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On the dynamical sliding mode control of nonlinear systems

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The consequences of the differential algebraic approach in the sliding mode control of nonlinear single-input single-output systems are reviewed in tutorial fashion. Input-dependent sliding surfaces, possibly including time derivatives of the input signal, are shown to arise naturally from elementary differential algebraic results pertaining to the Fliess's Generalized Controller Canonical Forms of nonlinear systems. This class of switching surfaces generally leads to chattering-free dynamically synthesized sliding regimes, in which the highest time derivative of the input signal undergoes all the bang–bang type discontinuities. Examples illustrating the obtained results are also included.

1. Introduction

Sliding mode control of dynamical systems has a long history of theoretical and practical developments. A rather complete chronological collection of journal articles and conference presentations has been gathered by Professor S. V. Emelyanov (1989, 1990 a), who is one of the founding fathers of the technique. Extensive surveys, with an enormous wealth of information, have been presented over the years by Utkin (1977, 1984, 1989). Several books have also been published on the subject: Emelyanov (1967), Itkis (1976), Bühler (1986), Utkin (1978, 1992). Contributed volumes by Zinober (1990) and Young (1993) reveal sliding mode control as an active discipline of research with enough theoretical maturity. A survey of the numerous industrial and laboratory applications of sliding regimes around the world is well beyond the scope of this article. In the following paragraphs we provide a necessarily incomplete overview of some of the contributions in sliding mode control for nonlinear dynamical systems. Many interesting developments in controller robustness, adaptive regulation, and observer design are not mentioned.

In recent years, the outstanding developments for nonlinear control systems based on differential geometric ideas (see the books by Isidori 1989, Nijmeijer and Van der Schaft 1990) have found immediate applications, and extensions, to sliding mode control, and closely related areas, such as high-gain, pulse-width-modulation and pulse-frequency modulation. Seminal work on sliding regimes for nonlinear systems is due to Luk'yanov and Utkin (1981). Starting with the contributions by Slotine and Sastry (1983) devoted to the field of robotics automation, the differential geometric approach to nonlinear systems control was exploited and put in perspective, within the sliding mode control area, by the independent work of several authors. An important contribution relating sliding mode systems to high-gain feedback controlled systems from a geometric standpoint was given by Marino (1985), Bartolini and Zolezzi (1986) presented interesting developments of sliding mode control as applied to robust linearization of nonlinear plants. A full case study of sliding mode control design for a

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nonlinear system was presented by Mathews *et al.* (1986). The sliding mode control of nonlinear multivariable systems was addressed by Fernandez and Hedrick (1987). A quite readable tutorial dealing with multivariable nonlinear systems was written by DeCarlo *et al.* (1988). Later, in a series of articles, Sira-Ramirez (1987, 1989 a, 1989 b, 1990) contributed some formalizations, application examples and generalizations, of sliding regimes in nonlinear systems. More recently, a rather complete picture of the nonlinear multivariable case has been provided by Kwatny and Kim (1990).

Recent developments in nonlinear systems include the use of differential algebra for the formulation, understanding, and conceptual solutions of long standing problems in automatic control. Developments in this area are fundamentally due to Professor Fliess (1986, 1988, 1989 a, 1989 b, 1989 c, 1990 a, 1990 b). Some other pioneering contributions were also independently presented by Pommaret (1983, 1986). Sliding mode control, and discontinuous feedback control, in general, have also benefited from this new trend. A seminal contribution in the use of differential algebraic results to sliding mode control was given by Fliess and Messager (1990). These results were extended and used in several case studies by Sira-Ramirez *et al.* (1992), Sira-Ramirez and Lischinsky-Arenas (1991) and by Sira-Ramirez (1992 a–1992 d). A most interesting article extending some of the ideas to multivariable linear systems and to the regulation of non-minimum phase linear systems is that of Fliess and Messager (1992). Extensions to pulse-width modulation control strategies from this viewpoint were also contributed by Sira-Ramirez (1991 a, 1992 e).

This article is an attempt to present, in tutorial fashion, some of the developments in sliding mode control theory that are a direct consequence of elementary results in the application of the differential algebraic viewpoint to control systems theory. It should be pointed out that some of the results obtained for sliding mode control via the use of differential algebra are closely related to previous ideas presented by Emelyanov (1987, 1990 b), from a quite different viewpoint, in his 'binary systems' formulation of control problems. Also, in a contribution by Bartolini and Pydynowsky (1991) smoothing of the input signals is achieved through continuous first-order estimators. It must be pointed out that in a paper by Kostyleva (1964) input dependent sliding surfaces were proposed for the sliding mode control of linear systems. Again, in these works, the basic developments are not drawn from differential algebra.

Section 2 of this article is devoted to presenting some simple examples which utilize sliding surfaces which not only depend on the state of the system but also on the system's inputs, thus resulting in dynamic sliding mode controllers. These examples try to motivate the need for a more general class of sliding surfaces which directly leads to such dynamical sliding modes and some of its advantageous properties. Section 3 presents some fundamental results of sliding mode control theory stemming from the differential algebraic approach to system dynamics.

2. Some motivating examples

In this section we provide simple, yet motivating examples which not only justify the differential algebraic approach in sliding mode controller design but they also point to the need and advantages associated to more general classes of sliding surfaces. In particular, we are interested in those sliding surfaces which include expressions in the input signal and (possibly) some of its time derivatives. The class of input-dependent sliding surfaces motivate our additional developments.

2.1. *An example of smoothing the banging of the input signal in discontinuous feedback control*

Let us begin by a simple example in which the smoothing properties of dynamical sliding regimes, arising from input-dependent switching surfaces, are clearly portrayed.

Consider the scalar system

$$\left.\begin{aligned} \dot{x} &= u \\ y &= x - X \end{aligned}\right\} \tag{2.1}$$

where y represents the scalar state error with respect to a pre-assigned constant reference value X. The variable u is the scalar input signal, constrained to take values in the discrete set $\{-U, U\}$, where $U > 0$.

It is well known that the following discontinuous feedback policy, given by:

$$u = -U\,\mathrm{sign}(y) \tag{2.2}$$

results in a sliding regime on the line $y = 0$. This is easily seen from the fact that the product $\sigma \mathrm{d}\sigma/\mathrm{d}t := y\,\mathrm{d}y/\mathrm{d}t = -U|y| \leqslant 0$. The required sliding surface is then represented as:

$$S = \{x\colon \sigma = x - X = 0\} \tag{2.3}$$

The ideal sliding dynamics are obtained from the condition $\mathrm{d}\sigma/\mathrm{d}t = 0$, i.e. $\mathrm{d}x/\mathrm{d}t = 0$, and $\sigma = 0$, i.e. $x = X$.

A simulation of the controlled system is shown in Fig. 1, with $X = 1$ and

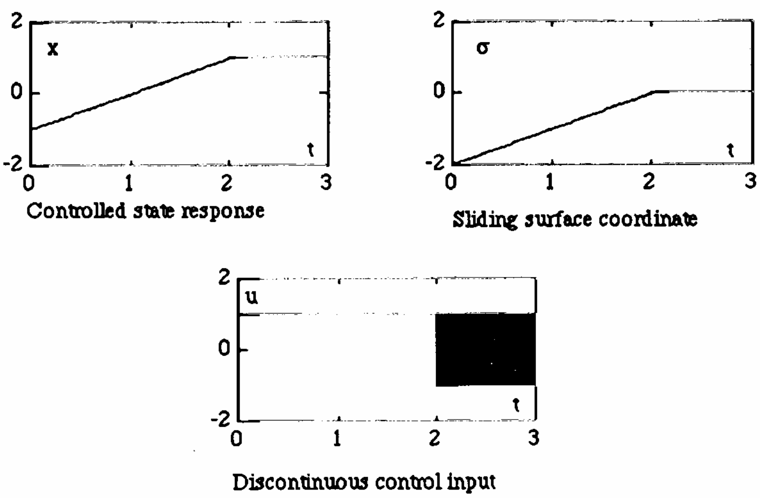

Figure 1. Simulation of (statically) sliding mode controlled responses of single integrator plant.

