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TRAPEZOIDAL PULSE-WIDTH-MODULATION CONTROL OF NONLINEAR MECHANICAL SYSTEMS*

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Abstract. A pulsed feedback control strategy is proposed for the robust stabilization of a class of multivariable mechanical systems in which rate constrained angular, or linear, velocity variables are regarded as control inputs. The trapezoidal character of the pulsed-width regulation policy complies with the physical limitation of having corresponding acceleration variables; i.e., applied torques, or forces, of bounded magnitude. This practical limitation would be clearly violated in the traditional mathematical formulation of (discontinuous) pulse-width-modulation control schemes using rectangular (velocity) control input pulses. The proposed approach is used in the approximate feedback control regulation of both a multivariable differentially flat system and also of a high frequency controlled non-differentially flat system.

Key Words—Trapezoidal pulse-width-modulation, differentially flat systems.

1. Introduction

Pulse-Width-Modulation (PWM) control of dynamical systems has been the subject of sustained theoretical and practical developments due to its inherent simplicity, robustness and widespread possibilities for inexpensive hardware implementation. Early work, in connection with the regulation of linear systems, is due to Nelson (1960), Kadota and Bourne (1961), Polak (1961), Skoog and Blankenship (1970), Tsypkin (1984) and LaCava et al. (1984). Developments casting PWM as a robust feedback control technique for nonlinear systems may be found in the work of Kuntsevich and Cherkhovoi (1971), Sira-Ramírez (1989; 1991; 1992), Sira-Ramírez and Lischinsky-Arenas (1990), Sira-Ramírez et al. (1993) and Taylor (1992). The technique has traditionally enjoyed a ubiquitous presence in applications to Power Electronics and Communication Systems (see Kassakian et al., 1991). The prevailing characteristic of such a control policy is the discontinuity of the applied feedback control input signal, constituted by pulses of varying width. As a consequence, the time derivatives of such a train of width-varying pulses, exhibit infinite magnitudes.

For a large class of mechanical systems, such as nonholonomically velocity

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constrained systems, velocity variables (whether angular or linear velocities) are sometimes considered as control inputs (see Bloch et al., 1992; Murray and Sastry, 1993). In the smooth feedback control case, this procedure not only simplifies the algebraic manipulations involved in the controller design, but it also results in a clear assessment of the fundamental structural obstructions to decouplability and feedback linearization. Computation of the required torques, or forces, is then carried out, if necessary, by means of straightforward differentiation and simple algebraic manipulations. This procedure, however, is not suitable for discontinuous feedback control techniques, such as sliding mode control and pulse-width-modulation (PWM), since the obtained feedback control expressions imply sudden changes, or steplike discontinuities, in the velocity variables. As a consequence, infinite applied forces or torques are required as ultimate control actions. In order to circumvent this difficulty, one may resort to an alternative procedure which still regards the velocity variables as control inputs but is now considering magnitude constraints on the accelerations. The rate of change of the involved velocity variables is thus purposefully limited. For the class of systems here treated, the proposed approach bestows physical realizability on the velocity control actions while retaining the essential simplicity of the traditional PWM control design procedure. This achievement, however, pays the cost of obtaining only stable convergence to the regulation objectives, rather than asymptotically stable behavior. From a practical viewpoint, the performance is, nevertheless, quite satisfactory.

In this article, a robust *trapezoidal* pulse-width-modulation scheme (TPWM) is proposed for the stabilization of a class of mechanical systems in which velocity variables are taken as control inputs for controller design purposes. The physical impossibility of having discontinuous velocity variables is thus overcome while allowing for physically meaningful feedback synthesized pulsed accelerations, i.e., applied torques, or forces.

Section 2 presents a fundamental stability result regarding a simple integrator system feedback regulated by means of a TPWM strategy. This development is later shown to be essential for the decoupled stabilization of multivariable nonlinear systems. Section 3 is devoted to treating two TPWM controller design examples. The first one is represented by a nonholonomically constrained system constituted by a "hopping robot." The regulation tasks are those of driving the length of the leg, and its angular position, to desired values while the robot is in the mid-air phase. This system has been shown to be *differentially flat* (see Fliess et al., 1992 a; b; 1993 a; b); i.e., it is linearizable by means of *dynamical endogenous feedback*. This fact is shown to facilitate greatly the TPWM controller design task. The second example deals with a single input non-differentially flat system constituted by a mass, sliding without friction, on an inverted pendulum. The suitable combination of high-frequency oscillatory control, already proposed in Fliess et al. (1993 a; b), and a TPWM strategy, results, respectively, in a differentially flat two-input system on which the link's angular velocity variable is physically synthesizable as a linearizing feedback policy. A stabilizing TPWM feedback controller is synthesized and regulates the mass position on the link towards a pre-specified small vicinity of the desired position. The performances of the proposed multivariable TPWM feedback controllers are evaluated when the systems are subject to unmodeled bounded stochastic disturbances. Section 4 contains the conclusions of the article.

2. Fundamental Background Results

In this section, we present several seemingly unrelated developments. In the first part, we study a simple scalar system regulated to the origin of the state space by means of a TPWM feedback strategy. The fundamental stability features of such a closed loop system are proved by using standard stability arguments on the exactly discretized system. The second part of the section revisits multivariable *differentially flat systems*. The problem of TPWM controller design for such class of systems is addressed, and it is readily acknowledged that such a design problem requires the concepts of *row relative degree* and *essential orders*. These concepts are also briefly revisited, and the connection of these tools with static, or dynamic, input-output decouplability of the given differentially flat system is addressed. Finally, resorting to the suitable definition of a set of scalar auxiliary output functions, on the decoupled multivariable flat system, it is shown that one can reduce the multivariable TPWM control design problem to a set of scalar TPWM controller design problems, precisely of the same type already treated at the beginning of this section.

2.1 Trapezoidal pulse-width-modulation control of a simple scalar system Consider the following scalar closed loop system characterized by a state variable s:

$$\left.\begin{aligned} \dot{s} &= v \\ v &= -W\mathrm{TPWM}(s) \end{aligned}\right\}, \tag{2.1}$$

$$\mathrm{TPWM}(s) = \begin{cases} \dfrac{1}{p\tau[s(t_k)]T}(t-t_k)\,\mathrm{sign}\,s(t_k) & \text{for } t_k \le t < t_k + p\tau[s(t_k)]T, \\ \mathrm{sign}\,s(t_k) & \text{for } t_k + p\tau[s(t_k)]T \le t < t_k + \tau[s(t_k)]T[1-p], \\ -\dfrac{1}{p\tau[s(t_k)]T}(t - t_k - \tau[s(t_k)]T)\,\mathrm{sign}\,s(t_k) & \\ \qquad \text{for } t_k + \tau[s(t_k)]T[1-p] \le t < t_k + \tau[s(t_k)]T, & \\ 0 & \text{for } t_k + \tau[s(t_k)]T \le t < t_k + T, \end{cases}$$

$t_k + T = t_{k+1}$ for $k = 0, 1, 2, \cdots$, where the function $\tau(s)$ represents the *duty ratio* function. Its sampled values, at every instant of time t_k, determines the width of the trapezoidal pulse for the current inter-sampling interval (see Fig. 1). The trapezoidal pulse width is determined at each sampling period as $\tau[s(t_k)]T$, where T denotes the sampling interval, or *duty cycle*, considered here to be constant (see also Sira-Ramírez et al. (1993) for pulse-frequency-modulation results). The duty ratio function is necessarily bounded by the closed interval $[0, 1]$. However, in order to avoid infinite slopes in the signal v, we need to hypothesize a minimum positive value, or constant lower bound, for the duty ratio function. Such a value will be denoted by the constant, $\tau_{\min}$. We also let the maximum control input rate to be specified by the constant, $A_{\max}$. The scalar p is then a positive real number defining the fraction of the pulse width $\tau[s(t_k)]T$, on which the signal v is allowed either to grow from zero to W, or to decrease from

