Elektrotehniški vestnik 60(2–3): 91–98, 1993
Electrotechnical Review, Ljubljana, Slovenija

A redundant PWM control policy for nonlinear systems regulation

Hebertt Sira-Ramírez[1], Orestes Llanes-Santiago[2]

[1] *Departamento Sistemas de Control, Universidad de Los Andes, Mérida - Venezuela*
[2] *Postgrado en Ingeniería de Control, Universidad de Los Andes, Mérida - Venezuela*

Abstract. This article proposes the use of a dynamically generated pulse-width-modulation (PWM) control scheme, defined on the basis of the zeroing of a suitable stabilizing input-dependent manifold which represents a known smooth feedback regulation scheme already available for the given nonlinear plant. The proposed control scheme constitutes a means of robustifying, via discontinuous feedback control of PWM type, any model-based smooth feedback control policy previously designed for the stabilization of the given nonlinear system. The approach naturally produces a redundant smoothed discontinuous feedback scheme of the PWM type, with several advantageous properties regarding insensitivity to external perturbation signals and to modelling errors, as inherited from the underlying PWM scheme. The scheme is also shown to be robust with respect to sudden failure of the smooth portion of the proposed feedback loop.

Key words: nonlinear systems, pulse-width-modulation control

Redudantno upravljanje nelinearnih regulacijskih sistemov s PŠM regulacijo

Povzetek. V prispevku je predlagana uporaba regulatorja z dinamično generiranim pulznoširinsko moduliranim (PŠM) izhodom, ki temelji na upravljivi, asimptotsko stabilni mnogoterosti določeni z obstoječo povratnozančno nelinearno progo. Predlagana regulacijska shema je dopolnitev zvezne, nelinearne regulirane proge, ki izboljša robustnost celega sistema. Ta podvojitev daje več prednosti glede občutljivosti na zunanje motnje in na nemodelirano dinamiko reguliranega sistema. Predlagana shema zagotavlja stabilnost sistema tudi v primeru nenadnega izpada zvezne povratne zanke.

Ključne besede: nelinearni sistemi, pulzno širinsko modulirane regulacije

1 Introduction

In recent times, the use of the *differential algebraic approach* (See Fliess, 1990) in the design, and analysis, of discontinuous feedback control policies, for nonlinear systems, has been shown to result in several advantageous properties of the closed loop dynamics and the involved controller. Fist of all, a unified approach has clearly emerged for the design of stabilizing discontinuous feedback strategies which includes: sliding modes, pulse-width- modulation and pulse-frequency-modulation (PFM) control of nonlinear systems (See Sira-Ramírez, 1992c). The prevailing characteristic of this unified approach is the use of *dynamical* discontinuous controllers providing the system with smoothed (i.e., continuous, as opossed to bang-bang) inputs. The original idea of using differential algebra in sliding mode control is due to Fliess and Massager (1990, 1991) and later developed by Messager (1992) for the linear systems case. Based on these works, Sira-Ramírez (1991, 1992a, 1993a) proposed a rather significant departure from traditional discontinuous feedback control design schemes by proposing input-dependent stabilizing surfaces. The use of such input- dependent stabilizing surfaces was shown to yield dynamical discontinuous feedback policies, with the effective elimination of bang-bang inputs and the associated chattering responses (Sira-Ramírez, 1992c). The possibilities of using discontinuous feedback control policies, such as sliding mode control, pulse-width-modulation, and pulse-frequency-modulation based strategies in non traditional application areas, such as chemical processes and mechanical systems, is also one of the developments that immediately emerged from this new viewpoint (see Sira-Ramírez, 1992b, 1992d).

Received 9 March 1993
Accepted 26 March 1993

In this article we propose the use of a redundant dynamical PWM control strategy based on an input-dependent stabilizing manifold directly suggested by a model-based designed smooth nonlinear feedback control law, assumed to ideally stabilize the nonlinear system according to a preselected stabilization criterion (optimal performance, pole placement, exact linearization, etc.).

Such a stabilizing surface coordinate prescription may be interpreted as a *feedback control implementation error* which needs to be zeroed. Thus, any deviation of the implemented input signal, from the required value, generated by the smooth state feedback control law, yields an error which immediately triggers a PWM based feedback control correction signal which forcefully imposes the designed feedback control law. The activation of the resulting dynamical PWM feedback controller mechanism, here proposed, complements the original feedback scheme in a redundant fashion. This feature results in the possibilities of tolerating sudden failures in the main smooth designed feedback loop. Feedback implementation error signals are frequently caused by the presence of independent perturabtion input signals to the actuator, by feedback designs carried out under unknown modelling errors and, also, because of sensor failures. The benefits of our proposed dynamical PWM feedback control scheme are: 1) enhanced robustness for the actual operation of the originally designed smooth feedback control law, 2) feedback correction based on smoothed discontinuous feedback control actions 3) redundancy, in the form of a dynamical PWM feedback law, of the originally designed smooth feedback control scheme and 4) robustness with respect to a class of sudden feedback signal failures.

