycle, T, (i.e., $\tau[s(t_k)]$ saturates to 1) as long as the value of the controlled state s is larger, in absolute value, than a given prespecified threshhold $1/\rho$ (see Figure 1). When the absolute value of the state, s, of the scalar system (1) is driven below the value $1/\rho$, the duty ratio, $\tau[s(t_k)]$, also starts decreasing, in a linear fashion, with respect to $|s|$. When the scalar system is, eventually, at rest in $s = 0$, no control pulses are then applied to the system.

Remark. The basic idea behind the above discontinuous PWM control law (1) is that large errors in the scalar state s should require larger corrective pulse widths during the fixed intersampling periods. Small errors, on the other hand, should be driven to zero, on the basis of the adopted sampling frequency with corrective pulse widths decreasing to zero until stabilization. In a sense, PWM policies are *proportional* feedback policies subject to saturation.

The following theorem establishes a sufficient condition for the asymptotic stability to zero of the PWM controlled system (1). The same theorem is also found in [1] and also in [2]. In [2] it is further established that the proposed convergence condition is also necessary. We furnish, however, a different proof of the sufficiency result.

Theorem 1. *The PWM controlled system (1) is asymptotically stable to* $s = 0$, *if:*

$$\rho W < a \tanh\left(\frac{aT}{2}\right). \tag{2}$$

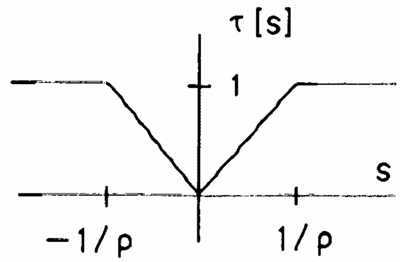

Figure 1. Duty ratio function for PWM controlled scalar system.

Proof. Due to the piecewise constant nature of the control inputs and the linearity of the underlying continuous system (1), it suffices to study the stability of the discretized version of (1) at the sampling instants. An exact discretization of the PWM controlled system (1) thus yields

$$s(t_k + T) = e^{-aT}s(t_k) - \frac{We^{-aT}}{a}(e^{a\tau[s(t_k)]T} - 1)\mathrm{sgn}[s(t_k)]. \tag{3}$$

Suppose the initial condition $s(t_0)$ is chosen deep into the region $|s| > 1/\rho$. The evolution of the sampled values of $s(t)$ obey, according to (1):

$$s(t_{k+1}) = e^{-aT}\left\{ s(t_k) - \frac{W}{a}(e^{aT} - 1)\mathrm{sgn}[s(t_k)] \right\}. \tag{4}$$

The absolute value of the incremental step, $\Delta s(t_k) := s(t_{k+1}) - s(t_k)$, is readily obtained from (4), from where it is easily found that

$$|\Delta s(t_k)| = (1 - e^{-aT}) \left| \left(s(t_k) + \frac{W}{a} \right) \right| < (1 - e^{-aT}) \left[|s(t_k)| + \frac{W}{a} \right]. \tag{5}$$

The condition $|\Delta s(t_k)| < 2/\rho$ is sufficient to ensure that the value of $s(t_k)$ will be eventually found within the bounded region $|s| < 1/\rho$, irrespectively of the initial condition value $s(t_0)$ given in $|s| > 1/\rho$. Sufficiency is clear from the facts that, in the region $|s| > 1/\rho$, the $|\Delta s(t_k)|$ decrease at each step and that one must guarantee that $s(t_k)$ does not persistently *jump* over the band $|s| < 1/\rho$, thus falling into a possible limit cycle behavior. From (5) and the fact that $|s(t_k)| > 1/\rho$, the condition $|\Delta s(t_k)| < 2/\rho$ is guaranteed if we let

$$(1 - e^{-aT}) \left[1 + \frac{\rho W}{a} \right] < 2, \tag{6}$$

which is just the expression (2) after some straightforward manipulations.