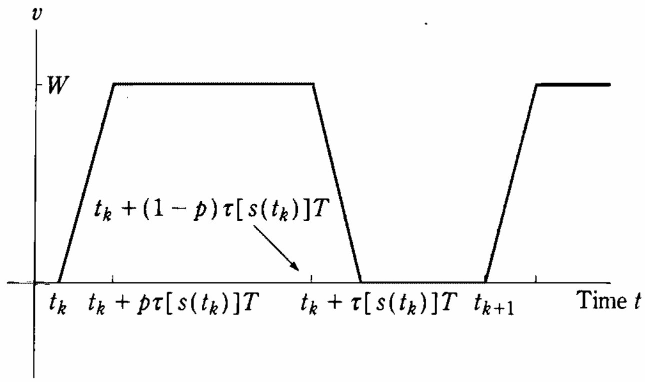

Fig. 1. Trapezoidal pulse-width modulated control signal.

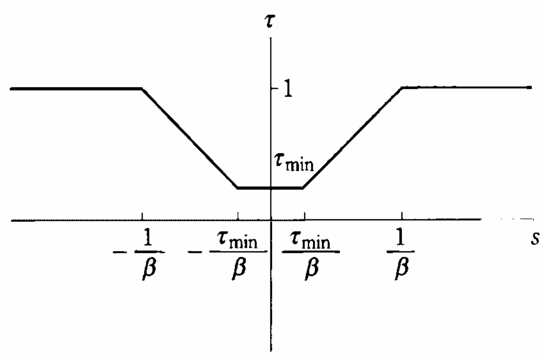

Fig. 2. Duty ratio function for scalar TPWM feedback strategy.

W to zero. This number p, evidently, must also be bounded away from 1. The minimum allowable value of the duty ratio function, $\tau_{\min}$, is evidently related to p, to the sampling interval T and to the gain W by the relation

$$\frac{W}{p\tau_{\min} T} \leq A_{\max}. \tag{2.2}$$

The positive constant gain W is a design parameter representing the maximum amplitude, in absolute value, of the control input signal v.

The duty ratio function, $\tau(s)$, is synthesized as a feedback function as follows (see Fig. 2):

$$\tau(s) = \begin{cases} 1 & \text{for} \quad |s| \geq \dfrac{1}{\beta}, \\ \beta|s| & \text{for} \quad \dfrac{\tau_{\min}}{\beta} < |s| < \dfrac{1}{\beta}, \\ \tau_{\min} & \text{for} \quad |s| \leq \dfrac{\tau_{\min}}{\beta}. \end{cases} \tag{2.3}$$

We remark that contrary to traditional PWM control schemes, we do not allow the duty ratio function to converge to zero, since this would imply an infinite slope for the velocity control variable v. This control input limitation in turn results only in a *stable* behavior of the controlled variable s around the value zero. For the above reason, the minimum value of the duty ratio function $\tau_{\min}$ must be specified in such a manner that the regulated trajectories for s stay in a small vicinity of zero while still complying with the limitation of the acceleration variable. The next paragraphs describes the stable features of the approach along with an amplitude estimate of the underlying *limit cycle* behavior exhibited by s.

Proposition 2.1. The closed loop system (2.1) is stable. Moreover, the trajectories of s are ultimately bounded by a vicinity of the origin given by

$$|s(t)| \leq \frac{\tau_{\min}}{\beta},$$

provided

$$(1-p)\beta WT < 1. \tag{2.4}$$

Proof. The proof of stability is quite straightforward by simply adopting the function

$$V(s) = \frac{1}{2}s^2$$

as a Lyapunov function candidate. According to (2.1), one has, along its solutions,

$$\dot{V} = \dot{s}s \leq 0.$$

Moreover, as long as $s \neq 0$, the sets where $\dot{V} = 0$ do not constitute trajectories of the system for an indefinite period of time. The system is, therefore, stable.

Consider now an *exact* discretization of the closed loop system (2.1),

$$s(t_{k+1}) = s(t_k) - (1-p)\tau[s(t_k)]WT\,\mathrm{sign}(s(t_k)). \tag{2.5}$$

Note that the quantity $(1-p)\tau[s(t_k)]WT$ represents, in absolute value, the magnitude of the steps undergone by the sampled scalar state $s(t_k)$. We denote by $\Delta s(t_k)$ such stepsizes; i.e.,

$$\Delta s(t_k) = |s(t_{k+1}) - s(t_k)|.$$

Assume now that condition (2.4) holds valid. Three situations are then possible:

1. For $|s(t_k)| > 1/\beta$, the discretized system (2.5) is described by

$$s(t_{k+1}) = s(t_k) - (1-p)WT\,\mathrm{sign}(s(t_k)). \tag{2.6}$$

Since $(1-p)\beta WT < 1$, it follows that $\Delta s(t_k) = (1-p)WT < 1/\beta$. As a consequence, the sampled values of s are guaranteed to reach the interval

$I_\beta = [-1/\beta, 1/\beta]$ in a finite number of steps. Any closed loop trajectory of s, starting on the region $|s(t_k)| > 1/\beta$, is, thus, eventually bounded by the interval $I_\beta = [-1/\beta, 1/\beta]$. If the first sampled value of s, say $s(t_N)$, belonging to the interval I_β, is also found in the smaller interval $I_{\tau_{\min}} \triangleq [-\tau_{\min}/\beta, \tau_{\min}/\beta]$, then, the third situation described below applies. Otherwise, if $s(t_N) \in I_\beta$ but $s(t_N) \notin I_{\tau_{\min}}$, the consideration that follows is valid.

2. Within the interval $\tau_{\min}/\beta < |s| < 1/\beta$, the closed loop trajectory of s is regulated by

$$s(t_{k+1}) = [1 - (1-p)\beta WT]s(t_k), \quad k \geq N. \tag{2.7}$$

Since condition (2.4) is valid, the only eigenvalue of (2.7) is strictly positive and smaller than 1. The scalar state $s(t_k)$, $k \geq N$ further decreases, in absolute value, without changing its sign. It follows that in a finite amount of steps, the sampled value of s, say $s(t_M)$ for some $M > N$, will be found within the interval $I_{\tau_{\min}}$.

3. The discretized system, governing the evolution of the sampled values, $s(t_k)$ of s on the interval $I_{\tau_{\min}}$, is given by

$$s(t_{k+1}) = s(t_k) - (1-p)\tau_{\min} WT \operatorname{sign}(s(t_k)), \quad k \geq M. \tag{2.8}$$

The stepsize $\Delta(s(t_k)) = (1-p)\tau_{\min} WT$ becomes constant in the interval $I_{\tau_{\min}}$. Moreover, condition (2.4) implies that

$$\Delta(s(t_k)) = (1-p)\beta\tau_{\min} WT < \tau_{\min}, \qquad \forall k > M;$$

in other words, the exhibited stepsize of the discretized system is strictly bounded by one-half of the width of the interval $I_{\tau_{\min}}$; i.e.,

$$(1-p)\tau_{\min} WT < \frac{\tau_{\min}}{\beta}.$$

As a consequence, the sampled values $s(t_k)$ of $s(t)$ become indefinitely confined to the interval $I_{\tau_{\min}}$ for $k > M$.