Section 2 presents the general feedback control scheme based on utilizing the designed smooth feedback control law as an input-dependent stabilizing scalar function which is to be zeroed by means of a PWM based control strategy. In this section we derive and analyze some of the advantageous features of such an approach. Section 3 is devoted to an application example drawn from the chemical process control area. Simulations studies which clearly portray the performance of the proposed controller under feedback signal failures are also included. Section 4 contains the conclusions and suggestions for further research.

2 Main results

2.1 A dynamical PWM controller based on prescribed smooth feedback control law

Consider a nonlinear n-dimensional single input smooth system of the form:

$$\dot{x} = f(x, u) \tag{1}$$

Suppose, furthermore, that a smooth feedback controller has been designed which locally stabilizes the trajectories of the control system (2.1) to a desirable constant equilibrium point $X(U)$, dependent upon a constant value of the input signal U. i.e., $f(X(U), U) = 0$. We assume, without any loss of generality that U is nonzero. The stabilizing feedback control law is assumed to be explicitly given by:

$$u = -k(x) \tag{2}$$

In other words, the closed loop system:

$$\dot{x} = f(x, -k(x)) \tag{3}$$

is assumed to locally exhibit desirable asymptotic stability features towards the equilibrium point. In equilibrium, the value of the feedback signal $-k(X(U))$ is compatible with the equilibrium value for u, i.e.; $U = -k(X(U)) \neq 0$.

Suppose now that an auxiliary input-dependent function of the form:

$$s(x, u) = u + k(x) \tag{4}$$

is synthesized, and that the condition $s(x, u) = 0$ is proposed as an input-dependent *stabilizing surface* on which the following discontinuous dynamics of the PWM type is imposed:

$$\dot{s}(x, u) = -W \ \mathrm{PWM}_\tau[s(x, u)] \ ; \ W > 0 \tag{5}$$

where:

$$\mathrm{PWM}_\tau[s(x,u)] = \begin{cases} \mathrm{sign}[s(x,u)] \ ; \ \text{for} \quad t_k \leq t < t_k + \tau[s(x(t_k), u(t_k))]T \\ 0 \ ; \ \text{for} \quad t_k + \tau[s(x(t_k), u(t_k))]T \leq t < t_k + T \end{cases}$$

$$k = 0, 1, 2, \ldots \quad ; \quad t_k + T = t_{k+1} \tag{6}$$

with:

$$\tau[s(x,u)] = \begin{cases} 1 & \text{for } |s(x,u)| \geq 1/\beta \\ \beta |s(x,u)| & \text{for } |s(x,u)| < 1/\beta \end{cases} \tag{7}$$

and W being a sufficiently large, strictly positive, constant quantity.

In Sira-Ramírez (1992b), the trajectories of the closed loop system (5)-(7) have been shown to be asymptotically stable to $s(x, u) = 0$, if, and only if, the following condition:

$$W\beta T < 2 \tag{8}$$

is satisfied. The appendix, however, presents a different proof of this result.

Ideally speaking, under stabilization to zero of the scalar function $s(x, u)$, one has: $u = -k(x)$ which is assumed to be a desirable stabilizing feedback law. The next step is concerned with computation of the discontinuous feedback control control law, of the PWM type, that accomplishes the asymptotic zeroing of the auxiliary output function $s(x, u)$.

Replacing (4) into (5) leads to the following dynamical PWM controller for the nonlinear system:

$$\dot{u} = -\left[\frac{\partial k(x)}{\partial x}\right] f(x, u) - W \ \mathrm{PWM}_\tau[u + k(x)] \tag{9}$$

Equation (9) represents a time-varying nonlinear first order differential equation for the control input signal u, with a discontinuous right hand side. The additional complication incurred in building such a dynamical discontinuous feedback controller is superseeded by the several advantages it bestows on the closed loop features of the controlled system.

A block diagram of the feedback controller (9) in shown in figure 1. A straightforward integration of the above expression (9) allows for the reinterpretation of the controller in terms of a redundant "hybrid" controller comprising the original feedback law (2) implemented in parallel to an integrated (i.e. smoothed) discontinuous feedback signal, of the PWM type, triggered by the incipient values of the feedback error signal $u + k(x)$. Indeed, integration of (9) yields:

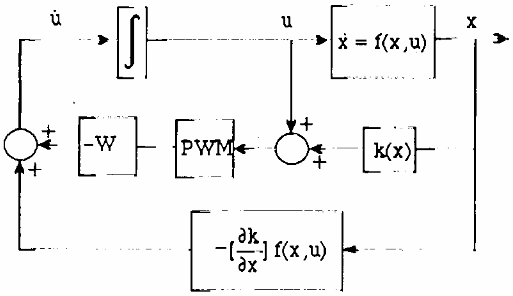

Figure 1. Dynamical PWM controller enforcing a designed smooth feedback control law.

$$u = -\int_0^t \Big[\frac{\partial k(x(\sigma))}{\partial x} f(x(\sigma), u(\sigma)) + W\ \mathrm{PWM}_\tau[u(\sigma) + k(x(\sigma))]\Big] d\sigma + u(0) =$$

$$= -k(x(t)) - \int_0^t W\ \mathrm{PWM}_\tau[u(\sigma) + k(x(\sigma))] d\sigma + [u(0) + k(x(0))] =$$

$$= -k(x(t)) - W \int_0^t \mathrm{PWM}_\tau[u(\sigma) + k(x(\sigma))] d\sigma + s(x(0), u(0))$$

A block diagram depicting this reinterpretation of the controller (9) in shown in figure 2.