$U = 1$. The controlled state response, the sliding surface coordinate response and the discontinuous (bang-bang) features of the resulting input signal u are separately portrayed in such a figure.

The effects of the above discontinuous feedback policy are summarized in two important features: (1) The condition $\sigma = y = 0$ is reached in finite time (given by $T = U^{-1}|x(0)|$, 2). After reaching the desired condition, the same is indefinitely guaranteed to hold. It may be easily proved that this condition can be sustained, in spite of the presence of bounded perturbations affecting the system behaviour through the input channel u.

Suppose we would like to trade the finite-time reachability of the zero state error condition by a smoother behaviour of the input variable u while still, possibly, being constrained to utilize auxiliary input signals (here denoted by v) taking values in the set $\{-U, U\}$. In order to achieve this purpose, let us propose the following asymptotically stable closed loop behaviour of the controlled scalar state:

$$\dot{x} = u = -\lambda(x - X) \tag{2.4}$$

If we now take as the sliding surface one representing a suitable input-dependent switching condition depicting the feedback input signal error:

$$S = \{(x, u): \sigma = u + \lambda(x - X) = 0\} \tag{2.5}$$

one ideally obtains the required closed loop behaviour whenever $\sigma \equiv 0$. A sliding regime guaranteeing such a condition can be established by requiring now that $\sigma \mathrm{d}\sigma/\mathrm{d}t \leq 0$. This may be accomplished by imposing on σ the discontinuous dynamics specified by $\mathrm{d}\sigma/\mathrm{d}t = -W \operatorname{sign}(\sigma)$, where $W > 0$ is an arbitrary positive real number. Using the new expression for σ, one obtains:

$$\dot{u} + \lambda u = -W \operatorname{sign}[u + \lambda(x - X)] \tag{2.6}$$

which is a differential equation with discontinuous right-hand side, whose solution represents the required control input variable. It is easy to see from the above equation (2.6) that the control input signal u is actually the outcome of a first-order low-pass filter with cut-off frequency represented by λ. Indeed, using the, by now popular, hybrid notation that merges frequency domain quantities with others in the time domain, one easily obtains:

$$u = -\frac{\lambda}{s + \lambda}\left[\left(\frac{W}{\lambda}\right) \operatorname{sign}[u + \lambda(x - X)]\right] = -\frac{\lambda}{s + \lambda} v \tag{2.7}$$

Thus, the input u may be synthesized as the output of a low-pass filter which accepts as an input a discontinuous (bang-bang) signal v, of amplitude W/λ. By virtue of the amplitude restriction on the ultimate (auxiliary) input signal v, mentioned above, this ratio is taken as:

$$\frac{W}{\lambda} = U \Leftrightarrow W = \lambda U \tag{2.8}$$

The diagram in Fig. 2 depicts the structure of the dynamical discontinuous feedback controller explicitly exhibiting the imbedded low-pass filter characteristics which are excited by a bang-bang input signal of amplitude U, as initially required. For a given fixed value of U, relation (2.8) establishes a trade-off between the exponential rate of approach of the controlled state x to its desired

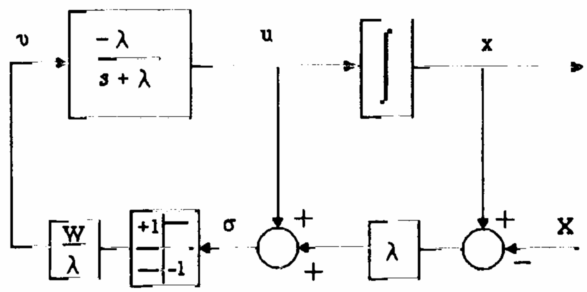

Figure 2. Filtering effect of dynamical sliding mode control of a single integrator plant.

value (alternatively, the cut-off frequency of the low pass filter, or filter bandwidth) and the design value of the amplitude W, which indirectly measures the reaching time of the condition $\sigma = 0$, through $T = W^{-1}|\sigma(0)|$. The faster it is desirable to reach $\sigma = 0$, the faster x will approach X, but then, the larger the cut-off frequency of the low pass filter, a larger number of harmonic components of the bang-bang signal v, and external noise, directly affect the input to the system.

A simulation of the dynamically discontinuously controlled system (2.1), (2.6) is shown in Fig. 3 with $X = 1$, $W = 1$ and $\lambda = 1$. The resulting input signal u is shown to be substantially smoothed out with respect to its previous behaviour when the static discontinuous controller was used. Further smoothing of the controlled scalar state x can be equally inferred from such a figure.

2.2. *Zeroing of input-dependent output signals via dynamical discontinuous control*

A rather general model for nonlinear single-input single-output nonlinear systems is constituted by the following n-dimensional single-input single-output

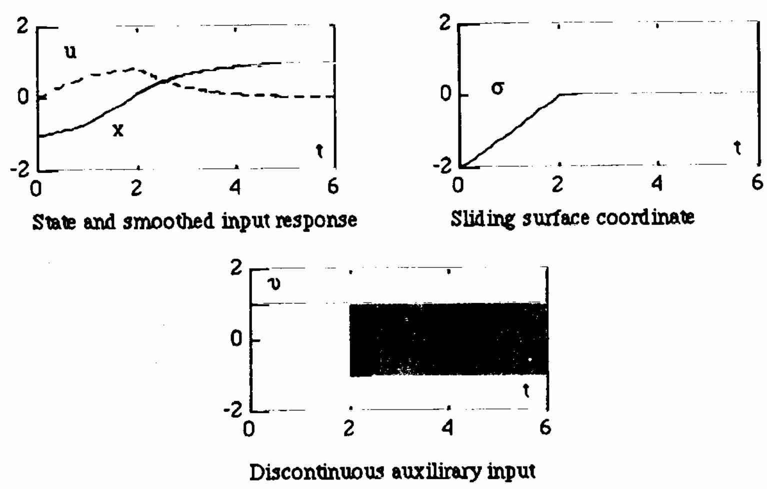

Figure 3. Simulation of dynamically sliding-mode controlled responses of single integrator plant.

analytic system, in Kalman form:

$$\left.\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned}\right\} \tag{2.9}$$

in which the output y is allowed to depend explicitly on the input variable u (such systems may be properly addressed as systems with *relative degree zero*). Suppose it is desired to 'zero-out' the scalar output variable y, possibly in finite time, through a discontinuous feedback control policy. This control task is possible by imposing, again, the following autonomous dynamics on the scalar output signal:

$$\dot{y} = -W \operatorname{sign}(y) \tag{2.10}$$

and computing the required control signal u. Using (2.9) and (2.10) one obtains:

$$\left[\frac{\partial h}{\partial x}\right] f(x, u) + \left[\frac{\partial h}{\partial u}\right] \frac{\mathrm{d}u}{\mathrm{d}t} = -W \operatorname{sign}[h(x, u)] \tag{2.11}$$

which may be locally rewritten as a first-order, time-varying, ordinary differential equation with discontinuous right-hand side:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\left[\frac{\partial h}{\partial u}\right]^{-1} \left\{\left[\frac{\partial h}{\partial x}\right] f(x, u) + W \operatorname{sign}[h(x, u)]\right\} \tag{2.12}$$

A block diagram depicting the dynamical discontinuous feedback control scheme summarized in (2.12) is shown in Fig. 4.

