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On the robust design of sliding observers for linear systems*

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Abstract: The restrictive character of well-known structural constraints, related to the matching conditions, for the sliding mode feedforward state reconstruction problem in linear, time-invariant, perturbed systems is critically reexamined from a new perspective. It is shown that, in generalized state space coordinates, a matched canonical state space realization exists which always allows discontinuous asymptotic stabilization of the observation error dynamics. The well-known structural conditions thus become largely irrelevant and robust asymptotic state estimation is shown to be feasible, for any perturbed observable system, by means of sliding mode observers.

Keywords: Observer theory; sliding modes; generalized state space.

1. Introduction

Sliding mode observation schemes for both linear and nonlinear systems have been of considerable interest in recent times. Discontinuous nonlinear observation schemes based on sliding modes share some of the fundamental robustness and insensitivity properties of sliding mode controllers (see [1]). A fundamental limitation found in the possibilities of sliding mode feedforward regulation of the observation error dynamics is represented by the necessity of satisfying some structural conditions of the 'matching' type. These conditions have been recognized in the work of Utkin [8], Walcott and Żak [9] and Dorling and Zinober [2]. Such structural constraints on the system and the observer have also been linked to *strictly positive real* conditions in [9] and in the work of Watanabe et al. [10]. More recently, a complete Lyapunov stability standpoint for the design of sliding observers, where the mentioned limitations are also apparent, was presented by Edwards and Spurgeon [3].

In this article we take a generalized state space approach to the problem of state reconstruction for any observable, perturbed, linear system. For the sake of simplicity, we constrain ourselves to scalar single output perturbed plants, but our results can easily be generalized to multivariable linear systems.

Using a *generalized controller canonical form*, similar to those developed by Fliess [5], we find that the structure of the perturbation input channels is largely irrelevant. It is shown that the influence of the external perturbations can always be conveniently placed in the range of the discontinuous feedforward output error injection signals and so asymptotic stability of the sliding mode feedforward-regulated observation error trajectories can be guaranteed.

In Section 2 we examine, from a classical state space representation viewpoint, the role of the matching conditions in sliding mode observer design. This section addresses the rather restrictive character of the structural conditions that guarantee the robust reconstruction of the system state vector components. In essence, these conditions imply that the perturbation input distribution map must be in the range of the feedforward output error injection map of the observer. Thus, the freedom in choosing the stability features of the sliding reconstruction error is severely curtailed. If the

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matching condition is not satisfied then the observation error is dependent upon the external perturbations and accurate state reconstruction is unfeasible.

In Section 3, using a matched generalized observer canonical form, based on the input–output description of the given system, we show that the matching conditions can always be satisfied while placing no restrictions on the stabilizability of the feedforward regulated error dynamics. The class of perturbed, time-invariant, linear systems whose state may be asymptotically estimated, in a robust fashion, by means of discontinuous feedforward injection is enlarged to include the whole class of observable linear systems. This result constitutes the 'dual', in a certain sense, to that recently published by Fliess and Messager [6], regarding sliding mode controllers for linear, time-invariant systems. Section 4 contains conclusions and suggestions for further research.

2. The role of matching conditions in sliding mode observers

Here we briefly present the classical approach to sliding mode observer design using the traditional Kalman state variable representation of linear, time-invariant systems. Within this constrained formulation robust observation schemes, using sliding mode observers, are feasible if and only if certain structural conditions are satisfied. The structural conditions restrict the systems input distribution map to be in the range of the observers feedforward output error injection map.

Consider an observable n-dimensional linear system of the form

$$\dot{x} = Ax + bu + \gamma\xi, \qquad y = cx, \tag{2.1}$$

where u and ξ are, respectively, the scalar control input signal and the (bounded) external perturbation input signal. The output y is also assumed to be a scalar quantity. All matrices have the appropriate dimensions. The column vector γ is referred to as the *perturbation input distribution map.*

An asymptotic observer for system (2.1), including an external feedforward compensation signal v, may be proposed as follows:

$$\dot{\hat{x}} = A\hat{x} + bu + h(y - \hat{y}) + \lambda v, \qquad \hat{y} = c\hat{x}. \tag{2.2}$$

The vector h is addressed as the *vector of observer gains* and the column vector λ is the *feedforward injection map.*

The *state reconstruction error*, defined as $e = x - \hat{x}$, obeys the following dynamical behaviour, obtained from equations (2.1) and (2.2):

$$\dot{e} = (A - hc)e + \gamma\xi - \lambda v, \qquad e_y = ce. \tag{2.3}$$

The signal $e_y = y - \hat{y}$ is the *output reconstruction error.*

Due to the observability assumption on the system, there always exist a gain vector h which assigns any arbitrary, prespecified, real-line symmetric set of n complex eigenvalues to the matrix $(A - hc)$.