By virtue of the above developments, we finally rewrite the dynamical controller (9) as:

$$\begin{aligned} u &= v - k(x) \\ \dot{v} &= -W\ \mathrm{PWM}_\tau[u + k(x)] \end{aligned} \qquad (10)$$

2.2 Some properties of the proposed dynamical PWM controller

The dynamical controller (10) exhibits several advantageous properties which are summarized below:

1)The discontinuities associated to the underlying PWM signals, are relegated to the first order time derivative of the control input signal u. Hence, the result-

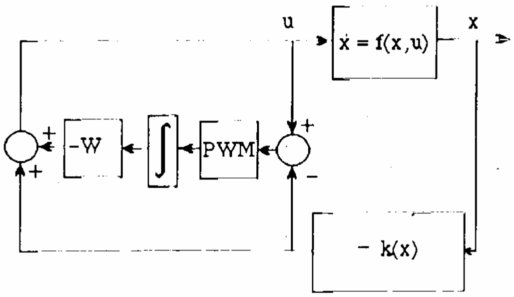

Figure 2. Reinterpretation of dynamical PWM controller enforcing a designed smooth feedback control law: $u = -k(x)$.

ing controller is, indeed, *continuous*. Bang-bang input signals, otherwise characteristic of PWM control schemes (see Sira-Ramírez, 1989a), are thus effectively suppressed by the dynamic nature of the proposed controller (see also Sira-Ramírez, 1992c).

2)Suppose that at certain time $t = T_f$, the smooth portion of the feedback loop, feeding the signal component $-k(x)$ to the control input signal u, fails for an indefinite period of time (see figure 3). Assume, furthermore, that at the failure time T_f the discontinuous portion of the controller was currently exhibiting a stable behaviour on $s(x, u) = 0$ (i.e., assume that, ideally, $s(xT_f), u(T_f)) = 0$). Suppose also that the system's state was already stabilized at its equilibrium value $x = X(U)$ and, hence, $u(T_f) = -k(X(U)) = U$. The feedback control law being enforced at any time $t > T_f$, after the feedback failure, safisfies:

$$u = v = -W \int_{T_f}^t \mathrm{PWM}_\tau[u(\sigma) + k(x(\sigma))] d\sigma \qquad (11)$$

i.e.,

$$\dot{v}(t) = \dot{u} = -W\ \mathrm{PWM}_\tau[\bar{u}(t) + k(x(t))] \qquad (12)$$

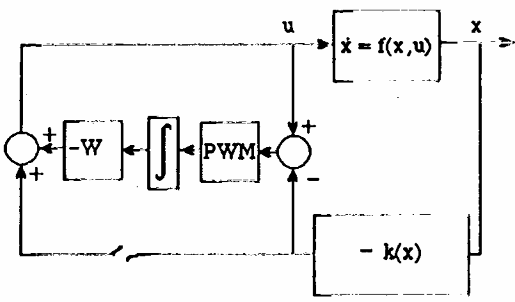

Figure 3. A feedback signal failure in the smooth portion of the redundant controller.

It follows that the value of $u(t)$ is instantaneously set to zero immediately after T_f. It easily follows from (11) and (12) that u evolves, for $t > T_f$, in such a fashion that the feedback error signal: $s(x,u) = u + k(x)$ is being constantly diminished in absolute value. The remaining, discontinuous, part of the failed controller locally forces the motions of the controlled system towards the input-dependent manifold: $s(x,u) = u + k(x) = 0$, thus asymptotically recovering the original feedback control law $u = -k(x)$.

Indeed, from (12) and the form of the PWM controller (6), it follows that either u remains constant, during the (OFF) intervals: $t_k + \tau[s(t_k)]T \leq t \leq t_k + T$, or, else, it evolves with constant slope, of value W, during the (ON) intervals: $t_k \leq t \leq t_k + \tau[s(t_k)]T$. On these last intersampling intervals, the control u exhibits a constant slope which is opposite in sign to that of the error signal $s(x,u)$ i.e., if the sampled value of $s(x,u)$, at time t_k, is positive ($u > -k(x)$), then u decreases with constant slope $-W$. Otherwise, if the sampled value of $s(x,u)$, at time t_K, is negative, ($u < -k(x)$) then u increases with constant slope $+W$. It follows that, if in absolute value, the rate of change of $-k(x)$, with respect to time, which is given by:

$$-\frac{dk(x)}{dt} = -\frac{\partial k(x)}{\partial x} f(x,u)$$

, is slower that the rate of change of the generated u ($+W$ or $-W$), then the control u asymptotically approaches, in a pulsed manner, the required condition: $u = -k(x)$ .i.e., the control u "catches up" with the designed feedback value $-k(x)$. Thus, for a sufficiently large value of the constant W, the surviving portion of the feedback control input signal approaches, in a pulsed manner, the right feedback law values.

