A Sliding Mode Strategy for Adaptive Learning in Adalines

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***Abstract*—A dynamical sliding mode control approach is proposed for robust adaptive learning in analog Adaptive Linear Elements (Adalines), constituting basic building blocks for perceptron-based feedforward neural networks. The zero level set of the learning error variable is regarded as a sliding surface in the space of learning parameters. A sliding mode trajectory can then be induced, in finite time, on such a desired sliding manifold. Neuron weights adaptation trajectories are shown to be of continuous nature, thus avoiding bang–bang weight adaptation procedures. Sliding mode invariance conditions determine a least squares characterization of the adaptive weights average dynamics whose stability features may be studied using standard time-varying linear systems results. Robustness of the adaptative learning algorithm, with respect to bounded external perturbation signals, and measurement noises, is also demonstrated. The article presents some simulation examples dealing with applications of the proposed algorithm to forward and inverse plant dynamics identification.**

I. Introduction

THE adjustment of learning parameters in perceptron based feedforward neural networks has been mainly explored form a discrete–time viewpoint. The celebrated Widrow-Hoff *Delta Rule* [10] constitutes a least mean square learning error minimization algorithm by which an asymptotically stable linear convergence dynamics is imposed on the underlying discrete–time error dynamics. Using quasi–sliding mode control ideas [7] a modification of the Delta Rule was proposed by Sira-Ramírez and Żak in [8] and in [11], whereby a switching weight adaptation strategy is shown to also impose a discrete–time asymptotically stable linear learning error dynamics. This algorithm is at the basis of recently proposed identification and control schemes, based on feedforward neural networks [1], [4]. To our knowledge, design of learning strategies in adaptive perceptrons, from the viewpoint of sliding mode control in continuous time, has not been addressed in the existing literature. However, the relevance of ordinary differential equations with discontinuous right hand sides was analyzed in the work of Li *et al.*, in [5], in the context of Analog Neural Networks of the Hopfield type with infinite gain nonlinearities. In that work, it is established under what circumstances sliding mode trajectories do not appear in such a class of neurons.

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In this paper, a continuous time sliding mode control approach is proposed for the robust adaptation of variable weights in Adalines, so that its scalar output variable tracks a bounded reference signal with a bounded first order time derivative. The zero level set of the learning error variable is regarded as the sliding surface coordinate function and a discontinuous law of adaptive weight variation is proposed which induces, in finite time, a sliding motion which robustly sustains the zero error condition. The sliding mode controlled weight adaptation trajectories are shown to be continuous, rather than bang–bang signals. The ideal sliding mode behavior, provides with a least squares characterization of the dynamical features of the average evolution of the vector of adaptive weights. The requirements for the stability of the average weight dynamics establishes essential connections with adaptive control issues, such as the persistency of excitation conditions.

A unique feature of the sliding mode approach lies in the enhanced insensitivity of the proposed adaptative learning algorithm, with respect to bounded external perturbation signals and measurement noises. For the case of performance under the influence of neuron input, and output, measurment noises, some of the geometric features of the proposed sliding mode algorithm are shown to be naturally linked to the well-known matching conditions.

Section II contains some definitions, assumptions and derivations of the main charcteristics of a sliding mode control approach to weight adaptation in Adalines. In this section, the robustness of the algorithm, with respect to bounded external perturbation inputs, and bounded measurement noises, is also demonstrated along with a derivation of the required matching condition. Section III contains some basic examples of relevant significance in the potential applications of the proposed adaptive learning strategy in automatic control applications. The examples include, both, identification of forward and inverse dynamics of unknown, externally perturbed, nonlinear plants. Section IV contains the conclusions.