The robust sliding mode observer design problem consists in specifying a vector of observer gains h, a feedforward injection map λ and a discontinuous feedforward injection policy v, based solely on output reconstruction error measurements e_y, such that the reconstruction error dynamics (2.3) is guaranteed to exhibit asymptotically stable behaviour to zero, inspite of all possible bounded values of the external perturbation input signal ξ.

Consider the time derivative of the output reconstruction error signal

$$\begin{aligned}\dot{e}_y &= c(A - hc)e + c\gamma\xi - c\lambda v \\ &= cAe - che_y + c\gamma\xi - c\lambda v.\end{aligned} \tag{2.4}$$

We assume, without loss of generality, that the quantity $c\lambda$ is nonzero and positive.

Let the perturbation input ξ be bounded in absolute value by a constant $M > 0$. Let also W be a positive scalar constant. Then a discontinuous feedforward input v of the form

$$v = W \operatorname{sign} e_y \tag{2.5}$$

is seen to create a *sliding regime* on a bounded region of the reconstruction error space. Such a region would be, necessarily, contained in the hyperplane $e_y = 0$.

As may be easily verified, on the region characterized by $e_y = 0$ and $|cAe| + |c\gamma\xi| \le (c\lambda)W$, the above choice of the feedforward signal v results in the *sliding condition* $e_y\dot{e}_y < 0$ (see [8]). Using the known bound M on the signal ξ, such a region is simply expressed as

$$|cAe| \leq (c\lambda)W - |c\gamma|M.$$

Thus, the discontinuous feedforward policy (2.5) drives the output observation error e_y to zero in finite time, irrespective of both the initial conditions for e and the values of the perturbation input ξ.

The *ideal sliding behaviour* (see [8]) of the state reconstruction error signal e is obtained from the following *invariance conditions*:

$$e_y = 0, \qquad \dot{e}_y = 0. \tag{2.6}$$

Conditions (2.6) imply a 'virtual' perturbation-dependent value of the output error feedforward injection signal v, which is addressed as the *equivalent feedforward* signal, and henceforth denoted by v_{eq}. This 'virtual' feedforward signal is useful in describing the average behaviour of the error system (2.3) when regulated by the feedforward signal v. Using (2.6) and (2.4) one readily obtains:

$$v_{eq} = \frac{cAe}{c\lambda} + \frac{c\gamma}{c\lambda}\xi. \tag{2.7}$$

Substituting the equivalent feedforward signal expression in (2.7) into the error equation (2.3), one obtains the following (redundant) *ideal sliding error dynamics*, taking place on a bounded region of $e_y = 0$:

$$\dot{e} = \left(I - \frac{\lambda c}{c\lambda}\right)Ae + \left(I - \frac{\lambda c}{c\lambda}\right)\gamma\xi. \tag{2.8}$$

Note that the matrix $S = [I - (\lambda c)/(c\lambda)]$ is a *projection operator* along the range space of λ onto the null space of c.

Thus, in general, the ideal sliding error dynamics will be dependent upon the perturbation signal ξ. However, under a structural constraint on the distributions maps γ and λ, known as the *matching conditions* it is possible to obtain an ideal sliding error dynamics (2.8) which is free of the influence of the perturbation signal ξ. One may establish that the ideal sliding error dynamics (2.8) are independent of ξ if and only if

$$\gamma = \mu\lambda \tag{2.9}$$

for some constant scalar μ. In other words, the sliding error dynamics is independent of ξ if and only if the range spaces of the maps γ and λ coincide.