Suppose now that the initial state, $s(t_0)$, of the scalar system (1), is found in the region $|s| < 1/\rho$. The exact discretization of the PWM controlled system is now given by

$$s(t_{k+1}) = e^{-aT}s(t_k) - \frac{We^{-aT}}{a}(e^{a\rho|s(t_k)|T} - 1)\mathrm{sgn}[s(t_k)]. \tag{7}$$

The absolute values of the incremental steps $\Delta s(t_k) = s(t_k + T) - s(t_k)$ are thus given by

$$|\Delta s(t_k)| = (1 - e^{-aT}) \left| \left[s(t_k) + \frac{W}{a} \frac{(e^{a(\rho|s(t_k)| - 1)T} - e^{-aT})}{(1 - e^{-aT})} \right] \right|$$

$$< (1 - e^{-aT}) \left[|s(t_k)| + \frac{W}{a} \frac{(e^{a(\rho|s(t_k)| - 1)T} - e^{-aT})}{(1 - e^{-aT})} \right]. \tag{8}$$

A sufficient condition for asymptotic stability of (7) to zero is given by $|\Delta s(t_k)| < 2 |s(t_k)|$. Notice, however, that from the fact that $\rho\ |s(t_k)| < 1$, (8) implies that

$$|\Delta s(t_k)| < (1 - e^{-aT}) \left[|s(t_k)| + \frac{W}{a} \right].$$

Hence, the above convergence condition is fully guaranteed if we let

$$|\Delta s(t_k)| < (1 - e^{-aT}) \left[|s(t_k)| + \frac{W}{a} \right] < 2\ |s(t_k)| \tag{9}$$

i.e., from the second inequality one has

$$\frac{W}{a} < \frac{2}{(1 - e^{-aT})} |s(t_k)| - |s(t_k)| = |s(t_k)| \tanh \left(\frac{aT}{2} \right),$$

which after multiplication of by ρ, and the fact that $\rho\ |s(t_k)| < 1$, results in

$$\frac{\rho W}{a} < \rho|s(t_k)| \tanh \left(\frac{aT}{2} \right) < \tanh \left(\frac{aT}{2} \right).$$

The result follows.

A computer simulation of system (1) is shown below in Figure 2. This figure depicts the state $s(t)$, the PWM control input signal $v(t)$ and the duty ratio function $\tau(s(t))$. The values of the involved constants were chosen as: $a = 4$. $\rho = 1.2$, $T = 0.2$, $W = 1$. Since $\rho W/a = 0.3 < \tanh(0.4) = 0.3799$ asymptotic stability of $s(t)$ to zero is guaranteed by Theorem 2.1.

2.2. Fliess' generalized controller canonical form

Consider the analytical n-dimensional state variable representation of a nonlinear system:

$$\dot{x} = F(x, u). \tag{10}$$

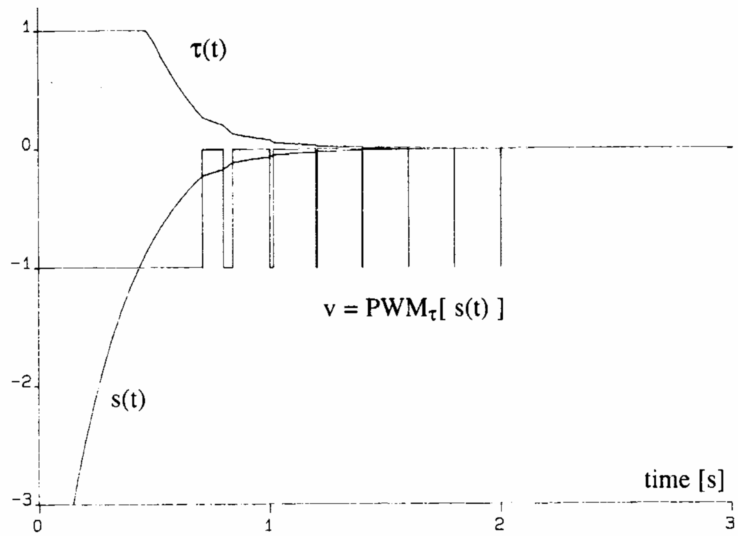

Figure 2. State Response, duty ratio function trajectory and PWM control signal for first order PWM controlled scalar system.

It is assumed that the nonlinear system (10) exhibits a constant equilibrium point of interest characterized by $F(X(U), U) = 0$. We refer to this equilibrium point as $(X(U), U)$.