II. A Sliding Mode Control Approach to Weight Adaptation in Adalines

Definitions and Basic Assumptions

Consider the perceptron model depicted in Fig. 1 where $x(t) = (x_1(t), \dots, x_n(t))$ represents a vector of bounded

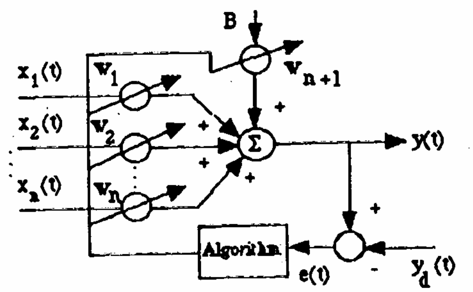

Fig. 1. Adaptive linear element.

time-varying inputs, assumed also to exhibit bounded time derivatives, i.e.,

$$\| x(t) \| = \sqrt{x_1^2(t) + \ldots + x_n^2(t)} \leq V_x \ \forall t; \quad \| \dot{x}(t) \| = \sqrt{\dot{x}_1^2(t) + \ldots + \dot{x}_n^2(t)} \leq V_{\dot{x}} \ \forall t \tag{1}$$

where V_x and $V_{\dot{x}}$ are known positive constants.

We denote by $\tilde{x}(t)$ the vector of augmented inputs, which includes a constant input of value $B > 1$, affecting the bias, or threshold weight w_{n+1} in the perceptron model, i.e.,

$$\tilde{x}(t) = \text{col}(x_1(t), \ldots, x_n(t), B) = \text{col}(x(t), B). \tag{2}$$

Remark 2.1: The scalar product $\tilde{x}^T(t)\tilde{x}(t) = B^2 + x^T(t)x(t) = B^2 + \| x(t) \|^2$ is bounded away from zero for all times.

The vector $\omega(t) = \text{col}(\omega_1(t), \ldots, \omega_n(t))$ represents the set of time-varying weights. It will be assumed that, due to physical constraints, the magnitude of the vector $\omega(t)$ is bounded $\| \omega(t) \| \leq W \ \forall t$, for some constant W. We also define the vector of augmented weights by including the bias weight component

$$\tilde{\omega}(t) = \text{col}(\omega_1(t), \ldots, \omega_n(t), \omega_{n+1}(t)) = \text{col}(\omega(t), \omega_{n+1}(t)). \tag{3}$$

Similarly, $\tilde{\omega}(t)$ is assumed to be bounded at each instant of time t by means of

$$\| \tilde{\omega}(t) \| = \sqrt{\omega_1^2(t) + \cdots + \omega_n^2(t) + \omega_{n+1}^2(t)} \leq \tilde{W} \ \forall t \tag{4}$$

for some constant $\tilde{W}$.

The scalar signal $y_d(t)$ represents the time-varying desired output of the perceptron. It will be assumed that $y_d(t)$ and $\dot{y}_d(t)$ are also bounded signals, i.e.,

$$| y_d(t) | \leq V_y \forall t \quad | \dot{y}_d(t) | \leq V_{\dot{y}} \ \forall t. \tag{5}$$

The output signal $y(t)$ is a scalar quantity defined as:

$$y(t) = \sum_{i=1}^{n} \omega_i(t)x_i(t) + \omega_{n+1}(t) = \omega^T(t)x(t) + \omega_{n+1}(t)\, B = \tilde{\omega}^T(t)\tilde{x}(t). \tag{6}$$

We define the learning error $e(t)$ as the scalar quantity obtained from

$$e(t) = y(t) - y_d(t). \tag{7}$$

Problem Formulation and Main Results

Using the theory of Sliding Mode Control of Variable Structure Systems (see [9]) we propose to consider the zero value of the learning error coordinate $e(t)$ as a time-varying sliding surface, i.e.,

$$s(e(t)) = e(t) = 0. \tag{8}$$

Condition (8) guarantees that the perceptron output $y(t)$ coincides with the desired output signal $y_d(t)$ for all time $t > t_h$ where t_h is addressed as the hitting time.