3)PWM controllers are known to be highly insensitive to external perturbation signals and to modelling errors (Sec Sira-Ramírez, 1389b). Thus, the above scheme always imposes, in a robust fashion, the "right feedback control law". Changes in state, due to external perturbation inputs to the system, result in corresponding changes in the feedback control law $-k(x)$, both, at the smooth and discontinuous portions of the proposed controller. If the designed smooth control law is known to enjoy robustness properties, with respect to a certain class of perturbation input signals, the proposed controller simply inherits those properties and results in a forceful imposition of the designed smooth control law.

3 An application example

3.1 Concentration control in an exothermic continuously stirred tank reactor

Consider the following nonlinear dynamical controlled model of an exothermic reaction occuring inside a CSTR (see Parrish and Bosilow, 1986), where the control objective is to regulate the outlet concentration through regulation, to a constant value, of the water jacket temperature:

$$\begin{aligned} \dot{x}_1 &= \frac{F}{V}(c_0 - x_1) - a x_1 e^{-\frac{b}{x_2}} \\ \dot{x}_2 &= \frac{F}{V}(T_0 - x_2) + \frac{aL}{c_p} x_1 e^{-\frac{b}{x_2}} - \frac{h}{V c_p}(x_2 - u) \\ y &= x_2 - T \end{aligned} \tag{13}$$

Where x_1 represents the product concentration. The state variable x_2 represents the reactor temperature. The control variable u is the water jacket temperature. F is the reactor throughput in lb/hr, c_0 is the inlet flow concentration in lb/lb, T_0 is the inlet flow temperature measured in deg.R, c_p is the material heat capacity in BTU/lb R while V and L are, respectively, the reactor holdup (in lb.) and the heat of the reaction (in BTU/lb). The constant h is the heat transfer parameter (in BTU/hr.R), b is the activation constant (in deg R) and a is the pre-exponential factor in hr^{-1}. A constant temperature T is to be stably maintained to indirectly control the product concentration x_1 to its constant equilibrium valve X_1.

A stable constant equilibrium point for this system is then given by:

$$\begin{aligned} x_2 = T \quad &; \quad x_1 = X_1(T) = \frac{c_0}{1 + \frac{V}{F} a e^{-\frac{b}{T}}} \; ; \\ u = U(T) &= T - \frac{c_p F}{h}(T_0 - T) - \\ &- \frac{aLV}{h} \frac{c_0 e^{-\frac{b}{T}}}{1 + \frac{V}{F} a e^{-\frac{b}{T}}} \end{aligned} \tag{14}$$

3.2 A smooth linearizing controller design for the CSTR system

It is easy to verify that the following smooth state feedback controller results in an exact input-output linearization of the given system (13):

$$\begin{aligned} u = &\left[1 + \frac{c_p}{h}(F - \lambda V)\right] x_2 - \frac{aLV}{h} x_1 e^{-\frac{b}{x_2}} + \\ &+ \left(\frac{\lambda V c_p T}{h} - \frac{F c_p}{h} T_0\right) \end{aligned} \tag{15}$$

where λ is a positive quantity regulating the exponential decay in the imposed linear asymptotically stable dynamics for the output y:

$$\dot{y} = -\lambda y \tag{16}$$

It may be verified, after some tedious but straightforward computations, that the closed loop system (13), (15) exhibits a globally asymptotically stable zero dynamics around the equilibrium point (14) (see also Sira-Ramírez, 1993b). Hence, the system is globally minimum phase and its output may be controlled to zero by exact cancellation of the nonlinearities.

3.3 Redundant dynamical sliding mode controller design for the CSTR system

According to the results of section 2 we choose as the scalar stabilizing relation the input-dependent auxiliary output function:

$$s(x,u) = u - \left[1 + \frac{c_p}{h}(F - \lambda V)\right] x_2 + + \frac{aLV}{h} x_1 e^{-\frac{b}{x_2}} - \left(\frac{\lambda V c_p T}{h} - \frac{F c_p}{h} T_0\right) \quad (17)$$