The proof of this result is as follows. If the matrix feeding the perturbations ξ into the (average) sliding error equations (2.8) is identically zero, then no perturbations are ever present in the error system. This would require the following identity to hold:

$$\left(I - \frac{\lambda c}{c\lambda}\right)\gamma = 0, \tag{2.10}$$

which simply means that γ may be expressed as $\gamma = \mu\lambda$, where $\mu = (c\gamma)/(c\lambda)$. On the other hand, if γ is a column vector of the form $\gamma = \mu\lambda$ then

$$\left(I - \frac{\lambda c}{c\lambda}\right)\gamma = \left(\lambda - \frac{\lambda c}{c\lambda}\lambda\right)\mu = (\lambda - \lambda)\mu = 0.$$

If the matching condition (2.9) is satisfied then the reconstruction error dynamics is specified by the following constrained dynamics:

$$\dot{e} = \left(I - \frac{\gamma c}{c\gamma}\right)Ae, \qquad e_y = ce = 0. \tag{2.11}$$

The resulting reduced-order unforced error dynamics obtained from (2.11) must be asymptotically stable. As may be easily seen, such a stability property is a structural property linked to the particular form of the maps A, c and γ. It can be shown that the asymptotic stability of (2.11) can be guaranteed if a *strictly positive real condition*, associated to the constrained system, is satisfied (see also [9]).

3. A generalized matched canonical form for perturbed linear systems

Suppose a linear system of the form (2.1) is given such that the matching condition discussed in the previous section does not yield an asymptotically stable reduced error system (2.11). By resorting to an input–output description of the perturbed system, one can find a canonical state space realization, in generalized state coordinates, which *always* satisfies the matching condition of the form (2.9) while producing a prespecified asymptotically stable constrained error dynamics. The state of the matched canonical realization can, therefore, be always robustly estimated.

By means of straightforward state vector elimination, the input–output representation of the linear, time-invariant, perturbed system (2.1) can be written in the form

$$y^{(n)} + k_n y^{(n-1)} + \cdots + k_2 \dot{y} + k_1 y = \beta_0 u + \beta_1 \dot{u} + \cdots + \beta_m u^{(m)} + \gamma_0 \xi + \gamma_1 \dot{\xi} + \cdots + \gamma_q \xi^{(q)}, \tag{3.1}$$

where ξ represents the bounded external perturbation signal and the integer q satisfies, without loss of generality, $q \leq n-1$.

The *generalized matched observer canonical form* (GOCF) of the above system is given by the following generalized state representation model (see [5]):

$$\begin{aligned}
\dot{\chi}_1 &= -k_1 \chi_n + \beta_0 u + \beta_1 \dot{u} + \cdots + \beta_m u^{(m)} + \lambda_1 \eta, \\
\dot{\chi}_2 &= \chi_1 - k_2 \chi_n + \lambda_2 \eta, \\
&\vdots \\
\dot{\chi}_{n-1} &= \chi_{n-2} - k_{n-1} \chi_n + \lambda_{n-1} \eta, \\
\dot{\chi}_n &= \chi_{n-1} - k_n \chi_n + \eta, \\
y &= \chi_n,
\end{aligned} \tag{3.2}$$

where η is an 'auxiliary' perturbation signal, modelling the influence of the external signal ξ on every equation of the proposed system realization.

The relation existing between the signal η and its generating signal ξ, is obtained by computing the input–output description of system (3.2) in terms of the perturbation input η. The input–output description of the hypothesized model (3.2) is then compared with that obtained for the original system (3.1). This procedure results in a scalar linear, time-invariant, differential equation for η which accepts as an input the signal ξ.

The models presented below constitute realizations of such an input–output description, according to the order q of the differential polynomial for ξ in (3.1).

For $q < n-1$, the perturbation input η is obtained as the output of the following dynamical system:

$$\begin{aligned}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= z_3, \\
&\vdots \\
\dot{z}_{n-1} &= -\lambda_1 z_1 - \lambda_2 z_2 - \cdots - \lambda_{n-1} z_{n-1} + \xi, \\
\eta &= \gamma_0 z_1 + \gamma_1 z_2 + \cdots + \gamma_{q-1} z_q.
\end{aligned} \tag{3.3}$$

For $q = n-1$ the state space realization corresponding to (3.3) simply reads

$$\begin{aligned}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= z_3, \\
&\vdots \\
\dot{z}_{n-1} &= -\lambda_1 z_1 - \lambda_2 z_2 - \cdots - \lambda_{n-1} z_{n-1} + \xi, \\
\eta &= (\gamma_0 - \gamma_{n-1}\lambda_1) z_1 + (\gamma_1 - \gamma_{n-1}\lambda_2) z_2 \\
&\quad + \cdots + (\gamma_{n-2} - \gamma_{n-1}\lambda_{n-1}) z_{n-1} + \gamma_{n-1} \xi.
\end{aligned} \tag{3.4}$$