Definition 2.2: A sliding motion is said to exist on a sliding surface $s(e(t)) = e(t) = 0$, after time t_h, if the condition $s(t)\dot{s}(t) = e(t)\dot{e}(t) < 0$ is satisfied for all t in some nontrivial semiopen subinterval of time of the form $[t, t_h) \subset (-\infty, t_h)$.

Basic Problem Formulation

It is desired to devise a dynamical feedback adaptation mechanism, or adaptation law, for the augmented vector of variable weights $\tilde{\omega}(t)$ such that the sliding mode condition of definition 1 is enforced.

Zero Adaptive Learning Error in Finite Time: Let "sign $e(t)$" stand for the signum function, defined as

$$\text{sign } e = \begin{cases} +1 & \text{for } e(t) > 0 \\ 0 & \text{for } e(t) = 0 \\ -1 & \text{for } e(t) < 0 \end{cases} \tag{9}$$

We then have the following result.

Theorem 2.3: If the adaptation law for the augmented weight vector $\tilde{\omega}(t)$ is chosen as

$$\dot{\tilde{\omega}}(t) = -\left(\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right) k \text{ sign } e(t) = -\left(\frac{\begin{bmatrix} x(t) \\ B \end{bmatrix}}{B^2 + x^T(t)x(t)}\right) k \text{ sign } e(t) \tag{10}$$

with k being a sufficiently large positive design constant satisfying

$$k > \tilde{W}V_{\dot{x}} + V_{\dot{y}} \tag{11}$$

then, given an arbitrary initial condition $e(0)$, the learning error $e(t)$ converges to zero in finite time t_h estimated by

$$t_h \leq \frac{| e(0) |}{k - \tilde{W}V_{\dot{x}} - V_{\dot{y}}} \tag{12}$$

and a sliding motion is sustained on $e = 0$ for all $t > t_h$.

Proof: See the Appendix.

Remark 2.4: Note that the proposed dynamical feedback adaptation law for the vector of weights in (10) results in a continuous regulated evolution of the vector of variable weights $\tilde{\omega}(t)$. The discontinuous feedback strategy (10) actually represents a least squares solution, with respect to $\dot{\tilde{\omega}}(t)$ of the following linear time-varying equation

$$\dot{\tilde{\omega}}^T(t)\tilde{x}(t) = -k \text{ sign } [y(t) - y_d(t)] \tag{13}$$

which yields the following suggestive regulated dynamics for the perceptron output signal $y(t)$

$$\dot{y} = \tilde{\omega}^T(t)\dot{\tilde{x}}(t) - k \text{ sign } (y(t) - y_d(t)) \tag{14}$$

where the signal $\tilde{\omega}^T(t)\dot{\tilde{x}}(t)$ acts as a bounded perturbation signal.

Note that if the quantity $\dot{\tilde{x}}(t)$ is measurable, one can obtain a more relaxed variable structure feedback control strategy than the one obtained in (10). Generally speaking, such an adaptive feedback strategy for the variable weights requires smaller design gains k to obtain a corresponding sliding motion. Since such a case is of some practical importance, we summarize its details in the following theorem.

Theorem 2.5: If the adaptation law for the augmented weight vector $\tilde{\omega}(t)$ is chosen as

$$\begin{aligned}\dot{\tilde{\omega}}(t) = &-\left(\frac{\tilde{x}(t)\dot{\tilde{x}}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\tilde{\omega}(t)\\ &-\left(\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right) k \text{ sign } e(t)\end{aligned} \tag{15}$$

with k being a positive design constant satisfying $k > V_{\dot{y}}$, then, given an arbitrary initial condition $e(0)$, the learning error $e(t)$ converges to zero in finite time t_h satisfying

$$t_h \leq \frac{|\, e(0)\,|}{k - V_{\dot{y}}} \tag{16}$$

and a sliding motion is sustained on $e = 0$ for all $t > t_h$.

Proof: See the Appendix.