The ideal sliding mode behaviour obtained on the input-dependent manifold $y = h(x, u) = 0$ is obtained as follows. Let the feedback law $u = \varphi(x)$ be the (unique) control law satisfying $h(x, \varphi(x)) \equiv 0$. Then $\varphi(x)$ also plays the role of the equivalent control and it is, evidently, a particular solution of (2.12), for suitable initial conditions. Indeed, the solutions of (2.12) locally yield $\mathrm{d}y/\mathrm{d}t = 0$, i.e. they yield constant output responses under ideal sliding mode conditions. If the initial value of the output is zero, then the dynamical controller locally induces the condition $y = 0$ on an open interval of time. This means, by virtue of the assumed uniqueness, that the actual (dynamically generated) applied control input u takes precisely the same values as $\varphi(x)$.

The ideally controlled dynamics are then obtained as

$$\left.\begin{aligned} \dot{x} &= f(x, \varphi(x)) \\ y &= 0 \end{aligned}\right\} \tag{2.13}$$

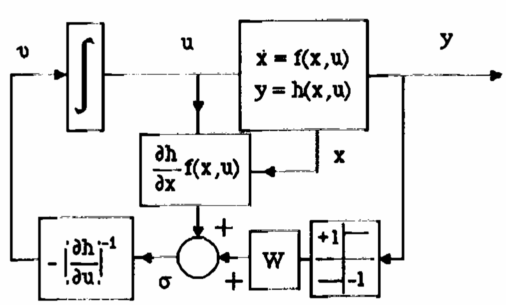

Figure 4. Dynamical sliding mode control scheme for zeroing of input-dependent outputs.

which, for obvious reasons, is assumed to be locally asymptotically stable to a desired equilibrium state.

Remark: An interesting feature of the above class of problems lies in the possibilities of robustly imposing ideally designed feedback control solutions to nonlinear plants. For instance, let $u = -k(x)$ be a desirable scalar feedback control law for the plant $dx/dt = f(x, u)$. Then, adopting as an output function the expression: $y = h(x, u) = u + k(x)$, the dynamical controller obtained from (2.12) imposes, in finite time, the required feedback control law on the given system. □

2.3. *A simple application example in rest-to-rest reorientation manoeuvres for single axis spacecraft*

Consider the nonlinear second-order plant representing the kinematic and dynamic equations of single-axis jet-controlled spacecraft with the attitude variable measured with respect to a skewed axis and specified in terms of the Cayley-Rodrigues parametrization (see Dwyer and Sira-Ramirez 1988):

$$\left.\begin{aligned} \dot{x} &= 0{\cdot}5(1 + x^2)\omega \\ \dot{\omega} &= \frac{1}{J} u \\ y &= x - X \end{aligned}\right\} \qquad (2.14)$$

where x represents the Cayley–Rodrigues orientation parameter, ω is the main axis angular velocity and u is the externally applied input torque. J is the moment of inertia of the spacecraft around its principal axes.

It is easy to show that the following nonlinear feedback control law, arising from extended linearization considerations, asymptotically stabilizes the system toward the desired reference attitude value $x = X$, with zero final angular velocity ω (see also Sira-Ramirez and Lischinsky-Arenas 1990):

$$u = -2J\{\zeta\omega_n\omega + \omega_n^2[\tan^{-1}(x) - \tan^{-1}(X)]\} \qquad (2.15)$$

where ω_n and ζ are positive design parameters with: $0 < \zeta < 1$.

Figure 5 depicts the simulated responses of the state variables x and ω as well as the required control input signal u, as computed from (2.15) with $\zeta = 0{\cdot}707$, $\omega_n = 2$ [rad s^{-1}], $X = 1{\cdot}5$ [rad].

One may, alternatively, take, as remarked above, an auxiliary output function y, for system (2.14), which is constituted by the control input error with respect to the required stabilizing feedback function, i.e.

$$y = u + 2J\{\zeta\omega_n\omega + \omega_n^2[\tan^{-1}(x) - \tan^{-1}(X)]\}$$

The discontinuous dynamical feedback controller induces a sliding regime on the input-dependent sliding surface S with coordinate function σ given, evidently, by:

$$S = \{(x, \omega, u): \sigma = u + 2J[\zeta\omega_n\omega + \omega_n^2(\tan^{-1}(x) - \tan^{-1}(X))] = 0\} \qquad (2.16)$$

A dynamical sliding mode controller, which robustly enforces the feedback control law (2.15) by zeroing the above input-dependent (auxiliary) output function y, is given, according to the previously stated results, by:

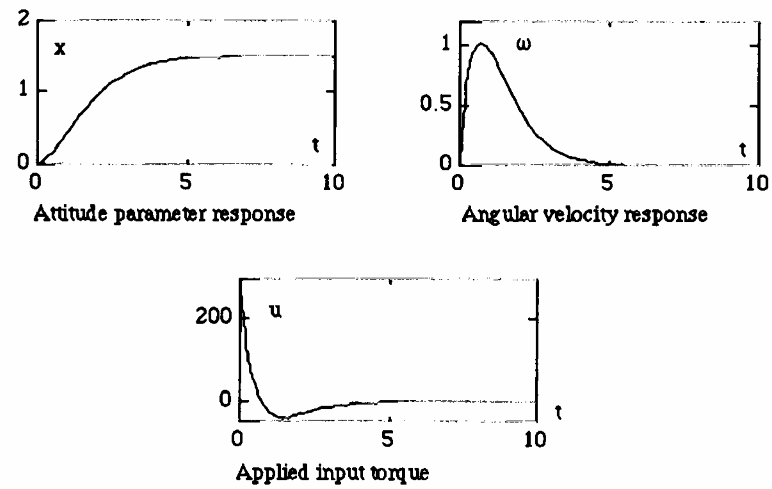

Figure 5. State variables responses and applied input torque for continuous feedback controlled single-axis spacecraft.

$$\dot{u} = -(2\zeta\omega_n u + J\omega_n^2\omega) - W\,\mathrm{sign}\,(y) \qquad (2.17)$$

Simulations were carried out for the dynamical sliding mode controlled system (2.14), (2.17) with a sliding surface given by (2.16). The spacecraft moment of inertia was taken as $J = 70\,\mathrm{N\,m\,s^{-2}}$. The desired attitude $X = 1{\cdot}5$ rad, and the controller design parameters were taken as: $\zeta = 0{\cdot}707$, $\omega_n = 2$, and $W = 40$.

Figure 6 depicts the dynamically sliding-mode controlled state variables responses for x and ω, the sliding surface coordinate σ and the smoothed externally applied input torque u, expressed in newton-metres.