2.2 Trapezoidal pulse-width-modulation control of multivariable nonlinear systems

2.2.1 Differentially flat systems In this section, we indicate how to utilize the basic result of Proposition 2.1 in TPWM feedback controller design for a class of multivariable nonlinear mechanical systems in which the velocity variables are assumed to act as control inputs to the system. For this class of systems, evidently, traditional PWM feedback control is not feasible, since it would imply infinite input accelerations, or forces. Hence, the need for TPWM controllers. The class of mechanical systems to be treated in Sec. 3 corresponds to those linearizable, by means of *endogenous* feedback, to decoupled controllable systems in Brunovsky's canonical form. This class of systems, addressed as *differentially flat* systems, constitutes the simplest extension, to the nonlinear case of the class of controllable linear systems. The fundamental characterization of this appealing class of systems has been the subject of extensive research carried out by Professor M. Fliess and his co-workers in Fliess et al. (1992 a; b; 1993 a; b).

A fundamental property of multivariable differentially flat systems resides in the existence of a so-called, *set of linearizing outputs*, or *set of flat outputs*, equal in number to the set of control inputs. This variables are such that every other variable in the system, such as states, regulated outputs and even input variables, are expressible as *differential functions* of such a set of linearizing outputs; i.e., they are all functions of the linearizing outputs and a finite number of their time derivatives.

The developments presented in the following paragraphs are taken from Martin (1992).

Consider the n-dimensional multi-input system with m inputs and an m-dimensional vector z of regulated outputs; i.e., the input-output system is *square*,

$$\left.\begin{aligned} \dot{x} &= f(x, u), \quad x \in \mathcal{R}^n, \quad u \in \mathcal{R}^m \\ z &= h(x), \quad z \in \mathcal{R}^m \end{aligned}\right\}. \tag{2.9}$$

Suppose that a *regular*, i.e., invertible, dynamic compensator can be designed for system (2.9), which is given by

$$\left.\begin{aligned} \dot{\theta} &= a(x, \theta, v) \\ u &= b(x, \theta, v), \quad \theta \in R^q, \quad v \in R^m \end{aligned}\right\}. \tag{2.10}$$

Assume, furthermore, that an invertible state coordinate transformation exists, of the form

$$\zeta = \phi(x, \theta), \tag{2.11}$$

such that the composite feedback system (2.9), (2.10),

$$\left.\begin{aligned} \dot{x} &= f(x, b(x, \theta, v)) \\ \dot{\theta} &= a(x, \theta, v) \\ z &= h(x) = \tilde{h}(x, \theta) \end{aligned}\right\} \tag{2.12}$$

is transformed into the following decoupled set of controllable linear systems, expressed in *Brunovsky's canonical form*:

$$\left.\begin{aligned} \dot{\zeta}_{i1} &= \zeta_{i2} \\ &\vdots \\ \dot{\zeta}_{i\gamma_i} &= v_i \\ y_i &= \zeta_{i1}, \quad i = 1, \cdots, m \end{aligned}\right\}, \tag{2.13}$$

where γ_i, $i = 1, \cdots, m$ are the *Brunovsky controllability indices* of the dynamically feedback compensated system. They satisfy $\sum_{i=1}^{m} \gamma_i = n + q$. It should now be evident that the set of *linearizing outputs*, or *flat outputs*, are constituted by the quantities $y_i = \zeta_{i1}$, $i = 1, \cdots, m$. Indeed, from the invertibility of the composite state coordinates transformation (2.11), one obtains,

$$\begin{aligned}(x, \theta) &= \phi^{-1}(\zeta) = \phi^{-1}(\zeta_{11}, \cdots, \zeta_{1\gamma_1}, \cdots, \zeta_{m1}, \cdots, \zeta_{m\gamma_m}) \\ &= \phi^{-1}(y_1, \cdots, y_1^{(\gamma_1 - 1)}, \cdots, y_m, \cdots, y_m^{(\gamma_m - 1)}); \end{aligned} \tag{2.14}$$

i.e., all the components of the augmented sate vector describing the dynamically compensated system (2.12) are expressible as differential functions of the linearizing outputs $y_1, \cdots, y_m$.

By virtue of the last fact, each and every one of the regulated outputs z are also differential functions of the linearizing outputs y. Indeed, in general terms, one has

$$\begin{aligned} z = h(x) = \tilde{h}(x, \theta) &= \tilde{h}[\phi^{-1}(\zeta_{11}, \cdots, \zeta_{1\gamma_1}, \cdots, \zeta_{m1}, \cdots, \zeta_{m\gamma_m})] \\ &= \tilde{h}[\phi^{-1}(y_1, \cdots, y_1^{(\gamma_1 - 1)}, \cdots, y_m, \cdots, y_m^{(\gamma_m - 1)})] \\ &= \psi(y_1, \cdots, y_1^{(\gamma_1 - 1)}, \cdots, y_m, \cdots, y_m^{(\gamma_m - 1)}). \end{aligned} \tag{2.15}$$

Each one of the new control inputs v_i, $i = 1, \cdots, m$ is also expressible as the highest time derivative of the corresponding linearizing output y_i, i.e., $v_i = \dot{\zeta}_{i\gamma_i} = y_i^{(\gamma_i)}$, $i = 1, \cdots, m$. It then easily follows, by virtue of (2.10), that the control input u can be expressed as

$$\begin{aligned} u &= b(\phi^{-1}(\zeta_{11}, \cdots, \zeta_{1\gamma_1}, \cdots, \zeta_{m1}, \cdots, \zeta_{m\gamma_m}),\ \dot{\zeta}_{1\gamma_1}, \cdots, \dot{\zeta}_{m\gamma_m}) \\ &= b(\phi^{-1}(y_1, \cdots, y_1^{(\gamma_1 - 1)}, \cdots, y_m, \cdots, y_m^{(\gamma_m - 1)}),\ y_1^{(\gamma_1)}, \cdots, y_m^{(\gamma_m)}); \end{aligned} \tag{2.16}$$

i.e.,

$$u = \vartheta(y_1, \cdots, y_1^{(\gamma_1)}, y_2, \cdots, y_2^{(\gamma_2)}, \cdots, y_m, \cdots, y_m^{(\gamma_m)}). \tag{2.17}$$

The *endogenous* character of the dynamical feedback is equivalent to the fact that the set of linearizing outputs y_i, $i = 1, \cdots, m$, are, in turn, differential functions of the state and input vector components; i.e., if and only if there exists a real analytic function, Ψ, of (x, u), and a finite number of their time derivatives, such that

$$y = \Psi(x, u, \dot{u}, \cdots, u^{(\gamma)}). \tag{2.18}$$

Suppose one is particularly interested in obtaining a multivariable feedback control strategy, possibly of the TPWM type, such that the components of the m-dimensional output vector z are driven in a stable manner towards compatible pre-specified equilibrium values, represented by the constant vector Z. The underlying input-output decoupled linearization problem requires, in general, an interplay between two fundamental concepts known as *row relative degree*, introduced in Isidori (1989) and the concept of *essential orders*, developed in Glumineau and Moog (1989). In certain cases, one may, however, utilize to advantage the differential flatness associated with the system in order to obtain a decoupled feedback linearizing design. We briefly explain the involved concepts and the associated design procedure in the following paragraphs.