Remark 2.6: As before, the proposed dynamical feedback adaptation law for the vector of weights in (15) results in a continuous weight evolution. Such a law of variation actually represents a minimum square error solution, with respect to $\dot{\tilde{\omega}}(t)$ of the following linear time-varying equation

$$\begin{aligned}\dot{y} &= \dot{\tilde{\omega}}^T(t)\tilde{x}(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t)\\ &= -k \text{ sign } [y(t) - y_d(t)].\end{aligned} \tag{17}$$

The proposed solution for $\dot{\tilde{\omega}}(t)$ in (17) is, necessarily, aligned with the augmented vector of inputs $\tilde{x}(t)$. Fig. 2 depicts the (instantaneous) geometric features at the basis of the proposed algorithms. The total disregard for the effect of the scalar signal $\dot{y}_d(t)$ in the above adaptation schemes, (10) and (15), arises from the implicit assumption that such a signal is not, generally speaking, measurable in practice. On the contrary, it will be shown in the next section that there is a large class of problems for which $\dot{\tilde{x}}(t)$ may be assumed to be measurable.

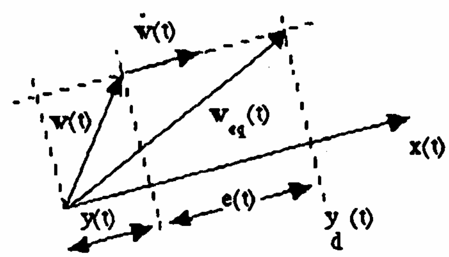

Fig. 2. Geometric interpretation of sliding mode learning algorithm.

Average Features of the Proposed Adaptation Mechanisms

We proceed, as it is customary in sliding mode control theory, to investigate the average behavior of the involved controlled variables. Such an analysis involves the consideration of the following invariance conditions

$$e(t) = 0; \dot{e}(t) = 0 \tag{18}$$

which are ideally satisfied after the sliding motion starts on the sliding surface and is indefinitely sustained thereon. Consideration of such invariance conditions naturally leads to propose the substitution of the discontinuous (bang-bang) input signals by a smooth input signal, known as the equivalent control input. This method has been rigorously validated in [9] as the Method of the Equivalent Control.

Consider the adaptation law (10) and the associated error equation (62) and substitute the discontinuous signal k sign $e(t)$ by its smooth equivalent value $v_{eq}(t)$.

$$\dot{e}(t) = -v_{eq}(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t). \tag{19}$$

The second condition in (18) implies that

$$v_{eq}(t) = \tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t) \forall\, t > t_h. \tag{20}$$

Upon use of (20), a virtual or equivalent variable weight adaptation law can also be associated with the actual discontinuous (bang-bang) policy described by (10). We denote such an equivalent adaptive weight vector by $\tilde{\omega}_{eq}(t)$. One obtains, for all $t > t_h$,

$$\begin{aligned}\dot{\tilde{\omega}}_{eq}(t) = &-\left(\frac{\tilde{x}(t)\dot{\tilde{x}}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\tilde{\omega}_{eq}(t)\\ &+\left(\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\dot{y}_d(t)\end{aligned} \tag{21}$$

i.e., the average variable weight vector trajectory satisfies a linear time–varying vector differential equation with forcing function represented by the bounded function $\dot{y}_d(t)$. Note that $\tilde{\omega}_{eq}(t)$ itself does not, necessarily, lie in the range of $\tilde{x}(t)$. The obtained expression (21) describes the projection, along the range of the vector of augmented inputs $\tilde{x}(t)$, of the derivative of the average regulated evolution of $\tilde{\omega}(t)$. We formalize this t in the following paragraphs after the following related ırk.