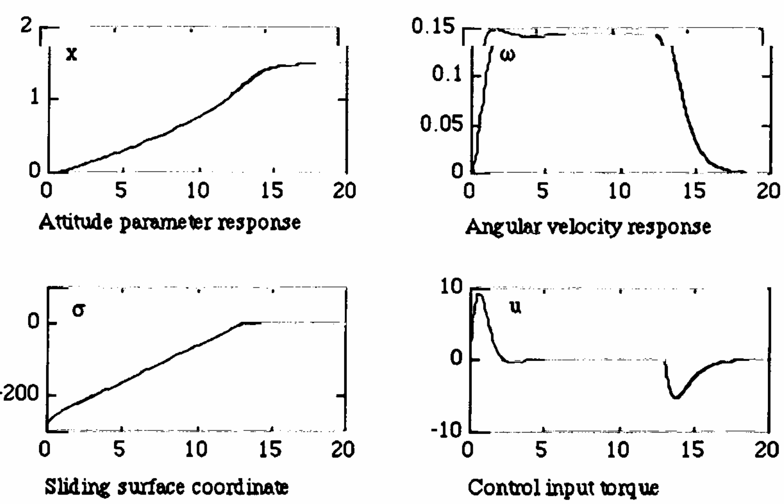

Figure 6. State variables responses, sliding surface coordinate evolution and applied input torque for dynamically sliding-mode controlled spacecraft.

In spite of the slower response of the dynamical sliding-mode controlled system, the applied input torque is considerably smaller than the one obtained with the continuous feedback control strategy represented by (2.15). This fact has a definite bearing on the stability and performance features of the closed loop system when amplitude control input torque restrictions are enforced. If, for instance, one limits the amplitude of the applied input torque to a reasonable value of, say, 2·5 N m, the (saturated) continuous feedback controller (2.15) leads to a stable, but quite degraded, response for the attitude parameter x, with exceedingly large overshoot. The dynamical sliding mode controller, on the other hand, still yields a perfectly asymptotically stable response with reasonably small overshoot. This is depicted in Fig. 7.

Remark: In all of the above examples the use of input-dependent discontinuity surfaces clearly results in dynamical sliding-mode controllers which exhibit a smoothing of the bang-bang control action traditionally associated with stabilizing schemes based on sliding regimes. These simple examples point to the need for a more general approach to discontinuous feedback control which naturally considers input-dependent sliding surfaces. This may be accomplished in two different ways. One is to resort to appropriate systems extensions (see Nijmeijer and Van der Schaft 1990) and use the conventional (input-free) design approach or to resort to the recently developed differential algebraic approach. Due to the theoretical richness of the developments found in the last approach, we shall choose this method in the hope of making it clear that from this new viewpoint discontinuous feedback control of nonlinear systems has a lot to gain from, both, the conceptual and the practical viewpoints. □

3. Some consequences of the differential algebraic approach in sliding mode control of nonlinear systems

In the previous section, some of the advantages of using input-dependent sliding surfaces were explored through quite simple illustrative examples. These examples point, essentially, to new possibilities of sliding-mode control when input-dependent switching surfaces are used. Such possibilities could also have been arrived at, by using the concept of the extended system (Nijmeijer and Van der Schaft 1990) in combination with traditional static sliding mode controller

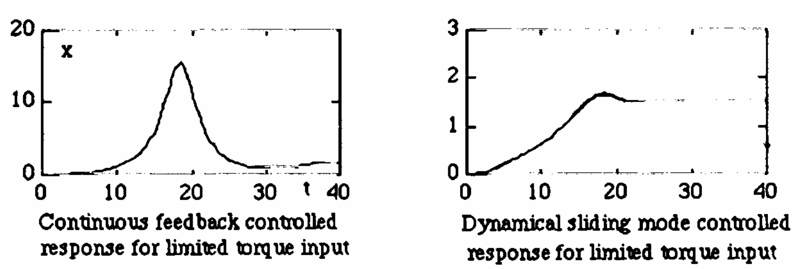

Figure 7. Continuous and dynamical sliding mode feedback controlled responses of attitude parameter subject to saturation of control input torque (torque saturation limits: $|u| \leq 2{\cdot}5$ N m).

design. However, input-dependent sliding surfaces may be seen as natural switching surfaces for nonlinear systems. This fact is a direct consequence of the differential algebraic approach, proposed by Fliess (1986, 1988, 1989 a, 1989 b, 1989 c, 1990 a, 1990 b), for the study of control systems. In this section we present some simple results of such a differential algebraic approach related to sliding mode control. The required background may be found in Fliess's numerous articles and outstanding contributions. However, we will try to be as self-contained as possible. The following developments closely follow those found in Fliess (1990 a).

3.1. *Fliess's generalized controller canonical forms*

One of the consequences of the many results drawn by Fliess (1990 a), is that a more general and natural representation of a nonlinear system requires implicit algebraic differential equations. Indeed, it may be easily shown, using elementary facts of *finitely generated differential algebraic extensions of fields* (see Fliess 1990 a) that a controlled dynamical system may always be implicitly defined by means of n polynomial differential equations of the form:

$$P_i(\dot{x}_i, x, u, \dot{u}, \ldots, u^{(\alpha)}) = 0; \quad i = 1, \ldots, n \tag{3.1}$$

It has been shown by Fliess and Hassler (1990) that such implicit representations are not entirely unusual in physical examples. The more traditional form of the state equations, known as *normal form* is recovered, in a local fashion, under the assumption that such polynomials locally satisfy the following rank condition:

$$\text{rank}\begin{bmatrix} \dfrac{\partial P_1}{\partial \dot{x}_1} & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \dfrac{\partial P_n}{\partial \dot{x}_n} \end{bmatrix} = n$$

The time derivatives of the x_i may then be locally solved for

$$\dot{x}_i = p_i(x, u, \dot{u}, \ldots, u^{(\alpha)}) = 0; \quad i = 1, \ldots, n \tag{3.2}$$

It should be pointed out that even if (3.1) is in polynomial form, it may happen, in general, that (3.2) is not. Representation (3.2) is now known as the *generalized state representation* of the nonlinear dynamics.

The following statement constitutes a direct application of the *theorem of the differential primitive element* (Kolchin 1973). This theorem plays a fundamental role in the study of systems dynamics from the differential algebraic approach (see Fliess 1990 a).

A system of the form (3.2) can always be canonically represented in terms of a *Global Generalized Controller Canonical Form* (GGCCF):

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} q_1 &= q_2 \\ \frac{\mathrm{d}}{\mathrm{d}t} q_2 &= q_3 \\ &\vdots \\ \frac{\mathrm{d}}{\mathrm{d}t} q_{n-1} &= q_n \\ C(\dot{q}_n, q, u, \dot{u}, \ldots, u^{(\alpha)}) &= 0 \end{aligned}\right\} \tag{3.3}$$

where C is a polynomial function of its arguments. If one can locally solve for the time derivative of q_n in the last equation, one locally obtains an explicit system of first-order differential equations, known as the *Local Generalized Controller Canonical Form* (LGCCF):

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} q_1 &= q_2 \\ \frac{\mathrm{d}}{\mathrm{d}t} q_2 &= q_3 \\ &\vdots \\ \frac{\mathrm{d}}{\mathrm{d}t} q_{n-1} &= q_n \\ \frac{\mathrm{d}}{\mathrm{d}t} q_n &= c(q, u, \dot{u}, \ddot{u}, \ldots, u^{(\alpha)}) \end{aligned}\right\} \tag{3.4}$$

The generalized phase variables $q = (q_1, \ldots, q_n)$ are obtained from the so-called *differential primitive element* ξ whose existence is always guaranteed. The quantities: ξ, $\mathrm{d}\xi/\mathrm{d}t, \ldots, \mathrm{d}^{(n-1)}\xi/\mathrm{d}t^{(n-1)}$ qualify as state variables since they do not satisfy any algebraic relation among themselves. One then simply lets: $q_1 = \xi$, $q_2 = \mathrm{d}\xi/\mathrm{d}t, \ldots, q_n = \mathrm{d}^{(n-1)}\xi/\mathrm{d}t^{(n-1)}$. It should be emphasized that (3.3) is obtainable from (3.2) by means of an invertible input-dependent state coordinate transformation of the form: $q = \Phi(x, u, \ldots, u^{(\alpha-1)})$.