2.2.2 Row relative degree, essential orders and decouplability We say that the component $z_i = h_i(x)$ of the output vector of the system (2.9) has *row relative degree* equal to the integer r_i, whenever the r_ith time derivative of z_i explicitely depends upon any, or several, of the components of the control input vector u. The vector formed with the row relative degrees $(r_1, \cdots, r_m)$ is briefly addressed as the *structure at infinity* of the underlying multivariable nonlinear system. The rank of the matrix relating the r_ith time derivatives of the output components z_i to the control input vector u contains the essential information that allows us to know whether or not the system is *decouplable* by means of static state feedback in a certain region, or in all of state space of the system. Such a matrix, addressed as the *decoupling matrix*, is obtained as

$$\frac{\partial}{\partial u}\begin{bmatrix} z_1^{(r_1)} \\ \vdots \\ z_m^{(r_m)} \end{bmatrix}. \tag{2.19}$$

If the decoupling matrix (2.19) is globally (resp. locally) full rank m, the system is said to be *globally (resp. locally) input-output decouplable by means of static state feedback*. In such a case, we say that the structure at infinity coincides with the *essential structure at infinity*. In other words, the essential structure is nothing else but the structure at infinity of the decoupled system. The two concepts are equivalent only for the particular case of statically decouplable systems.

Suppose, however, that the decoupling matrix (2.19) *is not* full rank. This means that dynamical input extensions (i.e., adding integration elements in front of some input channels) are necessary in order to obtain the decouplability rank condition of the matrix relating the differentials of the output vector components with the newly defined (extended) inputs. When a suitable extension of some of the input components results in the (static) decouplability of the extended system, the original system is said to be *decouplable by means of dynamical feedback*. It is clear that, in such a case, the new structure at infinity of the extended system no longer coincides with the structure at infinity of the original system. The essential structure is, however, still defined as the structure at infinity of the decouplable extended system and, therefore, it measures the *structural obstructions*, present in the original system, for achieving static input-output decouplability. The following result, taken from Glumineau and Moog (1989), summarizes the above discussion in general terms.

Theorem 2.2. (Glumineau and Moog, 1989) If there exists a static or dynamic state feedback solving the decoupling problem, then, for each one of the outputs of the system, it is verified that its (decoupled) relative degree is not inferior to its essential order. Moreover, there exists a (possibly extended) decoupled system, deduced from the original system, for which the essential orders coincide with the decoupled relative degrees.

2.2.3 Differential flatness and input-output decouplability It should be stressed that the *differential flatness* of a given system is not affected by the process of inducing dynamical extensions on some, or all, of the control input components. Therefore, if the system is differentially flat but not input-output decouplable by static state feedback, the decouplable extended system still remains differentially flat.

The simplest possible cases are represented by those systems in which the *regulated outputs* of the system z_i, $i = 1, \cdots, m$, entirely coincide with the set of linearizing, or flat, outputs, y_i, $i = 1, \cdots, m$. In this case, the system is naturally input-output decouplable by endogenous (static or dynamical) linearizing feedback. In such a particular case, the essential structure at infinity of the regulated outputs of the differentially flat system is easily assessed from the highest values of the orders of the time derivatives of the different linearizing output components over all the differential expressions defining the control input components.

Consider, then, that $z_i = y_i$, $i = 1, \cdots, m$. The TPWM feedback controller design problem is then reduced to obtain a set of scalar (i.e., decoupled) TPWM feedback control laws achieving the stabilization, to desired equilibrium values $Z_i = Y_i$, for the following linear controllable subsystems, obtained by regular dynamical feedback and diffeomorphic (composite) state coordinates transformation

$$\left.\begin{aligned} \dot{\zeta}_{i1} &= \zeta_{i2} \\ &\vdots \\ \dot{\zeta}_{i\gamma_i} &= v_i \\ y_i &= \zeta_{i1}, \quad i = 1, \cdots, m \end{aligned}\right\}. \tag{2.20}$$

To these design effects, consider the *error variables* $s_i = y_i - Y_i = \zeta_{i1} - Y_i$, $i = 1, \cdots, m$, with Y_i being the desired constant equilibrium value for the ith linearizing output y_i. Consider, also, the associated *auxiliary error variables*, σ_i, defined by

$$\begin{aligned} \sigma_i &= \lambda_{i1} s_i + \lambda_{i2} \dot{s}_i + \cdots + \lambda_{i(\gamma_i - 1)} s_i^{(\gamma_i - 2)} + s_i^{(\gamma_i - 1)} \\ &= \lambda_{i1}(\zeta_{i1} - Y_i) + \lambda_{i2}\zeta_{i2} + \cdots + \lambda_{i(\gamma_i - 1)}\zeta_{i(\gamma_i - 1)} + \zeta_{i\gamma_i}, \end{aligned} \tag{2.21}$$

where for each i, the set of constant coefficients, $\{\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{i(\gamma_i - 1)}, 1\}$, constitutes a *set of Hurwitz coefficients*; i.e., the roots of the associated characteristic polynomial $q_i(p)$, in the complex variable p,

$$q_i(p) = \lambda_{i1} + \lambda_{i2} p + \cdots + \lambda_{i(\gamma_i - 1)} p^{\gamma_i - 2} + p^{\gamma_i - 1}, \tag{2.22}$$

are all located in the open left portion of the complex plane.

Imposing on each of the auxiliary error variables, σ_i, $i = 1, \cdots, m$, the scalar TPWM dynamics,

$$\dot{\sigma}_i = -W_i \mathrm{TPWM}\sigma_i, \quad i = 1, \cdots, m \tag{2.23}$$

with $W_i > 0$, $\forall i$, one guarantees, by virtue of the fundamental result presented in Sec. 2.1, that the TPWM regulated trajectories of the linearizing output vector components converge to an arbitrary small vicinity of the prescribed equilibrium points Y_i, $i = 1, \cdots, m$.

Substitution of the auxiliary output error expressions on the imposed dynamics (2.23) results in the following TPWM feedback control policy in terms of the linearizing coordinates and their time derivatives,

$$\begin{aligned} v_i &= -\lambda_{i1}\zeta_{i2} - \lambda_{i2}\zeta_{i3} - \cdots - \lambda_{i(\gamma_i-1)}\zeta_{i(\gamma_i)} - W_i \mathrm{TPWM}(\zeta_{i1} - Y_i) \\ &= -\lambda_{i1}\dot{y}_i - \lambda_{i2}\ddot{y}_i - \cdots - \lambda_{i(\gamma_i-1)} y_i^{(\gamma_i-1)} - W_i \mathrm{TPWM}(y_i - Y_i), \\ & \qquad i = 1, 2, \cdots, m. \end{aligned} \tag{2.24}$$

Evidently, the TPWM controller expression (2.24), is placeable in terms of the composite state of the dynamically compensated system. In general, a dynamical endogenous TPWM feedback controller is obtained. The second example in Sec. 3 corresponds to the situation here described.

In the general case, however, the regulated output vector z does not coincide with the linearizing output vector y, but each and every one of its components are, nevertheless, differential functions of such a set of linearizing outputs. This is, precisely, the advantageous feature of exploiting the differential flatness property of the system in the process of designing multivariable TPWM feedback control policies, or any other kinds of feedback control policies. Although the general theoretical developments will not be presented here in detail, they can be easily inferred from the first example of Sec. 3. The fundamental assumption is referred to the possibilities of proposing, for each component of the vector of regulated outputs z, a suitable linear (Hurwitz) combination of the particular output error variable and its time derivatives. These auxiliary error functions must be such that the imposition of a set of first order scalar decoupled TPWM closed loop dynamics, of the form (2.23), yields an (invertible) multivariable relation with the highest order time derivatives of the set of linearizing output coordinates. From the obtained expressions, one then solves for the required dynamically, or statically, generated control inputs (for more details, in connection with the closedly related area of sliding mode controller design, the reader is referred to Sira-Ramírez (1994)).