Assumption 2.1. Suppose the components of the auxiliary perturbation distribution channel map $\lambda_1, \ldots, \lambda_{n-1}$, in equation (3.2), are such that the following polynomial in the complex variable s is Hurwitz:

$$p_r(s) = s^n + \lambda_{n-1} s^{n-2} + \cdots + \lambda_2 s + \lambda_1. \tag{3.5}$$

Equivalently, Assumption 2.1 implies that the output η of system (3.3) generating the auxiliary perturbation η, or, alternatively, that associated to system (3.4), is a *bounded signal* for every bounded external perturbation signal ξ. If, for instance, ξ satisfies $|\xi| < N$. Then, given N, the signal η satisfies $|\eta| \leq M$ for some positive constant M. An easy to compute, although conservative, estimate for M is given by $M = \sup_{\omega \in [0,\infty)} |G(j\omega)| N$, where $G(s)$ is the Laplace transfer function relating η to ξ in the complex frequency domain.

Remark. It should be stressed that the purpose of having a state space model for the auxiliary perturbation signal η, accepting as a forcing input the signal ξ, is the estimation, through η, of the influence of ξ on the proposed state realization (3.2) of the original system (2.1).

An observer for the system realization (3.2) is proposed as follows:

$$\begin{aligned}
\dot{\hat{\chi}}_1 &= -k_1 \hat{\chi}_n + \beta_0 u + \beta_1 \dot{u} + \cdots + \beta_m u^{(m)} \\
&\quad + h_1 (y - \hat{y}) + \lambda_1 v, \\
\dot{\hat{\chi}}_2 &= -k_2 \hat{\chi}_n + \hat{\chi}_1 + h_2 (y - \hat{y}) + \lambda_2 v, \\
&\vdots \\
\dot{\hat{\chi}}_{n-1} &= -k_{n-1} \hat{\chi}_n + \hat{\chi}_{n-2} + h_{n-1} (y - \hat{y}) + \lambda_{n-1} v, \\
\dot{\hat{\chi}}_n &= -k_n \hat{\chi}_n + \hat{\chi}_{n-1} + h_n (y - \hat{y}) + v, \\
\hat{y} &= \hat{\chi}_n.
\end{aligned} \tag{3.6}$$

Note that we have purposefully chosen exactly the same output error feedforward distribution map for the signal v, as that corresponding to the auxiliary perturbation input signal η in (3.2). As a consequence, our proposed canonical form (3.2) for the system is always matched to the observer. The crucial point is that the matched error feedforward distribution map can always be conveniently chosen to guarantee asymptotic stability of the ideal sliding error dynamics.

Use of (3.6) results in the following feedforward regulated reconstruction error dynamics:

$$\begin{aligned} \dot{\varepsilon}_1 &= -(k_1 + h_1)\varepsilon_n + \lambda_1(\eta - v), \\ \dot{\varepsilon}_2 &= \varepsilon_1 - (k_2 + h_2)\varepsilon_n + \lambda_2(\eta - v), \\ &\vdots \\ \dot{\varepsilon}_{n-1} &= \varepsilon_{n-2} - (k_{n-1} + h_{n-1})\varepsilon_n + \lambda_{n-1}(\eta - v), \\ \dot{\varepsilon}_n &= \varepsilon_{n-1} - (k_n + h_n)\varepsilon_n + (\eta - v), \\ \varepsilon_y &= \varepsilon_n, \end{aligned} \tag{3.7}$$

where ε_i represents the state estimation error components $\chi_i - \hat{\chi}_i$ for $i = 1, \ldots, n$.

In order to have a reconstruction error transient response associated to a preselected nth order characteristic polynomial, such as

$$p(s) = s^n + \alpha_n s^{n-1} + \cdots + \alpha_2 s + \alpha_1, \tag{3.8}$$

the gains h_i, $i = 1, \ldots, n$, should be appropriately chosen as $h_i = \alpha_i - k_i$, $i = 1, \ldots, n$.