Remark: We assume throughout that $\alpha \geqslant 1$. The case $\alpha = 0$ corresponds to that of exactly linearizable systems under state coordinate transformations and static-state feedback. One may still obtain the same smoothing effect of the dynamical sliding mode controllers we derive in this article by considering arbitrary prolongations of the input space. This is accomplished by succesively considering the 'extended system' (see Nijmeijer and Van der Schaft 1990) of the original one, and proceeding by using the same differential primitive element yielding the Generalized Controller Canonical Form of the original system. □

3.2. *Dynamical sliding regimes based on Fliess's GCCF*

The preceeding general results on canonical forms for nonlinear systems have an immediate consequence in the definition of sliding surfaces for stabilization and tracking problems in nonlinear systems.

Consider the following sliding surface coordinate function, expressed in the generalized phase coordinates q:

$$\sigma = c_1 q_1 + \cdots + c_{n-1} q_{n-1} + q_n \tag{3.5}$$

where the scalar coefficients c_i ($i = 1, \ldots, n-1$) are chosen in such a manner that the following polynomial, $p(s)$, in the complex variable s, is Hurwitz:

$$p(s) = c_1 + c_2 s + \cdots + c_{n-1} s^{n-2} + s^{n-1} \tag{3.6}$$

Imposing on the sliding surface coordinate function σ the discontinuous dynamics:

$$\dot{\sigma} = -W \operatorname{sign}(\sigma) \tag{3.7}$$

then, the trajectories of σ are seen to exhibit, in finite time T given by $T = W^{-1}|\sigma(0)|$, a sliding regime on $\sigma = 0$. Substituting on (3.7) the expression (3.5) for σ, and using (3.4), one obtains, after some straightforward algebraic manipulations, the following dynamical implicit sliding mode controller:

$$c(q, u, \dot{u}, \ldots, u^{(\alpha)}) = -c_1 q_2 - c_2 q_3 - \cdots - c_{n-1} q_n - W \operatorname{sign}[c_1 q_1 + \cdots + c_{n-1} q_{n-1} + q_n] \tag{3.8}$$

Evidently, under ideal sliding conditions $\sigma = 0$, the variable q_n no longer qualifies as a state variable for the system since it is expressible as a linear combination of the remaining states. The function $\operatorname{sign} \sigma$ is then ideally replaced by zero. The ideal (autonomous) closed loop dynamics may then be expressed in terms of a reduced state vector which only includes the remaining $n-1$ phase coordinates associated with the original differential primitive element. This leads to the following ideal sliding dynamics:

$$\left.\begin{aligned} \frac{d}{dt} q_1 &= q_2 \\ \frac{d}{dt} q_2 &= q_3 \\ &\vdots \\ \frac{d}{dt} q_{n-1} &= -c_1 q_1 - \cdots - c_{n-1} q_{n-1} \end{aligned}\right\} . \tag{3.9}$$

The characteristic polynomial of (3.9) is evidently given by (3.6) and, hence, the (reduced) autonomous closed loop dynamics are asymptotically stable to zero. Notice that by virtue of (3.5), the condition $\sigma = 0$, and the asymptotic stability of (3.9), that q_n also tends in an asymptotically stable fashion to zero.

The equivalent control, denoted by u_{EQ} is defined as a *virtual* feedback control action ideally achieving a smooth evolution of the system trajectories on the constraining sliding surface $\sigma = 0$, provided initial conditions are precisely set on such a switching surface. The equivalent control is formally obtained from the condition $d\sigma/dt = 0$. i.e.

$$c(q, u_{EQ}, \dot{u}_{EQ}, \ldots, u_{EQ}^{(\alpha)}) = c_1 c_{n-1} q_1 + (c_2 c_{n-1} - c_1) q_2 + \cdots + (c_{n-2} c_{n-1} - c_{n-3}) q_{n-2} + (c_{n-1} c_{n-1} - c_{n-2}) q_{n-1} \tag{3.10}$$

Since q asymptotically converges to zero, the solutions of the above time-varying implicit differential equation, describing the evolution of the equivalent control, asymptotically approach the solutions of the following autonomous implicit differential equation:

$$c(0, u, \dot{u}, \ldots, u^{(\alpha)}) = 0 \tag{3.11}$$

Equation (3.11) constitutes the zero-dynamics (See Fliess 1990 b) associated with the problem of zeroing the differential primitive element, considered now as an (auxiliary) output of the system. Notice that (3.10) may also be regarded as the zero-dynamics associated with zeroing of the sliding surface coordinate function σ. If (3.11) locally asymptotically approaches a constant equilibrium point $u = U$, then the system is said to be locally minimum phase around such an equilibrium point, otherwise the system is said to be non-minimum phase. The equivalent control is, thus, locally asymptotically stable to U, whenever the underlying input–output system is minimum phase.

One may be tempted to postulate, for the sake of physical realizability of the sliding mode controller, that a sliding surface σ is properly defined whenever the associated zero-dynamics are constituted by an asymptotically stable motion towards equilibrium. In other words, that the input-sliding surface system is minimum phase. It should be pointed out, however, that non-minimum phase systems might make perfect physical sense and that, in some instances, instability of a certain state variable, or input, does not necessarily means disastrous effects on the controlled system. The following example illustrates this fact.

Example 3.1. Control of a non-minimum phase system: Consider the problem of manoeuvring a motor-driven unicycle which advances with constant (ground) speed V on a plane equipped with cartesian coordinates, given by the ordered pairs (x, y), describing the position of the contact point. The control input is represented by the heading angle u, measured with respect to the x axis. The objective is to manoeuvre the unicycle to follow a circle of radius R, drawn on the plane, and centred at the origin O of coordinates (see Figure 8). For simplicity, we assume that u takes values in the interval $(-\infty, \pi/2)$ and, hence, only counter-clockwise solutions are considered.