Associated to the system (10) is FGCCF given by (see [16] for further details)

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\cdots \\
\dot{\xi}_{n-1} &= \xi_n \\
\dot{\xi}_n &= c(\xi, u, \dot{u}, \ldots, u^{(\alpha)})
\end{aligned} \tag{11}$$

Implicit in this representation is the assumption of the existence of an element, $\xi_1 = h(x)$, called the *differential primitive element*, which generates the *generalized state representation* (11) of system (10). Under these circumstances, system (10) is transformed into system (11) by means of an invertible input-dependent state coordinate transformation of the form

$$\xi = \Phi(x, u, \dot{u}, \ldots, u^{(\alpha-1)}) \tag{12}$$

given by

$$\Phi(x, u, \dot{u}, \ldots, u^{(\alpha-1)}) = [h(x), \dot{h}(x), \ldots, h^{(n-1)}(x, u, \dot{u}, \ldots, u^{(\alpha-1)})]'. \tag{13}$$

We assume that the equilibrium point $(X(U), U)$ of (8) is transformed, by (10), into the vector $\xi = 0$. i.e., $\Phi(X(U), U, 0, \ldots, 0) = 0$. From this assumption, it readily follows that $c(0, U, 0, \ldots, 0) = 0$.

Suppose that for a suitably designed feedback control policy the state of (9) is asymptotically driven to zero. The autonomous dynamics described by

$$c(0, u, \dot{u}, \ldots, u^{(\alpha)}) = 0 \tag{14}$$

constitutes the *zero dynamics* (see Fliess [18]). In order to guarantee an overall stable performance of the controlled system, it is of crucial importance to assess the stability of such a dynamics around its possible equilibrium points. Around the equilibrium point of interest $u = U$ we assume that the dynamical system (14) is asymptotically stable, i.e., the system $dx/dt = F(x, u)$ with output $y = h(x)$ is *minimum phase. Nonminimum phase systems* can also be controlled by PWM control policies by means of suitably defined *precompensators*. This topic, however, will be explored elsewhere (see also Fliess and Messager [19] for interesting related aspects in connection with sliding mode control of linear time invariant systems).

Motivated by the fact that for a large class of mechanical and electromechanical systems the integer α, in the corresponding GCCF (11), is equal to zero, we shall concentrate our developments, from now on, on such a particular class of systems. This class corresponds to *systems exactly linearizable by means of state coordinate transformations and static state feedback*. i.e., those systems in which the function c, in (11), is of the form $c(\xi, u)$, and for which the transformation (12) only involves state variables. We further assume that $\partial c/\partial u$ is nonidentically zero, at least, locally around the equilibrum point.

2.3. Static PWM control for exactly linearizable systems

Let $p_{n-1}(\lambda)$ be an $(n - 1)$th order Hurwitz polynomial with constant coefficients

$$p_{n-1}(\lambda) = \lambda^{n-1} + a_{n-1}\lambda^{n-2} + \ldots + a_2\lambda + a_1. \tag{15}$$

Consider now the following auxiliary output function of the system (11):

$$s(\xi) = \xi_n + a_{n-1}\xi_{n-1} + \ldots + a_2\xi_2 + a_1\xi_1. \tag{16}$$

If the condition $s = 0$ is achieved by means of suitable controls, the restricted motions of the system (11) satisfy the following asymptotically stable linear time-invariant dynamics:

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\ldots \\
\dot{\xi}_{n-1} &= - a_{n-1}\xi_{n-1} - \ldots - a_2\xi_2 - a_1\xi_1
\end{aligned} \tag{17}$$

The following proposition is a direct consequence of the preceding considerations and of Theorem 1.

Proposition 1. *A system of the form (10) is locally asymptotically stabilizable to the equilibrium point (U, X(U)) if the control action u is specified as a PWM control policy given by the solution of the following implicit (algebraic) equation*

$$c(\Phi(x), u) = -\sum_{i=1}^{n} (a_{i-1} + a a_i)\, h^{(i-1)}(x) - W\, \mathrm{PWM}_\tau \left[\sum_{i=1}^{n} a_i h^{(i-1)}(x) \right] \tag{18}$$

where $a_0 = 0$, and $a_n = 1$.

Proof. Imposing on the auxiliary output function $s(\xi)$, given in (16), the asymptotically stable discontinuous PWM controlled dynamics defined by (1), one immediately obtains, in terms of the transformed coordinates ξ, an *implicit* nonlinear equation for the required control input u. Rewriting the obtained expression in original state and input coordinates the static controller adopts the form (18).

Remark. From the assumption that $\partial c/\partial u$ is locally nonzero, it follows that (18) can be explicitly solved for u. We denote, in general, the solution for u in an equation of the form: $c(\xi, u) = z$, as $u = \gamma(\xi, z)$, i.e., $c(\xi, \gamma(\xi, z)) \equiv z$ for some given indeterminate z.