3. Trapezoidal Pulse-Width-Modulation Control of Some Mechanical Systems

3.1 The hopping robot Consider the dynamics of a hopping robot in flight phase (see Murray and Sastry, 1993). Let the state variables be defined as the length l of the leg and the angular position coordinates ψ and θ, respectively, of the unit mass body and of the leg with respect to the horizontal axis. The leg of the robot, of mass m_l, can rotate with respect to its attachment to the body. The total angular momentum is, however, conserved during such motions (such is the nature of the nonholonomic constraint). During the flight phase, the leg is also capable of extending and contracting within a given range, taken here, for simplicity, between 1 and $1 + L$ (see Fig. 3). The control inputs v_1 and v_2 are constituted, respectively, by the angular velocity of the leg's rotation and the rate of change of the length of the leg. The equations describing the dynamics are given by

$$\left.\begin{aligned} \dot{\psi} &= v_1 \\ \dot{l} &= v_2 \\ \dot{\theta} &= -\frac{m_l(1+l)^2}{1+m_l(1+l)^2} \cdot v_1 \end{aligned}\right\} . \tag{3.1}$$

The following state coordinate transformation:

$$x_1 = \psi, \quad x_2 = -\frac{m_l(1+l)^2}{1+m_l(1+l)^2}, \quad x_3 = \theta, \tag{3.2}$$

and the redefinition of the input variables,

$$\left.\begin{aligned} u_1 &= v_1 \\ u_2 &= -\frac{2m_l(1+l)}{(1+m_l(1+l)^2)^2} v_2 \end{aligned}\right\} \tag{3.3}$$

takes the system (3.1) into a *2-input, 1-chain, single generator chained system* form (see Murray and Sastry, 1993),

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_2 u_1 . \tag{3.4}$$

The system is evidently differentially flat, since all variables in the transformed system can be expressed as a differential function of the linearizing outputs

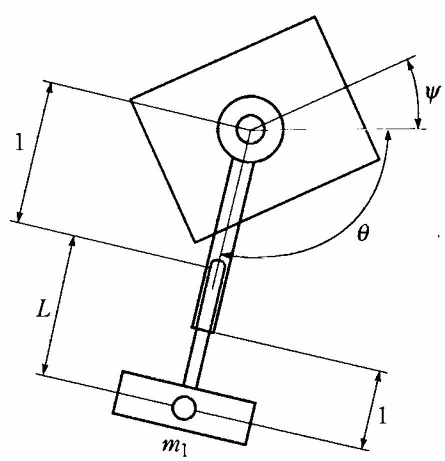

Fig. 3. A hopping robot.

$y_1 = x_1$ and $y_2 = x_3$ (i.e., as a function of the leg and the body angular position coordinates and a finite number of its time derivatives). Indeed,

$$\left.\begin{aligned} x_1 &= y_1 \\ x_2 &= \frac{\dot{y}_2}{\dot{y}_1} \\ x_3 &= y_2 \\ u_1 &= \dot{y}_1 \\ u_2 &= \frac{\ddot{y}_2\dot{y}_1 - \dot{y}_2\ddot{y}_1}{\dot{y}_1^2} \end{aligned}\right\}. \tag{3.5}$$

It is easy to see that the *row relative degrees* (see Isidori, 1989) of y_1 and y_2 are both equal to 1, while the *essential orders* (see Glumineau and Moog, 1989) are both equal to 2. The system is not decouplable by means of static state feedback, and a dynamic extension of order 1 is required on the transformed control input u_1 in order to make the structure at infinity of the extended system coincide with the essential structure. This extension is thus necessary for the appropriate definition of a dynamical feedback law which achieves decoupling of the system.

The dynamically extended version of the transformed system, which is now suitable for static linearly decoupling feedback, is given by

$$\left.\begin{aligned} \dot{x}_1 &= u_1, \quad & \dot{x}_2 &= u_2 \\ \dot{u}_1 &= v_1, \quad & \dot{x}_3 &= x_2 u_1 \end{aligned}\right\}, \tag{3.6}$$

where $v_1 = \dot{u}_1$ is a new control input to the system and the variable u_1 is just an additional state variable for the extended system. The extended control inputs, as differential functions of the linearizing outputs, are simply given by

$$v_1 = \ddot{y}_1, \quad u_2 = \frac{\ddot{y}_2\dot{y}_1 - \dot{y}_2\ddot{y}_1}{\dot{y}_1^2}. \tag{3.7}$$

Suppose it is desired to have the linearizing coordinate $y_1 = \theta$, adopt the constant value Θ, at the end of the flying phase, while the length l is driven to a constant value, say $L/3$. Appropriate error functions s_1 and s_2 may be defined as

$$\left.\begin{aligned} s_1 &= x_3 - \Theta = y_2 - \Theta \\ s_2 &= x_2 + \frac{m_l(1+L/3)^2}{1+m_l(1+L/3)^2} = \frac{\dot{y}_2}{\dot{y}_1} + \frac{m_l(1+L/3)^2}{1+m_l(1+L/3)^2} \end{aligned}\right\}. \tag{3.8}$$

While the first choice is clear, it is also easy to realize, from Eq. (3.8), that the only physically meaningful solution for l, from the condition $s_2 = 0$, is given by $l = L/3$. Note that the first regulated output $z_1 = y_2 = \theta$ coincides with a linearizing, or flat, output y_2 of the system, while the second regulated output,

$z_2 = x_2$, is a differential function of the flat output. In the extended system, the regulated output z_1 has relative degree two, while the regulated output z_2 has only relative degree 1. Thus, an imposed second order dynamics on z_1 and an imposed first order dynamics on z_2 already contain expressions involving the highest order time derivatives of the linearizing outputs y_1 and y_2. As it will be seen, such time derivatives of the flat outputs are solvable from these imposed relations.

In accordance with the particular form of the proposed regulated error functions, a second-order closed loop TPWM dynamics will be proposed for s_1. This is achieved by imposing a first-order TPWM dynamics on a (Hurwitz) linear combination of s_1 and $\dot{s}_1$. A first-order closed loop TPWM controlled dynamics suffices for the error coordinate s_2.

Define, then, the *auxiliary error coordinates* σ_1 and σ_2, as

$$\left.\begin{aligned} \sigma_1 &= \dot{s}_1 + \lambda s_1, \quad \lambda > 0 \\ \sigma_2 &= s_2 \end{aligned}\right\}. \tag{3.9}$$

A decoupled set of closed loop TPWM dynamics guarantees desirable stability features, for s_1 and s_2 after the auxiliary error functions σ_1 and σ_2 are approximately driven to zero, in finite time. Such imposed TPWM dynamics are given by

$$\left.\begin{aligned} \dot{\sigma}_1 &= - W_1 \text{TPWM}\sigma_1, \quad W_1 > 0 \\ \dot{\sigma}_2 &= - W_2 \text{TPWM}\sigma_2, \quad W_2 > 0 \end{aligned}\right\}. \tag{3.10}$$

The preceeding closed loop dynamics lead to error dynamics governed by

$$\left.\begin{aligned} \ddot{s}_1 &= - \lambda \dot{s}_1 - W_1 \text{TPWM}(\dot{s}_1 + \lambda s_1) \\ \dot{s}_2 &= - W_2 \text{TPWM} s_2 \end{aligned}\right\}. \tag{3.11}$$

After small amplitude stable oscillations occur around the zero level set of the auxiliary error functions $\sigma_1 = 0$ and $\sigma_2 = 0$, the s_1 and s_2 coordinates will approximately satisfy the following equations:

$$\left.\begin{aligned} \dot{s}_1 &= - \lambda s_1 \\ s_2 &= 0 \end{aligned}\right\}. \tag{3.12}$$

One may conclude that the proposed scheme guarantees a stable convergence of s_1 and s_2 to a small vicinity of zero. This accomplishes, in an approximate, but efficient manner, the proposed control objectives. The leg's angular coordinate θ is seen to converge towards the vicinity of the prescribed value Θ, while also closely achieving the required length $l = L/3$ for the rotating leg.