It is easy to see that the motions may be described by the following set of

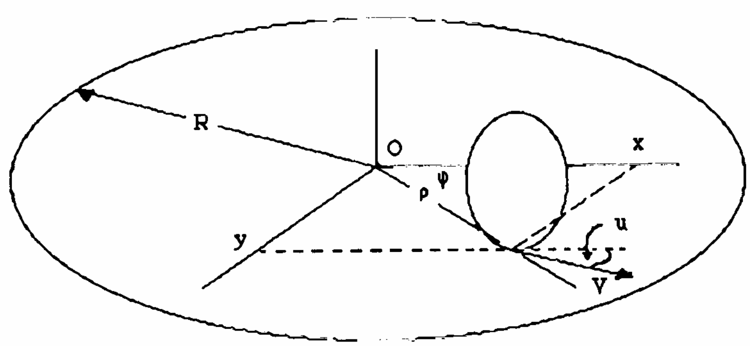

Figure 8. Geometry of the unicycle control problem of following a prescribed circular trajectory with constant velocity.

analytic differential equations:

$$\left.\begin{aligned} \dot{x} &= V\cos(u) \\ \dot{y} &= V\sin(u) \end{aligned}\right\} \tag{3.12}$$

or, in polar coordinates ρ, φ by:

$$\left.\begin{aligned} \dot{\rho} &= V\cos(u-\varphi) \\ \dot{\varphi} &= \frac{V}{\rho}\sin(u-\varphi) \end{aligned}\right\} \tag{3.13}$$

In spite of the analyticity of the expressions in the differential equations, the system may be reduced, by straightforward elimination, to an algebraic implicit differential equation:

$$\left\{\ddot{\rho} - \frac{1}{\rho}[V^2 - (\dot{\rho})^2]\right\}^2 - (\dot{u})^2[V^2 - (\dot{\rho})^2] = 0 \tag{3.14}$$

The condition

$$(\dot{\rho})^2 \neq V^2$$

must be enforced, so that the radial position coordinate does not become uncontrollable. The uncontrollable motions correspond to uniformly sustained purely radial motions from (or towards) the origin of coordinates. Moreover, notice that, unless u is allowed to become constant (i.e. unless $\mathrm{d}u/\mathrm{d}t = 0$), the implicit differential equation (3.14) does not have any real solutions if the following strict inequality:

$$(\dot{\rho})^2 < V^2 \tag{3.15}$$

is violated. □

Remark: The phenomenon of obtaining implicit differential equations and inequalities as the input–output description of a system, arising from a state elimination procedure, has been demonstrated to hold in full generality by Diop (1989). □

We consider the following position error: $\zeta = \rho - R$, with respect to the circle line.

The control task consists of stabilizing the value of ζ to zero and, thus, obtain a perfectly circular motion of radius R for the unicycle. Notice that under perfect tracking of the circle, $\mathrm{d}\rho/\mathrm{d}t = 0$ and the inequality (3.15) is always satisfied.

It is easy to see that $q_1 = \zeta = \rho - R$ qualifies as a differential primitive element. The GCCF for the system is, evidently, given by:

$$\left.\begin{aligned} \dot{q}_1 &= q_2 \\ \left\{\dot{q}_2 - \frac{1}{q_1 + R}[V^2 - (q_2)^2]\right\}^2 - (\dot{u})^2[V^2 - (q_2)^2] &= 0 \end{aligned}\right\} \tag{3.16}$$

The sliding surface candidate σ is constituted, in this case, by an appropriate stabilizing linear combination of the generalized state components:

$$\sigma = q_2 + c_1 q_1; \quad c_1 > 0 \tag{3.17}$$

Notice that in original coordinates, σ is an input dependent switching surface.

Under ideal sliding conditions $\sigma = 0$, the unicycle asymptotically approaches the circle of radius R. The dynamical sliding mode controller is obtained by imposing the discontinuous dynamics (3.7) on σ. Such a dynamical discontinuous controller is, implicitly, given, in terms of the transformed coordinates q_1, q_2, by:

$$\left\{c_1 q_2 + W \operatorname{sign}(q_2 + c_1 q_1) - \frac{1}{q_1 + R}[V^2 - (q_2)^2]\right\}^2 - (\dot{u})^2[V^2 - (q_2)^2] = 0 \tag{3.18}$$

The zero dynamics associated with the stabilized (closed loop) system are immediately obtained, according to (3.11), from (3.16) by letting q_1, q_2 and $\mathrm{d}q_2/\mathrm{d}t$ be zero:

$$(\dot{u})^2 = \frac{V^2}{R^2} \tag{3.19}$$

The imposed restrictions on the heading angle u dictate that the physically meaningful solution to the zero-dynamics implicit equation is given by: $\mathrm{d}u/\mathrm{d}t = -V/R$, which is, evidently, unstable.

Remark: The physical meaning of such unstable zero dynamics is quite clear: in order to maintain the motion of the unicycle on the prescribed circle, one must turn the heading of the unicycle at a fixed rate, which precisely coincides with the constant angular velocity $-V/R$ of the contact point moving, counter-clockwise along the circle, with fixed tangential velocity V. The fact that the heading angle is constantly decreasing, without bound, can hardly be considered to represent a physically harmful behaviour for the system or for the associated control task. □

An explicit representation of the system, which is necessarily local, may be obtained by solving with respect to $\mathrm{d}q_2/\mathrm{d}t$ from the second equation in (3.16):

$$\left.\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= \frac{1}{q_1 + R}[V^2 - (q_2)^2] \pm (\dot{u})[V^2 - (q_2)^2]^{1/2} \end{aligned}\right\} \tag{3.20}$$

It is easy to see, from equilibrium considerations, that the two possible solutions for $\mathrm{d}q_2/\mathrm{d}t$ represent the possibility of clockwise and counter-clockwise motions along the circle, in inverse correspondence with the sign adopted for the (unstable) zero dynamics above. We take the positive sign as the solution for $\mathrm{d}q_2/\mathrm{d}t$ in (3.20), since we have explicitly assumed that only counter-clockwise motions are allowed.

The explicit dynamical sliding mode controller is then readily obtained as

$$\dot{u} = -\frac{1}{[V^2 - (q_2)^2]^{1/2}}\left[c_1 q_2 + \frac{1}{q_1 + R}[V^2 - (q_2)^2] + W \operatorname{sign}(q_2 + c_1 q_1)\right] \tag{3.21}$$

or, by carefully taking into account the right angular relation, in original polar coordinates, as:

$$\dot{u} = \frac{1}{\sin(u - \varphi)} \left\{ c_1 \cos(u - \varphi) + \frac{V}{\rho} \sin^2(u - \varphi) + \frac{W}{V} \operatorname{sign}[V \cos(u - \varphi) + c_1(\rho - R)] \right\} \tag{3.22}$$

Simulations of the dynamically sliding-mode controlled unicycle were performed with the following parameters: $V = 5\,\mathrm{m\,s^{-1}}$, $R = 5\,\mathrm{m}$, $W = 10$, $c_1 = 2\,\mathrm{s^{-1}}$. The results are shown in Fig. 9.

The smooth trajectory on the plane is portrayed showing the asymptotic approach to the target circle. The sliding surface coordinate evolution is also shown in this figure and it is easily seen to comply with the imposed discontinuous dynamics. The angular position of the contact point of the unicycle on the plane exhibits an (unstable) ever-decreasing behaviour as pointed out above. The heading angle response, acting as an external control input, is also shown to grow without bound, asymptotically to a linear growth, as demanded by the nature of the equivalent control dynamics and their limiting behaviour, represented by the zero-dynamics.