As it can be easily seen from (18), the case of exactly linearizable systems results in a *static* PWM controller and, hence the proposed scheme yields discontinuous control actions. Thus, bang-bang feedback control signals are generated in the closed loop system.

2.4. Dynamical PWM control of nonlinear systems

Consider now the *Extended System*, associated with system (10) (see [17])

$$\begin{aligned} \dot{x} &= F(x, u) \\ \dot{u} &= \nu \end{aligned} \tag{19}$$

It is easy to see that if $\xi_1 = h(x)$ is a differential primitive element for (10), ξ_1 also qualifies as a differential primitive element for (19). Letting $c(\xi, u)$ become a new state variable ξ_{n+1}, it is follows readily that the GCCF of (19) is written as

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\cdots \\ \dot{\xi}_n &= \xi_{n+1} \\ \dot{\xi}_{n+1} &= \frac{\partial c(x, u)}{\partial x} F(x, u) \Bigg|_{\substack{x=\Phi^{-1}(\xi) \\ u=\gamma(\xi,\xi_{n+1})}} + \frac{\partial c(x, u)}{\partial u} \Bigg|_{\substack{x=\Phi^{-1}(\xi) \\ u=\gamma(\xi,\xi_{n+1})}} \nu \\ &=: f(\xi, \xi_{n+1}) + g(\xi, \xi_{n+1})\nu. \end{aligned} \tag{20}$$

Hence, if the original system is exactly linearizable by means of state coordinates transformations and static state feedback, so is the extended system with respect to the new auxiliary input ν. An equilibrium point of (20) is evidently given by $\nu = 0, u = U, x = X(U)$. We denote this equilibrium point by $((X(U), U), 0)$.

Notice that the state coordinate transformation taking (19) into the linearizable form (20) is given by

$$\hat{\xi} := \begin{bmatrix} \xi \\ \xi_{n+1} \end{bmatrix} = \begin{bmatrix} \Phi(x) \\ c(\Phi(x), u) \end{bmatrix} =: \hat{\Phi}(x, u) \tag{21}$$

which is evidently invertible whenever $\Phi(x)$ is invertible and $\partial c/\partial u$ is nonzero. It is easy to see that $\hat{\xi} = 0$ is an equilibrium point of (20).

Let $p_n(\gamma)$ be an n-th order Hurwitz polynomial with constant coefficients

$$p(\lambda) = \lambda^n + \alpha_n \lambda^{n-1} + \ldots + \alpha_2 \lambda + \alpha_1. \tag{22}$$

If one considers now the following auxiliary output function of the system (20)

$$\sigma(\xi, \xi_{n+1}) = \xi_{n+1} + \alpha_n \xi_n + \ldots + \alpha_2 \xi_2 + \alpha_1 \xi_1. \tag{23}$$

Then the condition $\sigma = 0$ implies that the restricted motions of the system (20) satisfy the following asymptotically stable linear time-invariant dynamics

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\ldots \\ \dot{\xi}_n &= -\alpha_n \xi_n - \ldots - \alpha_2 \xi_2 - \alpha_1 \xi_1. \end{aligned} \tag{24}$$

The following proposition is a direct consequence of the preceding considerations and of Theorem 2.

Proposition 2. *A system of the form (10) is locally asymptotically stabilizable to the equilibrium point $((X(U), U), 0)$ if the control action u is specified as a dynamical PWM control policy given by the solution of the following explicit first order nonlinear differential equation with discontinuous right-hand side*

$$f(\hat{\Phi}(x, u)) + g(\hat{\Phi}(x, u))\dot{u} = -\sum_{i=1}^{n}(\alpha_{i-1} + a\alpha_i)\, h^{(i-1)}(x) - (\alpha_n + a)\, h^{(n)}(x, u)$$

$$- W\, \mathrm{PWM}_\tau \left[\sum_{i=1}^{n} \alpha_i h^{(i-1)}(x) + h^{(n)}(x, u) \right] \tag{25}$$

where $a_0 = 0$.

Proof. As in Proposition 1, above, imposing on the auxiliary output function $\sigma(\xi, \xi_{n+1})$, given in (23), the asymptotically stable discontinuous PWM controlled dynamics, defined by (1), one immediately obtains, in terms of the transformed coordinates (ξ, ξ_{n+1}), a nonlinear algebraic equation for the required control input ν. Rewriting the obtained static controller expression in terms of the original state and input coordinates (x, u), the controller adopts the dynamical form of equation (25).