By virtue of the differential flatness of the system, the regulated dynamics (3.10) can be immediately translated into required autonomous dynamics for the linearizing outputs $y_1 = x_1$ and $y_2 = x_3$. Indeed, in terms of y_1 and y_2 the Eq. (3.10) result in the following nonlinear set of differential equations with right-hand sides specified by TPWM feedback policies:

$$\left.\begin{aligned} \ddot{y}_1 &= \frac{1}{\dot{y}_2}\cdot\Big\{[-\lambda\dot{y}_2 - W_1\,\mathrm{TPWM}(\dot{y}_2+\lambda(y_2-\Theta))]\dot{y}_1 \\ &\quad +\dot{y}_1^2 W_2\,\mathrm{TPWM}\left(\frac{\dot{y}_2}{\dot{y}_1}+\frac{m_l(1+l/3)^2}{1+m_l(1+L/3)^2}\right)\Big\} \\ \ddot{y}_2 &= -\lambda\dot{y}_2 - W_1\,\mathrm{TPWM}[\dot{y}_2+\lambda(y_2-\Theta)] \end{aligned}\right\}. \tag{3.13}$$

Using the highest order derivatives $\ddot{y}_1$ and $\ddot{y}_2$, obtained from the previous set of differential equations, on the expressions for the (extended) control inputs given in Eq. (3.7), one immediately obtains, in transformed coordinates, the required extended control input v_1 as $v_1 = \dot{u}_1 = \ddot{y}_1$ and the (static) control input u_2 as

$$\left.\begin{aligned} \dot{u}_1 &= -\frac{1}{x_2}\cdot\left\{\left[\lambda x_2 - W_2\,\mathrm{TPWM}\left(x_2+\frac{m_l(1+L/3)^2}{1+m_l(1+L/3)^2}\right)\right]u_1 \right. \\ &\quad \left. + W_1\,\mathrm{TPWM}[x_2u_1+\lambda(x_3-\Theta)]\right\} \\ u_2 &= -W_2\,\mathrm{TPWM}\left(x_2+\frac{m_l(1+L/3)^2}{1+m_l(1+L/3)^2}\right) \end{aligned}\right\}. \tag{3.14}$$

The multivariable TPWM controller thus includes a first order dynamical TPWM compensator for the control input u_1 controlling the leg's angle and a static feedback TPWM regulator for the control input u_2 regulating the leg's length. The dynamical and statical controllers can also be expressed in the original system coordinates as

$$\left.\begin{aligned} \dot{v}_1 &= -\frac{1+m_l(1+l)^2}{m_l(1+l)^2}\left[\left(\lambda-\frac{m_l(1+l)^2}{1+m_l(1+l)^2}+W_2\,\mathrm{TPWM}\sigma_2\right)v_1 \right. \\ &\quad \left. - W_1\,\mathrm{TPWM}\sigma_1\right] \\ v_2 &= -W_2\frac{(1+m_l(1+l)^2)^2}{2m_l(1+l)}\cdot\mathrm{TPWM}\sigma_2 \end{aligned}\right\} \tag{3.15}$$

with

$$\left.\begin{aligned} \sigma_1 &= -\left(\frac{m_l(1+l)^2}{1+m_l(1+l)^2}\right)v_1+\lambda(\theta-\Theta) \\ \sigma_2 &= -\frac{m_l(1+l)^2}{1+m_l(1+l)^2}+\frac{m_l(1+L/3)^2}{1+m_l(1+L/3)^2} \end{aligned}\right\}. \tag{3.16}$$

3.2 Simulation Results Computer simulations were carried out for the system and the designed dynamical TPWM control policy. In order to test the robustness of the proposed controller, an (unmodeled) computer generated stochastic perturbation signal η was added to the hopping robot plant model. The perturbed model used for the simulations was then take to be

$$\dot{\psi} = v_1 ,$$

$$\dot{l} = v_2 + \eta ,$$

$$\dot{\theta} = -\frac{m_l(1+l)^2}{1+m_l(1+l)^2} \cdot v_1 + \eta ,$$

where η is the hypothesized zero-mean pseudo white noise.

The required leg's angular position was set to be $\Theta = -2\pi/3$ [rad] $\cong -2.095$ [rad]. The leg's length was taken as $L = 1$ [mt], so that the desired final length was $L/3 = 0.333$ [mt]. The controller constants were chosen as

$$\lambda = 5\,[\mathrm{s}^{-1}], \quad W_1 = 8\,[\mathrm{rad/s}^2], \quad W_2 = 0.2, \quad \beta_1 = 1.4\,[\mathrm{s/rad}], \quad \beta_2 = 60,$$

$$\tau_{\min 1} = 0.05, \quad \tau_{\min 2} = 0.05, \quad T = 0.1\,[\mathrm{s}].$$

The value of p was set to be $p = 1/6$ for both TPWM signals. Figure 4 shows the TPWM controlled trajectory of the perturbed evolution of the angle θ and the leg's length l. These variables are seen to approach the prescribed equilibrium values, respectively, in a stable manner and achieving the desired performance in finite time. The original control input signals v_1 and v_2 are also shown in this figure. The control v_1 is the output of a dynamical TPWM compensator and, therefore, exhibits a "smoother" behavior than the control signal v_2, which is the output of a static feedback controller. The signal v_2 thus exhibits a characteristic pulsed behavior.

The body angle ψ is seen to exhibit a stable response towards an arbitrary equilibrium point. Figure 4 also shows a sample of the perturbation signal η. The peak-to-peak amplitude bound for this signal was allowed to be 5.

3.3 Regulation of a sliding mass position on an inverted pendulum

The example presented in this section is treated also in Fliess et al. (1993 b), where classical linear controllers of the proportional and proportional derivative type are proposed.

Consider a rectilinear link, free to rotate on a vertical plane around a fixed point, on which a known mass M slides without friction. The distance from the mass position on the link to the center of rotation of the link is denoted by q, the angular position of the link is u measured with respect to the vertical (see Fig. 5). It is desired to regulate the position of the sliding mass to a constant value L. The control input is assumed to be the angular velocity variable $v = \dot{u}$.

A model for the system (see Fliess et al., 1993 b; Bressan and Rampazzo, 1993) is given by

$$\left.\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\cos(u) + qv^2 \\ \dot{u} &= v \end{aligned}\right\}, \tag{3.17}$$

where it is assumed that the values of the mass M and of the gravity acceleration have been normalized to unit values. System (3.1) *is not* a differentially flat