3.3. *Higher-order sliding regimes*

Consider (3.1), with σ as an output. We may rewrite such implicit dynamics as the following Global Generalized Observability Canonical Form (GGOCF) (see Fliess 1988):

$$\left.\begin{aligned} \dot{\sigma}_1 &= \sigma_2 \\ \dot{\sigma}_2 &= \sigma_3 \\ &\vdots \\ P(\sigma_1, \ldots, \sigma_\rho, \dot{\sigma}_\rho, u, \dot{u}, \ldots, u^{(\gamma)}) &= 0 \\ \sigma &= \sigma_1 \end{aligned}\right\} \cdot \tag{3.23}$$

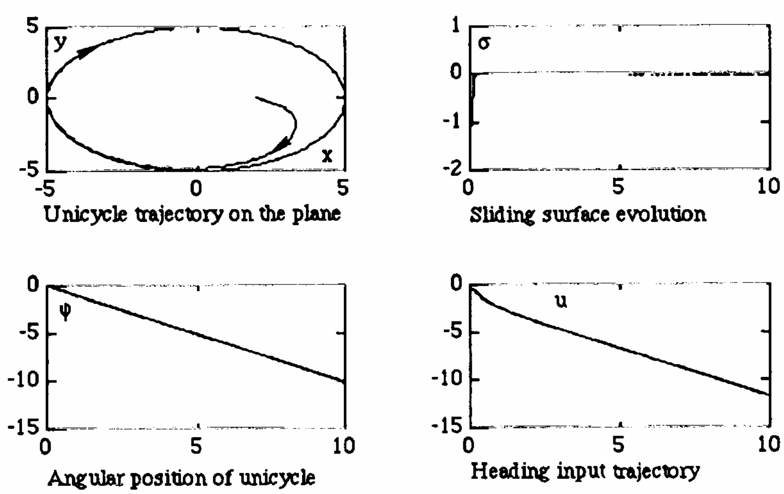

Figure 9. Simulations of dynamically sliding-mode controlled unicycle.

As before, an explicit LGOCF can be obtained for the element σ whenever the condition:

$$\frac{\partial P}{\partial(\dot{\sigma}_\rho)} \neq 0$$

is valid.

$$\left.\begin{aligned} \dot{\sigma}_1 &= \sigma_2 \\ \dot{\sigma}_2 &= \sigma_3 \\ &\vdots \\ \dot{\sigma}_\rho &= p(\sigma_1, \ldots, \sigma_\rho, u, \dot{u}, \ldots, u^{(\gamma)}) \\ \sigma &= \sigma_1 \end{aligned}\right\} \tag{3.24}$$

Definition: An element σ admits a *ρ–h order sliding* regime if the GOCF (3.24), associated with σ, is ρth order. □

One defines a ρth order sliding surface candidate as any arbitrary (algebraic) function of σ and its time derivatives, up to $(\rho - 1)$st order. For obvious reasons, the most convenient type of function is represented by a stabilizing linear combination of σ and its time derivatives.

$$s = m_1\sigma_1 + m_2\sigma_2 + \cdots + m_{\rho-1}\sigma_{\rho-1} + \sigma_\rho \tag{3.25}$$

A first-order sliding motion is then imposed on such a linear combination of generalized phase variables by means of the discontinuous sliding mode dynamics:

$$\dot{s} = -M\,\mathrm{sign}\,(s); \quad M > 0 \tag{3.26}$$

This results in the implicit dynamical higher-order sliding mode controller:

$$\begin{aligned} p(\sigma_1, \ldots, \sigma_\rho, u, \dot{u}, \ldots, u^{(\gamma)}) = &-m_{\rho-1}\sigma_\rho - \cdots - m_2\sigma_3 - m_1\sigma_2 \\ &- M\,\mathrm{sign}\,[m_1\sigma_1 + \cdots + m_{\rho-1}\sigma_{\rho-1} + \sigma_\rho] \end{aligned} \tag{3.27}$$

As previously discussed, s goes to zero in finite time and, provided the coefficients in (3.25) are properly chosen, an ideally asymptotically stable motion can be then obtained for σ, as governed by the following autonomous linear dynamics:

$$\left.\begin{aligned} \dot{\sigma}_1 &= \sigma_2 \\ \dot{\sigma}_2 &= \sigma_3 \\ &\vdots \\ \dot{\sigma}_{\rho-1} &= -m_1\sigma_1 - m_2\sigma_2 - \cdots - m_{\rho-1}\sigma_{\rho-1} \\ \sigma &= \sigma_1 \end{aligned}\right\} \tag{3.28}$$

Remark: It should be pointed out that a definition of static *higher-order sliding modes* is also available from the work of Korovin (1992). Our definition of higher-order sliding regimes ensures the possibilities of creation of a sliding regime, on a given surface candidate, only in an asymptotic fashion, by discontinuously controlling higher-order derivatives of the sliding surface candidate. □

Example 3.2: In Example 3.1, the first-order sliding regime obtained for σ is actually a second-order sliding regime for the radial position error: $q_1 = \zeta = \rho - R$. As is easily seen from (3.13), such an error quantity does qualify as a sliding surface candidate and, hence, a non-smoothed first-order sliding regime could have also been created on it. □

Example 3.3. Continuously stirred tank biological reactor: The following differential equations describe a simplified model of methanol growth in a continuously stirred tank biological reactor which utilizes *methylomonas* organisms (see Hoo and Kantor 1986, and Sira-Ramírez 1992 e). Let x_1 represent the density of methylomonas cells and let x_2 represent the methanol concentration:

$$\left.\begin{aligned} \dot{x}_1 &= A_\mu \varphi(x_2) x_1 - u x_1 \\ \dot{x}_2 &= -A_\sigma \varphi(x_2)\, x_1 + u(A_f - x_2) \\ y &= x_2 \end{aligned}\right\} \tag{3.29}$$

where

$$\varphi(x_2) = \frac{x_2}{B + x_2} \tag{3.30}$$

The control input u represents the dilution rate of the substrate and A_f is the feed substrate concentration, assumed to be constant. A_μ and A_σ are known constants.

For constant values $u = U$, of the dilution rate, the system exhibits two constant equilibrium points. One of the equilibrium points is located at $(0, A_f)$, which is of no physical interest, and the second one is given by:

$$X_1(U) = \frac{A_f A_\mu - (A_f + B)U}{A_\sigma} A_\mu; \quad X_2(U) = \frac{BU}{A_\mu - U} \tag{3.31}$$

The equilibrium value U, for the dilution rate, must necessarily satisfy the following relation:

$$U < \frac{A_f A_\mu}{(A_f + B)}$$

in order to have physically meaningful (i.e. positive and finite) equilibrium values for x_1 and x_2.

Suppose it is desired to regulate the methanol concentration x_2 to its equilibrium point $X_2(U)$ for a given U.

The methanol concentration error, $\sigma = x_2 - X_2(U)$, evidently qualifies as a sliding surface candidate, since its time derivative is dependent on the control input u. The resulting static 'first-order' sliding-mode controller does not seem to have much practical sense, since a discontinuous dilution rate u, i.e. one including arbitrarily large frequency switchings, is difficult, if not impossible, to achieve:

$$u = \frac{1}{(A_f - x_2)} [A_\sigma \varphi(x) - W \operatorname{sign}(x_2 - X_2(U))] \tag{3.32}$$

A simulation of the static sliding-mode controlled biological tank reactor is shown in Fig. 10. The system, and design, parameter values used for the simulation were chosen as:

$$A_f = 1{\cdot}8,\ A_\mu = 0{\cdot}504,\ A_\sigma = 1{\cdot}32,\ B = 8{\cdot}49 \times 10^4$$
$$U = 0{\cdot}4,\ X_2(U) = 3{\cdot}3 \times 10^{-3},\ W = 10$$

The state trajectory response for x_1 is sufficiently smooth and is seen to converge slowly to its equilibrium value $X_1(U) = 0{\cdot}6849$, while the trajectory of x_2 exhibits significant chattering around its pre-assigned equilibrium point. The feedback control input also exhibits a chattering response, thus making the feasibility of the controller quite questionable from practical grounds.