Notice that since $g = \partial c/\partial u$ is assumed to be nonzero, the controller (25) is locally well defined and no *impasse points* needs to be considered (see Fliess and Hassler [20]).

3. An illustrative example

3.1. PWM control of a single link rigid robotic manipulator

Consider the following nonlinear dynamical model of a single link robotic manipulator (Khalil [21]):

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{L}\sin x_1 - \frac{k}{M}x_2 + \frac{1}{ML^2}u \end{aligned} \tag{26}$$

where x_1 is the link angular position, x_2 is the angular velocity and u represents the applied torque. The mass M is assumed to be concentrated at the tip of the manipulator. The constant k is the viscous damping coefficient.

It is desired to synthesize, both, a static and a dynamical PWM feedback control policy which drives the angular position of the system to a constant desired angular position x_{1d}.

Static PWM Controller Design

Let $\xi_1 = x_1 - x_{1d}$. It is easy to see that ξ_1 qualifies as a differential primitive element for (26) and that to obtain the FGCCF of (26) it simply requires the use of the following (trivial) state coordinate transformation : $\xi_1 = x_1 - x_{1d}$, $\xi_2 = x_2$:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= -\frac{g}{L}\sin(\xi_1 + x_{1d}) - \frac{k}{M}\xi_2 + \frac{1}{ML^2}u. \end{aligned} \tag{27}$$

Let $s(\xi) = \xi_2 + a_1\xi_1$ with $a_1 > 0$, be an auxiliary output function for system (27). Notice that if $s(\xi)$ is stabilized to zero, the constrained dynamics evolves according to $d\xi_1/dt = -a_1\xi_1$ (or: $dx_1/dt = -a_1(x_1 - x_{1d})$) thus achieving the desired regulation task. Imposing on the auxiliary output function $s(\xi)$ the asymptotically stable dynamics of the PWM controlled system (1) one obtains the following static PWM controller

$$u = ML^2 \left[-aa_1\xi_1 + \left(\frac{k}{M} - a - a_1 \right) \xi_2 + \frac{g}{L} \sin(\xi_1 + x_{1d}) - W \, \mathrm{PWM}_\tau(\xi_2 + a_1\xi_1) \right] \tag{28}$$

which, in original coordinates is rewritten as

$$u = ML^2 \left\{ -aa_1(x_1 - x_{1d}) + \left(\frac{k}{M} - a - a_1 \right) x_2 + \frac{g}{L} \sin(x_1) - W \, \mathrm{PWM}_\tau[x_2 + a_1(x_1 - x_{1d})] \right\} \tag{29}$$

Simulations were run for the above PWM controlled manipulator (26), (29), with the following parameters: $M = 1$ [Kg], $L = 1.0$ [m], $k = 0$, $x_{1d} = 4$ [rad], $g = 9.8$ [m/s^2]. The auxiliary output function was synthesized with $a_1 = 1.25$ [s^{-1}]. The static PWM controller parameters were chosen as: $a = 4$ [s^{-1}], $W = 1.0$ [rad/s], $T = 0.2$ [s], $\rho = 0.2$ [s/rad]. Figure 3 depicts the state trajectories of the controlled system clearly showing convergence to the desired angular position. Figure 4 shows the evolutions of the auxiliary output function, of the PWM control signal, constituting (29) and of the duty ratio function $\tau(s(t))$. Figure 5 shows a magnified view of a portion of the discontinuous (PWM) applied torque input signal u, as generated by (29), and of the PWM signal.

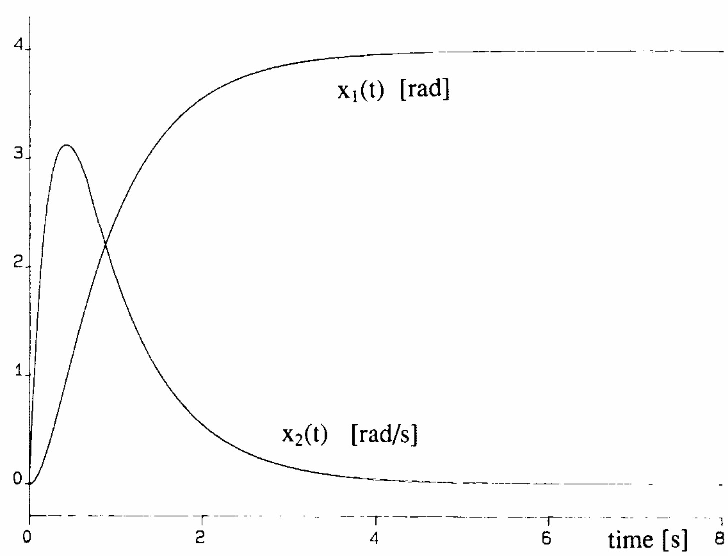

Figure 3. Angular position and angular velocity of static PWM controlled robotic manipulator.