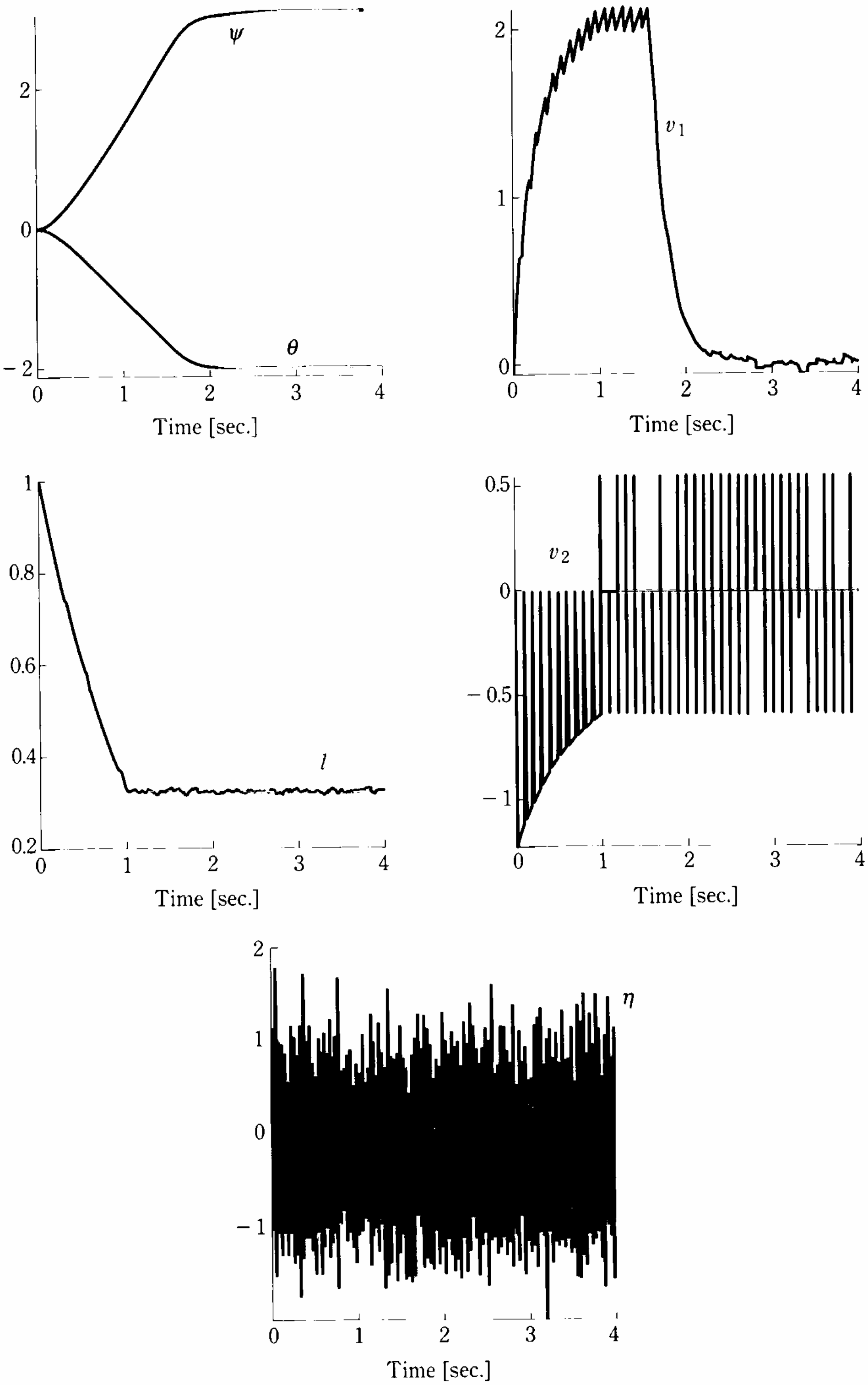

Fig. 4. TPWM regulated trajectories, control input signals and perturbation noises for the hopping robot.

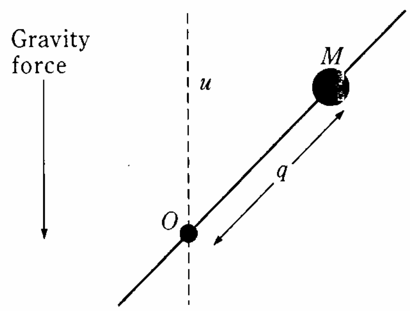

Fig. 5. Sliding mass on inverted pendulum.

system. However, as it has been already shown in Fliess et al. (1993 b), the high-frequency vibratory control input defined by

$$v = v_2 + v_1 \sin\left(\frac{t}{\varepsilon}\right), \tag{3.18}$$

where ε is a small real number, yields an *average* description of the system, given by

$$\left.\begin{aligned} \dot{\bar{q}} &= \bar{p} \\ \dot{\bar{p}} &= -\cos(\bar{u}) + \bar{q}\left(v_2^2 + \frac{1}{2}v_1^2\right) \\ \dot{\bar{u}} &= v_2 \end{aligned}\right\}. \tag{3.19}$$

The average system (3.19) is now a multivariable nonlinear system, which is also differentially flat. Indeed, if one sets as linearizing output coordinates the average values of the mass position and the angular link position $(y_1, y_2) = (\bar{q}, \bar{u})$, then, all variables in the average system are expressible as differential functions of these coordinates. Indeed,

$$\left.\begin{aligned} \bar{p} &= \dot{y}_1 \\ v_2 &= \dot{y}_2 \\ v_1 &= \sqrt{2\left(\frac{\ddot{y}_1 + \cos(y_2)}{y_1} - (\dot{y}_2)^2\right)} \end{aligned}\right\}. \tag{3.20}$$

The essential order of the linearizing output y_1 is 2, while that of the linearizing output y_1 is just one. The essential orders thus coincide with the row relative degrees. The system is, therefore, decouplable and linearizable by means of endogenous static state feedback.

The desired equilibrium point for the average system (3.19) is given by

$$\bar{q} = L, \quad \bar{p} = 0, \quad v_2 = 0, \quad v_1 = \sqrt{\frac{2\cos(\bar{U})}{L}},$$

where $\bar{U}$ is an arbitrarily desired angular position for the link.

The regulated outputs z_1 and z_2 are here taken to coincide with the linearizing outputs y_1 and y_2. In order to obtain a TPWM feedback control strategy for the auxiliary control inputs v_1 and v_2, one defines the error coordinates s_1 and s_2 as $s_1 = z_1 - L = y_1 - L$ and $s_2 = z_2 - \bar{U} = y_2 - \bar{U}$. The auxiliary error variables σ_1 and σ_2 are defined, in this case, as $\sigma_1 = \dot{s}_1 + \lambda_1 s_1$ and $\sigma_2 = s_2$. One then imposes on these auxiliary variables the following scalar TPWM closed loop dynamics:

$$\left.\begin{aligned}\dot{\sigma}_1 &= -W_1\,\mathrm{TPWM}\sigma_1\\ \dot{\sigma}_2 &= -W_2\,\mathrm{TPWM}\sigma_2\end{aligned}\right\}. \tag{3.21}$$

The TPWM regulated trajectories of the auxiliary error coordinates, σ_1 and σ_2, are thus guaranteed to be stabilized to an arbitrary small neighborhood of zero, where they exhibit a stable oscillatory behavior whose amplitudes are inversely related to the magnitude limitations imposed on the rates of the velocity control variables v_1 and v_2.

In terms of the error variables, the imposed multivariable closed loop dynamics are given by

$$\left.\begin{aligned}\ddot{s}_1 &= -\lambda_1\dot{s}_1 - W_1\,\mathrm{TPWM}(\dot{s}_1 + \lambda_1 s_1)\\ \dot{s}_2 &= -W_2\,\mathrm{TPWM}(s_2)\end{aligned}\right\}. \tag{3.22}$$

The TPWM feedback control laws v_1 and v_2 are then immediately obtained

$$\left.\begin{aligned}v_1 &= \sqrt{2\left\{\frac{-\lambda_1\dot{y}_1 - W_1\,\mathrm{TPWM}[\dot{y}_1 + \lambda_1(y_1 - L)] + \cos(y_2)}{y_1} - [W_2\,\mathrm{TPWM}(y_2 - \bar{U})]^2\right\}}\\ v_2 &= -W_2\,\mathrm{TPWM}(y_2 - \bar{U})\end{aligned}\right\}. \tag{3.23}$$

In terms of the average state coordinates of the system, the required TPWM feedback is then given by

$$\left.\begin{aligned}v_1 &= \sqrt{2\left\{\frac{-\lambda_1\bar{p} - W_1\,\mathrm{TPWM}[\bar{p} + \lambda_1(\bar{q} - L)] + \cos(\bar{u})}{q} - [W_2\,\mathrm{TPWM}(\bar{u} - \bar{U})]^2\right\}}\\ v_2 &= -W_2\,\mathrm{TPWM}(\bar{u} - \bar{U})\end{aligned}\right\}. \tag{3.24}$$

3.4 Simulation results The performance of the average feedback controller, given by Eq. (3.24) was tested on a perturbed version of the given system (3.17). The perturbed model was taken to be

$$\left.\begin{aligned}\dot{q} &= p\\ \dot{p} &= -\cos(u) + qv^2 + \eta\\ \dot{u} &= v\end{aligned}\right\}, \tag{3.25}$$

where η was synthesized as a computer generated pseudo white noise process.