The concentration error σ is seen to satisfy a second-order algebraic differential equation of the form:

$$\begin{aligned}\ddot{\sigma} = &-A_\sigma[\varphi'(\sigma + X_2(U))\dot{\sigma} + \varphi(\sigma + X_2(U))\\ &\times(A_\mu\varphi(\sigma + X_2(U)) - u)]\left(\frac{u(A_f - \sigma - X_2(U))}{A_\sigma\varphi(\sigma + X_2(U))} - \dot{\sigma}\right)\\ &+ \dot{u}\,(A_f - \sigma - X_2(U)) - u\dot{\sigma}\end{aligned} \tag{3.33}$$

where $\varphi'(.)$ stands for $d\varphi(.)/d(.)$.

The LGOCF, which in this case is also a GGOCF, associated with the concentration error σ is then given by:

$$\left.\begin{aligned}\dot{\sigma}_1 &= \sigma_2\\ \dot{\sigma}_2 &= A_\sigma[\varphi'(\sigma_1 + X_2(U))\sigma_2 + \varphi(\sigma_1 + X_2(U))\\ &\quad\times(A_\mu\varphi(\sigma_1 + X_2(U)) - u)]\left(\frac{u(A_f - \sigma_1 - X_2(U))}{A_\sigma\varphi(\sigma_1 + X_2(U))} - \sigma_2\right)\\ &\quad+\dot{u}(A_f - \sigma_1 - X_2(U)) - u\sigma_2\\ \sigma &= \sigma_1\end{aligned}\right\} \tag{3.34}$$

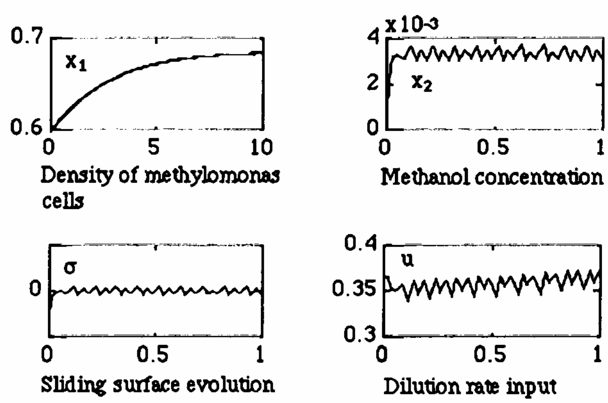

Figure 10. First-order sliding-mode-controlled continuously stirred tank biological reactor.

A second-order sliding regime may now be created for σ using the sliding surface:

$$s = \sigma_2 + m_1\sigma_1 \tag{3.35}$$

Notice that, expressed in terms of the state variables, such a sliding surface is actually an input-dependent switching function. Indeed, one obtains the following alternative expression for s:

$$s = -A_\sigma\varphi(x_2)x_1 + u(A_f - x_2) + m_1(x_2 - X_2(U)) \tag{3.36}$$

Imposing the discontinuous dynamics $ds/dt = -M\,\text{sign}(s)$, on the second-order sliding surface candidate s, yields the following dynamical sliding mode controller:

$$\begin{aligned}
&\dot{u}\,(A_f - \sigma_1 - X_2(U)) = \\
&- A_\sigma[\varphi'(\sigma_1 + X_2(U))\sigma_2 + \varphi(\sigma_1 + X_2(U))(A_\mu\varphi(\sigma_1 + X_2(U)) - u)] \\
&\times \left(\frac{u(A_f - \sigma_1 - X_2(U))}{A_\sigma\varphi(\sigma_1 + X_2(U))} - \sigma_2\right) + (u - m_1)\sigma_2 - M\,\text{sign}(\sigma_2 + m_1\sigma_1)
\end{aligned} \tag{3.37}$$

which, expressed now in terms of the state variables of the system, reads

$$\begin{aligned}
\dot{u} = \frac{1}{A_f - x_2}\{&[-A_\sigma\varphi(x_2)x_1 + u(A_f - x_2)](A_\sigma\varphi'(x_2)x_1 + u - m_1) \\
&+ A_\sigma\varphi(x_2)x_1(A_\mu\varphi(x_2) - u) \\
&- M\,\text{sign}[-A_\sigma\varphi(x_2)x_1 + u(A_f - x_2) + m_1(x_2 - X_2(U))]\}
\end{aligned} \tag{3.38}$$

The dynamical controller (3.38) exhibits a singularity (impasse point) at $x_2 = A_f$. The desired value $X_2(U)$ must then be chosen far away from A_f. If, however, trajectories must necessarily cross through this singularity, then suitable discontinuities must be appropriately devised on the control input prescription (see Abu el Ata-Doss *et al*. 1992 for details).

The simulations shown in Fig. 11 depict the higher-order sliding mode

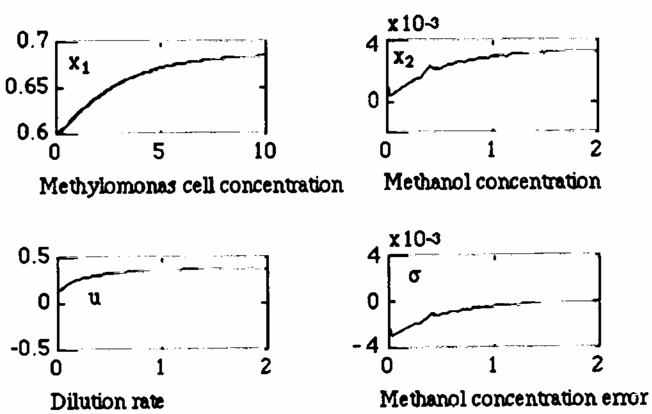

Figure 11. Second-order sliding-mode-controlled continuously stirred tank biological reactor.

controlled state responses x_1 and x_2 converging towards their equilibrium points, the smoothed nature of the dilution rate u, acting now as a dynamically generated feedback input and, finally, the asymptotic convergence of the concentration error σ, to zero. □

4. Conclusions and suggestions for further research

The differential algebraic approach to system dynamics provides both theoretical and practical grounds for the development of sliding mode control of nonlinear dynamical systems. More general classes of sliding surfaces, which include the presence of inputs and, possibly, their time derivatives, were shown to allow naturally for chattering-free sliding mode controllers of dynamical nature. Although equivalent smoothing effects can be similarly obtained by simply resorting to appropriate systems extensions, or prolongations of the input space, the theoretical simplicity, and conceptual advantages, stemming from the differential algebraic approach, bestow new possibilities to the broader area of discontinuous feedback control. For instance, the same smoothing effects and theoretical richness, can be used for the appropriate formulation and the attack of many potential application areas based on pulse-width-modulated control strategies (see Sira-Ramirez 1992 e). The less explored pulse-frequency-modulated control techniques have also been shown to benefit from this new approach (Sira-Ramirez 1992 f).

Discontinuous feedback controller design will undoubtedly be enriched by the differential algebraic approach. For instance, it has been shown, in a most elegant manner, by Fliess and Messager (1992), that non-minimum phase linear systems can be asymptotically stabilized using dynamical precompensators and sliding mode controllers. Such results could be extended to the nonlinear systems case with, possibly, some significant additional efforts. This topic, as well as possible extensions of the theory to nonlinear multivariable systems and to infinite dimensional systems, deserves some attention in the forseeable future.

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