By imposing the PWM controlled dynamics (5) on the scalar quantity $s(x,u)$, one obtains the following dynamical discontinuous feedback controller:

$$u = v + \left[1 + \frac{c_p}{h}(F - \lambda V)\right] x_2 - - \frac{aLV}{h} x_1 e^{-\frac{b}{x_2}} + \left(\frac{\lambda V c_p T}{h} - \frac{F c_p}{h} T_0\right)$$

$$\dot{v} = -W\ \mathrm{PWM}_\tau \left[u - \left[1 + \frac{c_p}{h}(F - \lambda V)\right] x_2 + + \frac{aLV}{h} x_1 e^{-\frac{b}{x_2}} - \left(\frac{\lambda V c_p T}{h} - \frac{F c_p}{h} T_0\right)\right] \quad (18)$$

3.4 Simulations

Simulations were performed for the system (13) with the dynamical PWM controller (18). The numerical values adopted for the system parameters were taken from Parrish and Basilow, (1986):

F = 2000 lb/hr ; c_0 = 0.50 lb/lb ; V = 2400 lb.;
$a = 7.08 \times 10^{10}$ hr^{-1} ; b = 15080 deg.R ;
T_0 = 5320 deg.R ; L = 600 BTU/lb. ;
c_p = 0.75 BTU/lb.R ; h = 15000 BTU/hr.R;

For such values of the parameters, the equilibrium point (14) of the system results in:

$$x_2 = T = 600 \text{ deg.R} ;$$
$$u = U(T) = 107.679 \text{ deg.R} ;$$
$$X_1(T) = 0.246 \text{ lb/lb.}$$

The values of the dynamical PWM controller parameters, W and λ, were set to be:

$$W = 30; \quad \lambda = 2hr^{-1}$$

Figure 4 shows the state responses of the dynamically PWM controlled system asymptotically converging to their corresponding equilibrium points. Figure 5 depicts the continuous trajectory of the dynamically generated control input signal u, along with the time evolution of the input-dependent stabilizing surface coordinate function $s(x,u)$. The value of $s(x,u)$ is seen

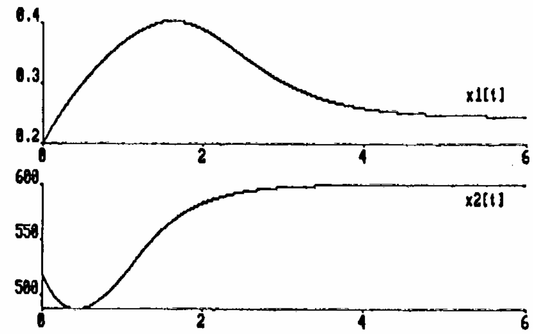

Figure 4. State trajectory responses of dynamically PWM controlled system.

to asymptotically converge to zero. Figure 6 shows the evolution of the duty ratio function τ and of the underlying discontinuous (PWM) signal adscribed to the dynamical controller.

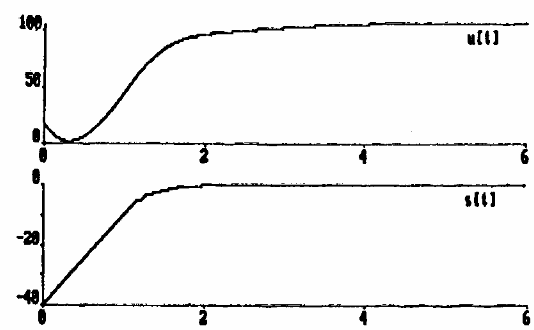

Figure 5. Continuous trajectory of the Dynamical PWM generated control input input signal and the corresponding evolution of the stabilizing surface coordinate fauntion, $s(x,u)$.

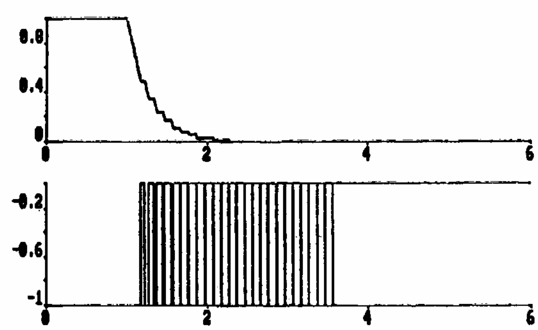

Figure 6. Evolution of the duty ratio function of the dynamically PWM controller and the associated PWM signal.

In order to check the robustness of the proposed dynamical PWM control scheme, with respect to sudden failures in the originally designed smooth feedback loop, we also simulated the performance of the system with the following dynamical discontinuous feedback controller:

$$
\begin{aligned}
u &= v + \kappa \left\{ \left[1 + \frac{c_p}{h}(F - \lambda V)\right] x_2 - \right. \\
&\left. - \frac{aLV}{h} x_1 e^{-\frac{b}{x_2}} + \left(\frac{\lambda V c_p T}{h} - \frac{F c_p}{h} T_0\right) \right\} \\
\dot{v} &= -W \, \mathrm{PWM}_\tau \left[u - \left[1 + \frac{c_p}{h}(F - \lambda V)\right] x_2 + \right. \\
&\left. + \frac{aLV}{h} x_1 e^{-\frac{b}{x_2}} - \left(\frac{\lambda V c_p T}{h} - \frac{F c_p}{h} T_0\right) \right]
\end{aligned}
\tag{19}
$$

where the variable κ, simulating the feedback loop failure, was allowed to be:

$$
\kappa = \begin{cases} 1 & \text{for } t \le T_f \\ 0 & \text{for } t > T_f \end{cases} \tag{20}
$$

with $T_f = 3.5$.