The TPWM feedback control policy (3.24) was implemented by directly feeding back the actual state variables q, p and u rather than the averaged variables.

The desired equilibrium value for the mass position was set to be $L = 1$, and the desired link angular position $\bar{U} = 1$. The design values of the constant parameters specifying the static TPWM control laws were chosen as

$$W_1 = 0.3, \quad W_2 = 0.3, \quad \beta_1 = 20, \quad \lambda = 0.83, \quad \beta_2 = 100,$$

$$\tau_{\min 1} = 0.06, \quad \tau_{\min 2} = 0.06, \quad T = 0.1.$$

The value of p was chosen, for both controllers, as $p = 1/6$.

In Fig. 6, the feedback regulated mass position q, its linear velocity p, and the link angular position trajectories u are shown to converge to a small vicinity of their respective pre-specified equilibrium values. The high frequency control input v, including TPWM components, is also shown in this figure along with a sample trajectory of the perturbation signal η. This computer generated signal was specified with a peak-to-peak amplitude bound of 2 units.

4. Conclusions

In this article, a new class of pulsed feedback control strategies, without steplike discontinuities, has been proposed for the regulation of nonlinear multivariable systems. The feedback technique, addressed as "trapezoidal pulse width modulation" control, has been shown to be suitable for the regulation of a large class of nonlinear multivariable systems with limited control input rates.

A fundamental result on the stability features of the trapezoidal pulsed regulation, of a single integrator scalar system, provides the basis for suitable error stabilization in more complex systems, such as multivariable nonlinear systems.

The proposed scheme is specially suitable for the solution of stabilization and tracking, problems defined on nonlinear mechanical systems in which angular, or linear, velocity variables are naturally regarded as control input variables to the system. The limited slope assumption on the generated feedback input signals corresponds to magnitude acceleration constraints and, hence, it also naturally handles realistic torque, or force, magnitude limitations. The proposed pulsed feedback controller also represents a "smoothed" approximation strategy for traditional pulse-width modulation feedback schemes of discontinuous nature.

Due to the lack of asymptotic stability features of the fundamental scheme, feedback TPWM regulation can only achieve regulation, to arbitrarily small neighborhoods of pre-specified constant equilibrium points. From a practical viewpoint, however, the precision features of the corresponding regulated position variables are, not surprisingly, quite satisfactory. This is basically due to the averaging effects of the integration of the induced small-amplitude velocity limit cycles.

Two application examples of physical flavor were presented, along with encouraging computer simulations. These included unmodeled digitally generated stochastic perturbation signals.

The important issue of TPWM controller design for parameter uncertain,

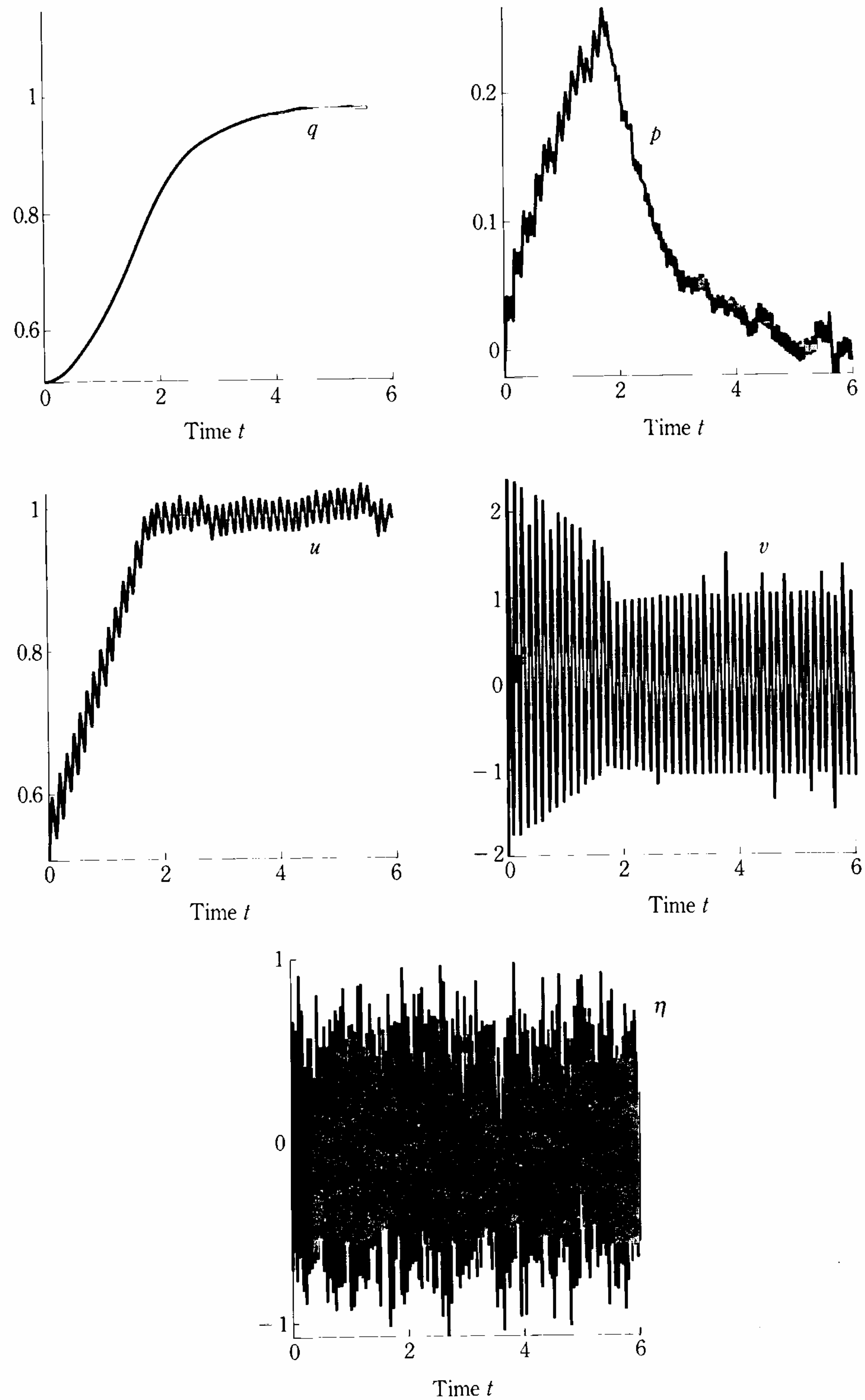

Fig. 6. TPWM regulated trajectories, control input signal and perturbation noises for the sliding mass stabilization problem.

multivariable, flat systems requires the development of suitable adaptive feedback control techniques. This area requires further developments, and it certainly constitutes an interesting topic for further research.

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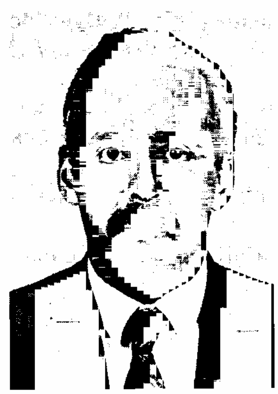

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