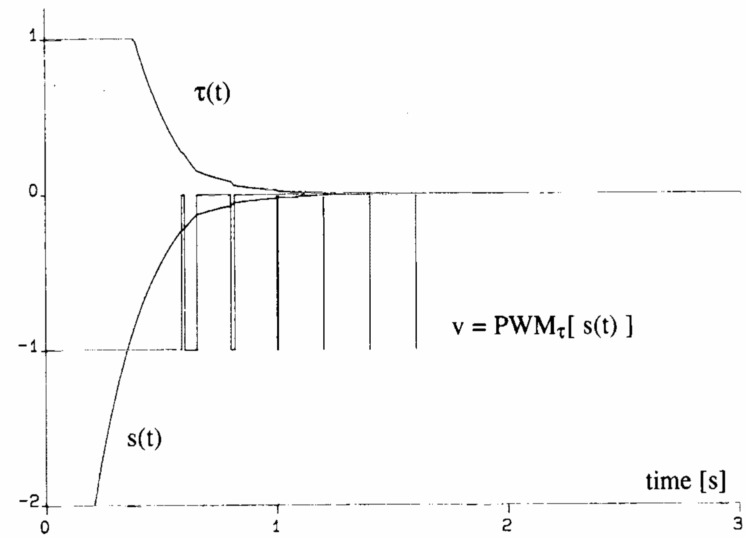

Figure 4. Duty ratio function trajectory, PWM control signal and auxiliary output function trajectory for static PWM controlled robotic manipulator.

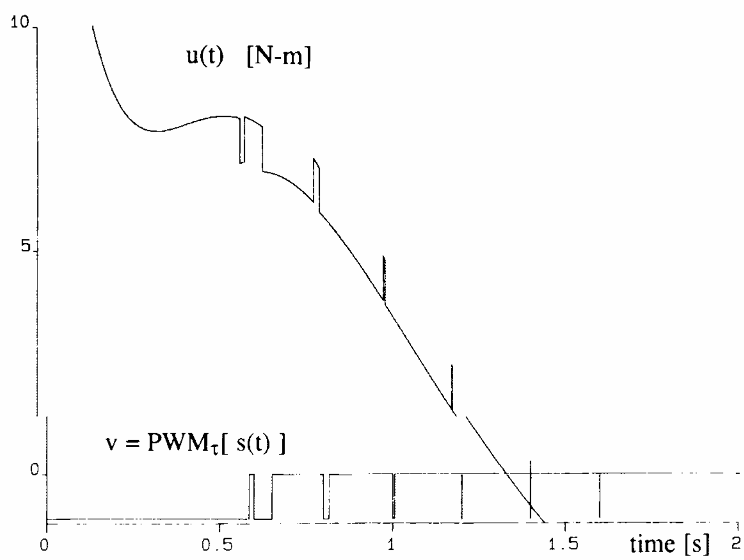

Figure 5. Discontinuous control input torque signal (magnified view of only a portion) and PWM control signal for static PWM controlled robotic manipulator.

Dynamical PWM Controller Design

Consider now the extended system of (26)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{L}\sin x_1 - \frac{k}{M}x_2 + \frac{1}{ML^2}u$$
$$\dot{u} = \nu. \tag{30}$$

The resulting FGCCF of the extended system is obtained as

$$\dot{\xi}_1 = \xi_2$$
$$\dot{\xi}_2 = \xi_3$$
$$\dot{\xi}_3 = -\frac{g}{L}\xi_2 \cos(\xi_1 + x_{1d}) - \frac{k}{M}\xi_3 + \frac{1}{ML^2}\nu \tag{31}$$

with $\xi_1 = x_1 - x_{1d}$, $\xi_2 = x_2$, $\xi_3 = -(g/L)\sin x_1 - (k/M)x_2 + [1/(ML^2)]\,u$.