Figures 7 shows the state responses of the dynamically PWM controlled system subject to the sudden failure of the form (20). The PWM controller is seen to re-stabilize the state trajectories to their corresponding equilibrium points. Figure 8 depicts the corresponding trajectory of the failed control input, showing, at failure time T_f, the instantaneous resetting to the value zero of the control input signal and its subsequent asymptotic recovery to its pre-failure equilibrium value, thus achieving asymptotic stabilization of the system. Notice that the surviving portion of the controller still generates a continuous feedback control signal in spite of its underlying (PWM) discontinuous nature. In figure 8 it is also shown the behaviour of the input dependent stabilizing scalar function $s(x, u)$ before and after the smooth feedback loop signal failure. Figure 9 depicts the corresponding trajectories of the duty ratio function and of the underlying PWM signal beng internally generated by the (failed) dynamical discontinuous feedback controller arrangement.

4 Conclusion

A robust redundant feedback control scheme, based on dynamical PWM control, has been proposed for nonlinear systems for which a smooth feedback control policy is already known. The proposed scheme utilizes the designed smooth feedback control policy as a scalar stabilizing condition and proceeds to forcefully impose this desirable relation by means of a discontinuous feedback policy of the PWM type. The resulting dynamical controller is then reinterpreted in terms of two subsystems. One being the smooth, static, portion of the controller represented by the originally designed stabilizing feedback control law, and the second one being a parallel

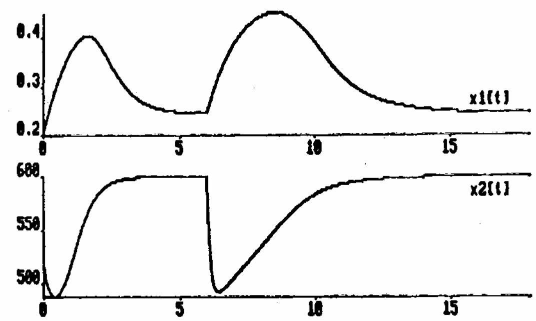

Figure 7. State trajectory responses of dynamically PWM controlled system subject to a sudden failure in the smooth portion of the feedback controller.

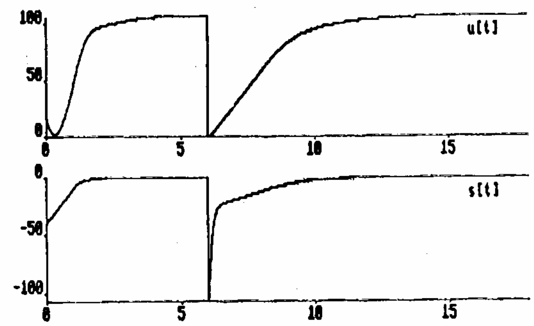

Figure 8. Trajectories of the control input signal u and of the corresponding stabilizing surface coordinate function $s(x, u)$ when PWM controlled system is subject to a sudden feedback loop failure.

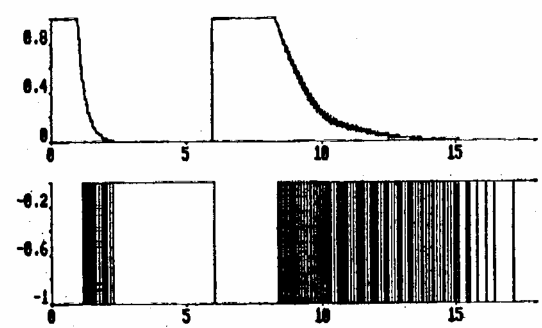

Figure 9. Behavior of the duty ratio function and of the associated PWM signal when the system is subject to a sudden feddback loop failure.

regulator based on dynamically generated (i.e smoothed) discontinuous PWM control actions. The scheme was shown to be advantageous in several respects, among which, we found local robustness with respect to sudden failures in the static portion of the proposed feedback controller. An application example, drawn from a non-traditional application area for PWM control, was also presented. The basic features of the proposed redundant dynamical discontinuous feedback control scheme were illustrated by means of simulations.

Dynamical PWM control of nonlinear systems has been extended, in a unifying manner, to sampled sliding modes and to pulse-frequency-modulation based schemes (see Sira- Ramírez, 1992c). Such unified treatment involves a systematic use of generalized canonical forms of nonlinear systems as proposed by Fliess (1990). The redundant feedback controllers, here described, can also be extended, in a rather similar manner, to the above mentioned classes of discontinuous feedback control policies. Extension to multi-input systems should have little or no difficulties, provided decoupled PWM controlled motions of the form (5) are imposed on the several input-dependent stabilizing manifolds representing every component of the designed multivariable smooth feedback controller.