ımark 2.7: The first invariance condition $e(t) =$ $)\tilde{x}(t) - y_d(t) = 0$ also leads to some minimum norm ion for the weight adaptation trajectory $\tilde{\omega}(t)$. Such a ion is given by

$$\hat{\omega}_{eq}(t) = \frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}\, y_d(t). \tag{22}$$

solution (22) is, evidently, aligned with $\tilde{x}(t)$ for all t. easy to verify that, in general, the time varying vector $t)$ of (22) is not a solution of the linear time-varying :d differential equation (21), but only its instantaneous :ction onto the range of $\tilde{x}(t)$.

efinition 2.8: A matrix $M(t)$ is said to be a time-varying :ction operator, along the range space $\mathcal{V}(t)$ of a nonzero or function $v(t)$, onto its (instantaneous) perpendicular :rplane, if $M(t)$ satisfies

$M(t)z(t) = 0 \; \forall \; z(t) \in \mathcal{V}(t)$

$M(t)\zeta(t) = \zeta(t) \; \forall \; \zeta(t)$ s.t. $v^T(t)\zeta(t) = 0$

roposition 2.9: Let $\tilde{\mathcal{X}}(t)$ denote the, 1-D time-varying e space of the vector function $\tilde{x}(t)$. The matrix

$$M(t) = \left(I - \frac{\tilde{x}(t)\tilde{x}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right) \tag{23}$$

time-varying projection operator along $\tilde{\mathcal{X}}(t)$ onto its ıntaneous orthogonal hyperplane.

he proof of this proposition is immediate upon verification ıe two given conditions given in definition 2 for $v(t) =$

roposition 2.10: The projection of the vector $\dot{\tilde{\omega}}_{eq}(t)$, onto hyperplane representing the ideal sliding condition $e(t) =$ zero, i.e., the projection of the vector $\tilde{\omega}(t)$, onto such a -varying hyperplane, remains constant.

Proof: See the Appendix.

he same proposition holds valid for the actual (discontin- ;) sliding mode controlled trajectories of the adaptation ghts, given by (10) and (15).

ccording to the results of proposition 2, the equivalent ght adaptation velocity vector satisfies the property : $\dot{\tilde{\omega}}(t) \in$). This is in full accordance with the form of the proposed al discontinuous adaptation law represented by (10) and . This result has an important bearing on the stability ures of the adaptive algorithm. Namely, the boundedness he vector of variable weights, after sliding occurs, is usively dependent upon the variations of the input vector and those of the desired output signal $y_d(t)$.

he following proposition follows readily from the fact that the discontinuous strategy (21), the equivalent input v_{eq} is ained from the invariance condition $\dot{e}(t) = 0$, and the error ation (A.8), as

$$\dot{e}(t) = -v_{eq}(t) - \dot{y}_d(t) \tag{24}$$

$\forall \; t \; > \; t_h$

$$v_{eq}(t) = -\dot{y}_d(t). \tag{25}$$

Proposition 2.11: The equivalent adaptation law corresponding to the discontinuous strategy (15) results in the same expression as in (21).

Requirements for the Stability of the Average Controlled Weights Dynamics

Definition 2.12: (see [2]) Denote by $F(t)$ the time varying matrix

$$F(t) = -\frac{\tilde{x}(t)\dot{\tilde{x}}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}. \tag{26}$$

The differential equation $\dot{\tilde{\omega}}_{eq}(t) = F(t)\, \tilde{\omega}_{eq}(t)$ is said to be uniformly stable if there exists a positive constant γ such that, for all t_0 and all $t > t_0$, the state transition matrix $\Phi(t, t_0)$, corresponding to the matrix $F(t)$, satisfies

$$\| \Phi(t, t_0) \| < \gamma. \tag{27}$$

This definition allows us to formulate the following propostion

Proposition 2.13: Suppose the system $\dot{\tilde{\omega}}_{eq}(t) = F(t)\tilde{\omega}_{eq}(t)$ is uniformly stable and let $\dot{y}_d(t)$ be absoultely integrable. Then, the solutions of (21) are bounded

Proof: See the Appendix.