The feedforward output error injection signal v is chosen as a discontinuous regulation policy:

$$v = W \operatorname{sign} \varepsilon_y = W \operatorname{sign} \varepsilon_n, \tag{3.9}$$

where W is a positive constant. From the last equation in (3.7), we see that, for a sufficiently large gain W, the proposed choice of the feedforward signal v results in a sliding regime on a region properly contained in the set expressed by

$$\varepsilon_n = 0, \quad |\varepsilon_{n-1}| \leq W - M. \tag{3.10}$$

The equivalent feedforward signal v_{eq} is obtained from the *invariance conditions* (see also [1])

$$\varepsilon_n = 0, \qquad \dot{\varepsilon}_n = 0. \tag{3.11}$$

One obtains from (3.11) and the last of (3.7),

$$v_{eq} = \eta + \varepsilon_{n-1}. \tag{3.12}$$

The equivalent feedforward signal is, generally speaking, dependent upon the perturbation signal η. It should be remembered that the equivalent feedforward signal v_{eq} is a *virtual* feedforward action that need not be synthesized in practice, but one which helps to establish the salient features of the *average* behaviour of the sliding mode regulated observer.

The resulting dynamics governing the evolution of the error system on the sliding region is then ideally described by

$$\begin{aligned} \dot{\varepsilon}_1 &= -\lambda_1 \varepsilon_{n-1}, \\ \dot{\varepsilon}_2 &= \varepsilon_1 - \lambda_2 \varepsilon_{n-1}, \\ &\vdots \\ \dot{\varepsilon}_{n-1} &= \varepsilon_{n-2} - \lambda_{n-1}\varepsilon_{n-1}, \\ \varepsilon_y &= \varepsilon_n = 0. \end{aligned} \tag{3.13}$$

The resulting ideal sliding error dynamics exhibits, in a natural manner, a feedforward error injection structure of the 'auxiliary output error' signal ε_{n-1}, through the design gains $\lambda_1, \ldots, \lambda_{n-1}$. As a result, the roots of the characteristic polynomial in equation (3.5) determining the behaviour of the homogeneous, reduced-order, system (3.13), are completely determined from a suitable choice of the components of the feedforward vector, $\lambda_1, \ldots, \lambda_{n-1}$.

An asymptotically stable behaviour to zero of the estimation error components $\varepsilon_1, \ldots, \varepsilon_{n-1}$ is therefore achievable as the output observation error ε_n undergoes a sliding regime on the relevant portion of the 'sliding surface' $\varepsilon_n = 0$. The states of the estimator (3.6) are then seen to converge asymptotically towards the corresponding components of the state vector of the system realization (3.2).

The characteristic polynomial (3.5) of the reduced-order observation error dynamics (3.13) entirely coincides with that of the transfer function relating the auxiliary perturbation model signal η to the actual perturbation input ξ. Hence, appropriate choice of the design parameters $\lambda_1, \ldots, \lambda_{n-1}$ not only guarantees asymptotic stability of the sliding error dynamics, but also ensures boundedness of the auxiliary perturbation signal η, for any given bounded external perturbation ξ.

Remark. In general, the observed states of the matched realization are different from those of the particular realization (2.1), originally given for the system. The state χ in (3.2) may even be devoid of any physical meaning. A linear relationship can always be established between the originally given state vector x of system (2.1) and the state χ, reconstructed from the canonical form (3.2). However, generally speaking, such a relationship entitles a *perturbation-dependent* state coordinate transformation and cannot be used in practice. Nevertheless, it should be emphatically stressed that the observer state vector $\hat{\chi}$, which asymptotically converges to the state vector χ of the matched canonical realization, is as good as any other state vector for the purposes of any kind of feedback control.

4. Conclusions

In this article we have shown that by using generalized state space representations of linear systems, in observer canonical form, the methodology to be used for asymptotic state reconstruction via the use of sliding observers becomes particularly clear.

The adopted approach also allows us to establish that structural conditions of the matching type, relating the perturbation input distribution channel and the feedforward injection map, are largely irrelevant for robust state reconstruction using sliding observers. In other words, the class of linear systems for which robust sliding mode state reconstruction can be obtained, independently of any *matching conditions*, comprises the entire class of observable linear systems. This remark is of particular practical interest when the designer has the freedom of proposing a convenient state space representation for an unmatched system. This is in total agreement with the corresponding result found in [6], regarding the robustness of the sliding mode control of perturbed controllable linear systems expressed in *generalized observability canonical form.*

Sliding mode observer theory for linear systems may also be examined from an algebraic viewpoint using *module theory* (see [4]). The conceptual advantages of using a module theoretic approach to sliding mode control were recently addressed by Fliess and Sira-Ramírez [7]. The module theoretic approach can give further generalizations and insights related to the results presented in this article.

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