5 Acknowledgement

This work was supported by the Consejo de Desarrollo Cientifico, Humanistico y Technologico of the Universidad de Los Andes, under Research Grant I-358-91.

6 References

[1] M. Fliess, 1990, Generalized Controller Canonical Forms for Linear and Nonlinear Dynamics, *IEEE Transactions on Automatic Control*, AC-35, 994-1001.

[2] M. Fliess and F. Messager, 1990, Vers une stabilisation non lineaire discontinue, *Analysis and Optimization of Systems*, A. Bensoussan and J. L. Lions (eds.), Lecture Notes on Control and Information Sciences, Vol. 144, 778-787. Springer-Verlag: New York; 1991, "Sur la commande en régime glissant," *C. R. Academie Sciences Paris*, 313, Serie I, 951-956.

[3] F. Messager, 1992, Sur la stabilisation discontinue des systemes, Thése Doctoral. Université de Paris-Sud, Centre D'Orsay.

[4] J. Parrish and C. Bosilow, 1986, Nonlinear Inferential Control of Reactor Effluent Concentration from Temperature and Flow Measurements. *Proc. of the American Control Conference*, Vol. 2, 1027-1033, Seattle, WA, June 18 - 20.

[5] H. Sira-Ramírez, 1989a, A Geometric Approach to Pulse-Width-Modulated Control in Nonlinear Dynamical Systems, *IEEE Transactions on Automatic Control* AC-34, 184-187;

[6] H. Sira-Ramírez, 1989b, Invariance Conditions in Nonlinear PWM Controlled Systems, *International Journal of Systems Science*, 20, 1679-1690;

[7] H. Sira-Ramírez, 1991, Nonlinear dynamically feedback controlled descent on a non atomosphere-free planet: A differential algebraic approach, *Control-Theory and Advanced Technology*. 7, 301-320;

[8] H. Sira-Ramírez, 1992a, Asymptotic output stabilization for nonlinear systems via dynamical variable structure control, *Dynamics and Control*. 2, 45-58;

[9] H. Sira-Ramírez, 1992b, Dynamical pulse-width-modulation control of nonlinear systems, *Systems and Control Letters* 18, 223-231;

[10] H. Sira-Ramírez, 1992c, Dynamical Discontinuous Feedback Control of Nonlinear Systems, *Proc. IFAC Nonlinear Control Systems Conference* (NOLCOS-92) pp. 471-476, Bordeaux (France) June 24-26;

[11] H. Sira-Ramírez, 1992d, Dynamical Sliding Mode Control Strategies in the Regulation of Nonlinear Chemical Processes, *International J. of Control*. 56, 1- 21;

[12] H. Sira-Ramírez, 1993a, Dynamical variable structure control strategies in asymptotic output tracking problems, *IEEE Transactions on Automatic Control* (to appear).;

[13] H. Sira-Ramírez, 1993b, Pulse-Frequency-Modulation Control of Nonlinear Systems, *Control-Theory and Advanced Technology* (submitted for publication).

Appendix

In this appendix we give a proof, based on Lyapunov stability arguments, of the fact that the scalar dynamical PWM controlled system (5) - (7) is asymptotically stable to zero if and only if the condition (8) is valid. A different proof of this result appears in Sira-Ramirez (1992b)

Proposition

The dynamical PWM controlled scalar system:

$$\dot{s} = -W \ \mathrm{PWM}_\tau[s]; \quad W > 0 \tag{21}$$

where:

$$\mathrm{PWM}_\tau[s] = \begin{cases} \mathrm{sign}[s] \ ; \ \text{for} \quad t_k \le t < t_k + \tau[s(t_k)]T \\ 0 \ ; \ \text{for} \quad t_k + \tau[s(t_k)]T \le t < t_k + T \end{cases}$$

$$k = 0, 1, 2, \ldots \quad ; \quad t_k + T = t_{k+1} \tag{22}$$

with

$$\tau[s] = \begin{cases} 1 & \text{for} \quad |s| \ \ge \ 1/\beta \\ \beta|s| & \text{for} \quad |s| \ < \ 1/\beta \end{cases} \tag{23}$$

is asymptotically stable to zero if, and only if, the condition: $\beta WT < 2$, is satisfied.

Proof

Suppose the initial state of (21) is chosen such that the condition: $|s| \ge 1/\beta$ is valid. Consider then the following Lyapunov function candidate for the system (21), (22):

$$V(s) = s^2 \tag{24}$$

(notice that, in this case $\tau[s(t_k)] = 1$, according to (23)).