Definition 2.14: The system $\dot{\tilde{\omega}}_{eq}(t) = F(t)\tilde{\omega}_{eq}(t)$, is exponentially stable if there exists positive constants γ and λ such that, for all $t > t_0$, the state transition matrix $\Phi(t, t_0)$, associated with $F(t)$, satisfies

$$\| \Phi(t, t_0) \| \leq \gamma e^{-\lambda(t-t_0)}. \tag{28}$$

Proposition 2.15: The matrix $F(t)$ is bounded if $\dot{\tilde{x}}(t)$ is bounded

Proof: See the Appendix.

It is well known [2] that if the matrix $F(t)$is bounded, then exponential stability is equivalent to the uniform integrability, over arbitrary intervals of time, of the norm of the corresponding transition matrix. The following theorem is proved in [2].

Theorem 2.16: Let $\dot{\tilde{x}}(t)$ be bounded on $(-\infty, +\infty)$ and let M be a constant, independent of t_0 and t_1, then, the system $\dot{\tilde{\omega}}_{eq}(t) = F(t)\tilde{\omega}_{eq}(t)$ is exponentially stable if and only if

$$\int_{t_0}^{t_1} \| \Phi(t, t_0) \| \,\mathrm{d}\, t \leq M \,\forall\, t_1 > t_0. \tag{29}$$

The next result touches upon a special form of the well knwon condition of persistency of excitation, of common occurrence in linear and nonlinear adaptive control schemes [6].

Theorem 2.17: Let $\dot{\tilde{x}}(t)$ be bounded on $(-\infty, +\infty)$, moreover, assume that the following form of the persistency of excitation condition holds uniformly in t.

There exists positive constants δ and ϵ, such that the following matrix condition is satisfied

$$\int_t^{t+\delta} \Phi(t, \sigma) \left[\frac{\tilde{x}(\sigma)\tilde{x}^T(\sigma)}{(\tilde{x}^T(\sigma)\tilde{x}(\sigma))^2}\right] \Phi^T(t, \sigma)\, \mathrm{d}\sigma \geq \epsilon\, I \,\forall\, t > t_0. \tag{30}$$

Then, the equivalent adaptation law (21) uniformly yields a bounded trajectory for the vector of weights $\tilde{\omega}_{eq}(t)$, for every bounded signal $\dot{y}_d(t)$, if, and only if, the autonomous system $\dot{\tilde{\omega}}_{eq}(t) = F(t)\tilde{\omega}_{eq}(t)$ is exponentially stable.

Proof: See the Appendix.

Condition (30) admits the following scalar form ([6])

$$\int_t^{t+\delta} z^T\Phi(t,\sigma)\left[\frac{\tilde{x}(\sigma)\tilde{x}^T(\sigma)}{(\tilde{x}^T(\sigma)\tilde{x}(\sigma))^2}\right]\Phi^T(t,\sigma)\, z\, d\sigma = \int_t^{t+\delta} \left| \frac{z^T\Phi(t,\sigma)\,\tilde{x}(\sigma)}{\tilde{x}^T(\sigma)\tilde{x}(\sigma)} \right|^2 d\sigma \geq \epsilon \,\forall\, t > t_0,;\ \| z \| = 1 \quad (31)$$

which is a condition on the energy, averaged over all directions of a unit sphere, of the nonsingularly transformed input vector, $\tilde{\chi}(\tau) = \Phi(\tau,t)\,\tilde{x}(t)/(\tilde{x}^T(t)\tilde{x}(t))$.. This means that the vector function $\tilde{\chi}(\tau)$ is quite an "active" time–varying vector, so that the integral of the matrix $\tilde{\chi}(t)\tilde{\chi}^T(t)$ is uniformly positive definite over any interval of finite length δ.