Let the auxiliary output function $\sigma(\hat{\xi})$ be defined as $\sigma(\hat{\xi}) = \xi_3 + a_2\xi_2 + a_1\xi_1$, with a_2 and a_1 positive constants, chosen in the standard second order system form, with damping factor ζ and natural frequency ω_n: $a_2 = 2\zeta\omega_n$ and $a_1 = \omega_n^2$. Notice that if $\sigma(\hat{\xi})$ is stabilized to zero, the constrained dynamics evolves in accordance to the asymptotically stable second order dynamics: $d\xi_1/dt = \xi_2$; $d\xi_2/dt = -2\zeta\omega_n\xi_2 - \omega_n^2\xi_1$, thus, achieving the desired control task: $\xi_2 = x_2 \to 0$ and $\xi_1 = x_1 - x_{1d} \to 0$. Imposing on $\sigma(\hat{\xi})$ the same asymptotically stable dynamics of the PWM controlled system (1) one obtains the following static PWM controller for the extended system

$$\nu = ML^2\left[-\left(a + \frac{k}{M} + 2\zeta\omega_n\right)\xi_3 - (2\zeta\omega_n a + \omega_n^2)\,\xi_2 - a\omega_n^2\xi_1 + \frac{g}{L}\xi_2\cos(\xi_1 + x_{1d}) - W\,\mathrm{PWM}_\tau(\xi_3 + 2\zeta\omega_n\xi_2 + \omega_n^2\xi_1)\right] \tag{32}$$

which, in original coordinates, is rewritten as a dynamical PWM controller of the form:

$$\dot{u} = ML^2\left\{-\left(a + \frac{k}{M} + 2\zeta\omega_n\right)\left(-\frac{g}{L}\sin x_1 - \frac{k}{M}x_2 + \frac{1}{ML^2}u\right) - (2\zeta\omega_n a + \omega_n^2)\,x_2 - a\omega_n^2(x_1 - x_{1d}) + \frac{g}{L}x_2\cos(x_1) - W\,\mathrm{PWM}_\tau\left[-\frac{g}{L}\sin x_1 - \frac{k}{M}x_2 + \frac{1}{ML^2}u + 2\zeta\omega_n x_2 + \omega_n^2(x_1 - x_{1d})\right]\right\} \tag{33}$$

Simulations were run for the dynamically PWM controlled manipulator (26, 33) with the same physical parameter values as before. The auxiliary output function σ was synthesized such that the corresponding characteristic polynomial of the linearized system is $p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$, with $\zeta = 0.8$ and $\omega_n = 2.0$ [rad/s]. The static PWM controller parameters were chosen in the same manner as in the previous example. Figure 6 depicts the state trajectories of the controlled system clearly showing convergence to the desired angular position and to zero angular velocity. Figure 7 shows the evolution of the auxiliary output function σ of the PWM control signal, constituting (33), and of the duty ratio function $\tau(\sigma(t))$. Figure 8 shows the substantially smoothed out (chattering-free) applied torque input signal u, as generated by the dynamical PWM controller (33).

4. Conclusions

A general stabilizing design procedure, based on dynamical PWM feedback control policies has been presented for nonlinear systems describing some linearizable mechanical systems. A stabilizing discontinuous static controller of the PWM type is proposed for an elementary scalar system. Based on this simple result, a static PWM controller design can be obtained, for general high order nonlinear systems, by the zeroing of a suitably chosen auxiliary scalar output function. Zeroing of such an auxiliary output function induces an asymptotically stable motion for the constrained dynamics characterized by a linear time-invariant system with eigenvalues placeable at will. The results are easily implemented on a dynamical extension of the original nonlinear system which now results in a dynamical PWM feedback controller. In the dynamical controller case, the discontinuities, associated