The time derivative of the function $V(s)$ results then in:

$$\dot{V}(s) = -W|s| < 0 \tag{25}$$

i.e., the scalar controlled function s decreases, in absolute value, independently of the numerical values of W, T, β, and of the initial value for s, as long as the condition: $|s| \geq 1/\beta$ remains valid. Since the right hand side of the controlled system in decided precisely at the sampling times t_k, the value of s may enter a limit cycle type of behaviour if, and only if, the step by which $s(t)$ decreases (if it is positive), or increases (if it is negative) during the sampling interval, is such that $s(t_k) = -s(t_{k+1})$, with values of a satisfying: $|s(t_k)| > 1/\beta$. The condition $|s| \geq 1/\beta$ may then be violated in finite time, independently of the inital value of s, if and only if such a limit cycle behaviour does not exist. It is easy to see, by straightforward exact discretization of (21), that, for $|s(t_k)| > 1/\beta$, one has:

$$s(t_{k+1}) = s(t_k) - WT\text{sign}[s(t_k)]$$

Thus:

$$|s(t_{k+1}) - s(t_k)| = WT$$

i.e., the limit cycle condition prevails if and only if:

$$\begin{aligned} |s(t_{k+1}) - s(t_k)| = & \quad |2s(t_{k+1})| = |2s(t_k)| = \\ = & \quad 2|s(t_k)| = WT > 2/\beta \end{aligned}$$

Therefore, for any given initial value of s, in the region $|s| \geq 1/\beta$, a limit cycle behaviour of the sampled values of s does not exist, and hence the condition: $|s(t_k)| > 1/\beta$ may always be violated in finite time, if, and only if, the following condition is valid:

$$\beta WT < 2$$

Suppose now that $|s| < 1/\beta$. The time derivative of the Lyapunov function candidate (24) now results in:

$$\dot{V}(s) = \begin{cases} -W|s| \ ; \text{ for } \quad t_k \leq t < \\ \qquad t_k + \tau[s(t_k)]T \\ \\ 0 \ ; \text{ for } \quad t_k + \tau[s(t_k)]T \leq \\ \qquad t < t_k + T \end{cases}$$

$$t_k + T = t_{k+1} \tag{26}$$

and, therefore, during the intersampling interval, the scalar value of the state s, of the controlled system, decreases (if it is initially positive) or increases (if it is initially negative) during an interval of time equals to: $\tau[s(t_k)]T$, and then proceeds to remain constant, at the attained value, during the rest of the intersampling interval. The corresponding discrete time evolution of the state, in the region: $|s| < 1/\beta$, is obtained from (21), (22) by means of exact discretization as:

$$\begin{aligned} s(t_{k+1}) &= s(t_k) - W\beta|s(t_k)|T\text{sign}[s(t_k)] = \\ &= s(t_k) - \beta WTs(t_k) = (1 - \beta WT)s(t_k) \end{aligned}$$

The trajectories of the sampled system are asymptotically stable to zero, if, and only if, the eigenvalues of the characteristic polynomial of the above linear discrete-time system are, in absolute value, strictly smaller than 1. Hence, once the system state is found in the region $|s| < 1/\beta$, the controlled trajectory of s converges to zero if, and only if:

$$|1 - \beta WT| < 1$$

which, due to the positivity of β, W and T, is equivalent to the announced result.

Hebertt Sira-Ramírez obtained the Degree of Ingeniero Electricista from the Universidad de Los Andes in Merida - Venezuela in 1970. He pursued graduate studies from 1972 to 1977 at the Massachusetts Institute of Technology, Cambridge, where he obtained the M.S. and Ph.D. degrees in E.E in 1974 and 1977, respectively. He is currently a Full Professor in the Control Systems Department of the Systems Engineering School of the Universidad de Los Andes, where he has also held the positions of Head of the Control Systems Department from 1978 to 1980 and was elected Vice-president of the University for the term 1980-1984. He was appointed Head of the Graduate School in Control Engineering in 1992. Dr. Sira-Ramírez is a Senior Member of the IEEE, where he serves as a member of the IEEE International Committee. He is also a member of the IFAC, SIAM, Sigma Xi, The Planetary Society and of the Venezuelan College of Engineers.

Dr. Sira-Ramírez received the IEEE Distinguished Lecturer Award in 1993 and the Venezuelan College of Engineers Award for Scientific Research in 1987 as well as the Senior Researcher Scholarship Award from the Venezuelan National Council for Scientific and Technological Research in 1990. Dr. Sira-Ramírez is interested in the theory and applications of discontinuous feedback control strategies for nonlinear dynamic systems.

Orestes Llanes-Santiago received the degree of Ingeniero Electricista from the Instituto Superior Politecnico "Jose A. Echeverria" in 1981, in La Habana, Cuba. he received the M.Sc. degree in Automatic Control Engineering in 1991 from the Universidad de Los Andes, Merida, Venezuela, where he is currently working towards the Ph.D. degree in Automatic Control Engineering. Mr. Llanes-Santiago is an Assistant Professor of the Electric Faculty of the Instituto Superior Politecnico "Jose A. Echeverria". His research interests are concerned with the theory and applications of nonlinear dynamical systems. He is a student member of the Institute of Electrical and Electronic Engineers (IEEE).