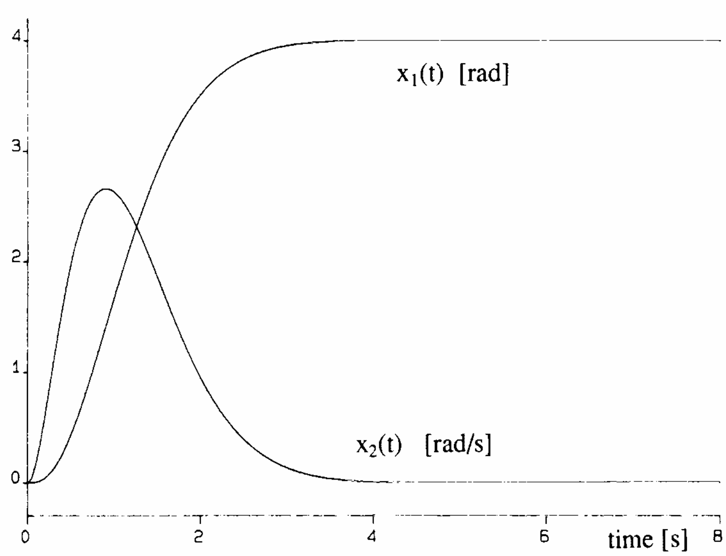

Figure 6. Angular position and angular velocity of dynamical PWM controlled robotic manipulator.

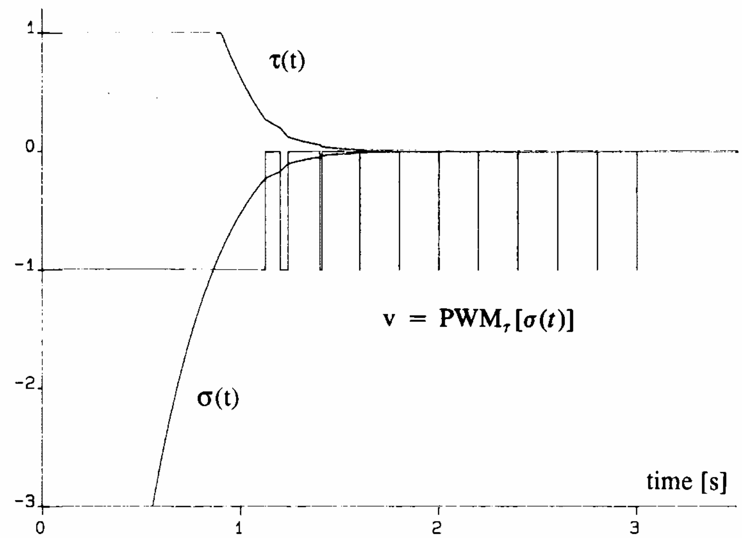

Figure 7. Duty ratio function trajectory, PWM control signal and auxiliary output function trajectory for dynamical PWM controlled robotic manipulator.

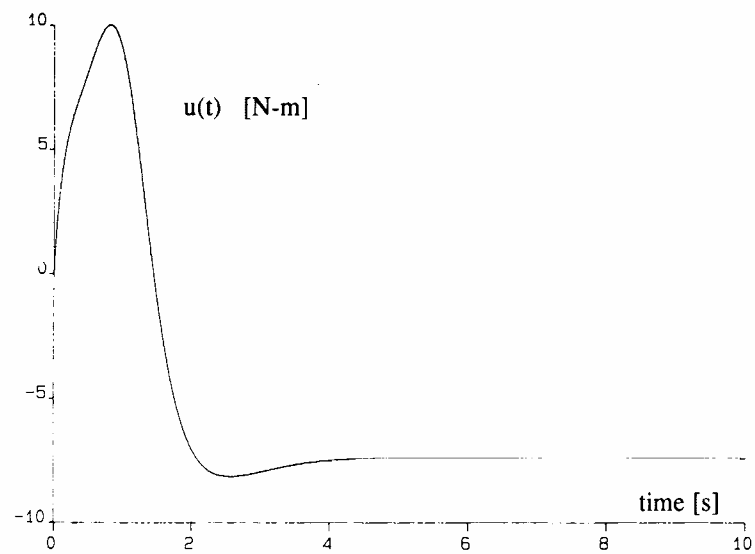

Figure 8. Chattering free control input torque signal for dynamical PWM controlled robotic manipulator.

to the PWM regulator, take place on the one-dimensional state space of the dynamical controller and not in the state space of the system. The resulting integrated control actions are, thus, *continuous* with substantially reduced (smoothed out) chattering.

In order to establish the salient stability features of the actual closed loop PWM controlled system, no need arises to resort to *average* controlled system considerations, nor high sampling frequency assumptions. As a matter of fact, the sampling frequency plays no crucial role in the stabilization features of the system, aside from the verification of a simple algebraic condition. New applications areas, such as nonlinear chemical process control, nonlinear electromechanical systems control etc., in which PWM control was not traditionally feasible can now benefit from the inherent robustness and high performance characteristics of this class of discontinuous control strategies.

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