Robustness Features with Respect to External Perturbations

Inputs with bounded additive noise: Consider a vector-valued norm-bounded external perturbation input, denoted by $\xi(t) = (\xi_1(t),\ldots,\xi_n(t))$, which additively affects the values of the input vector $x(t)$ to the perceptron. It is assumed that the perturbation input $\xi(t)$ is not "larger" than the input $x(t)$, i.e.,

$$\| \xi(t) \| = \sqrt{\xi_1^2(t) + \ldots + \xi_n^2(t)} \leq V_\xi \ < \ V_x \ \forall t. \quad (32)$$

The time derivatives of the components of $\xi(t)$ are assumed to be also bounded

$$\| \dot{\xi}(t) \| = \sqrt{\dot{\xi}_1^2(t) + \ldots + \dot{\xi}_n^2(t)} \leq V_{\dot{\xi}} \ \forall t. \quad (33)$$

We define the augmented external perturbation input vector as

$$\tilde{\xi}(t) = (\xi_1(t),\ldots,\xi_n(t),0). \quad (34)$$

This means that it is implicitly assumed that the constant input B to the bias weight $\omega_{n+1}(t)$ is a fixed value which does not contain the influence of perturbation signals.

The perturbed learning error $\hat{e}(t) = y(t) - y_d(t)$ is now given by

$$\hat{e}(t) = [\tilde{x}(t) + \tilde{\xi}(t)]^T\tilde{\omega}(t) - y_d(t). \quad (35)$$

Note that, in spite of the fact that the perturbed input signal $\tilde{x}(t) + \tilde{\xi}(t)$ is actually available for measurement, its time derivative $\dot{\tilde{x}}(t) + \dot{\tilde{\xi}}(t)$ is not. This means that such time derivatives can not be used in the weight adaptation law. Hence, only an adaptation law of the type proposed in (10) can be actually devised for sliding mode creation on the zero learning error hyperplane.

By virtue of the above considerations, we shall center our attention on the perturbed adaptation law:

$$\dot{\tilde{\omega}}(t) = -\left(\frac{\tilde{x}(t) + \tilde{\xi}(t)}{[\tilde{x}(t) + \tilde{\xi}(t)]^T[\tilde{x}(t) + \tilde{\xi}(t)]}\right) k \ \mathrm{sign}\ \hat{e}(t). \quad (36)$$

The weight adaptation law (36) results, as it is easily verified, in the following discontinuous perturbed learning error dynamics

$$\dot{\hat{e}}(t) = -k \ \mathrm{sign}\ \hat{e}(t) + \tilde{\omega}^T(t)[\dot{\tilde{x}}(t) + \dot{\tilde{\xi}}(t)] - \dot{y}_d(t). \quad (37)$$

The robustness result is summarized in the following theorem whose proof is rather similar to that of Theorem 1.

Theorem 2.18: Consider the sliding mode creation problem on the zero learning error hypersurface of an adaline including a perturbed input vector. If the adaptation law for the augmented weight vector $\tilde{\omega}(t)$ is chosen as in (36) with k being a positive constant satisfying

$$k > \tilde{W}(V_{\dot{x}} + V_{\dot{\xi}}) + V_{\dot{y}} \quad (38)$$

then, given an arbitrary initial condition $\hat{e}(0)$, the perturbed learning error $\hat{e}(t)$ converges to zero in finite time $\hat{t}_h$, estimated by

$$\hat{t}_h \leq \frac{| e(0) |}{k - \tilde{W}(V_{\dot{x}} + V_{\dot{\xi}}) - V_{\dot{y}}} \quad (39)$$

in spite of all possible assumed (bounded) values of the perturbation inputs and its time derivatives. Moreover a sliding motion is sustained on $\hat{e}(t) = 0$ for all $t > \hat{t}_h$.

The equivalent input $v_{eq}(t)$ is now defined, from (37) as

$$v_{eq}(t) = \tilde{\omega}^T(t)\ [\dot{\tilde{x}}(t) + \dot{\tilde{\xi}}(t)] - \dot{y}(t). \quad (40)$$

The equivalent feedback adaptation law is obtained by substituting the discontinuous term in the adaptation law (36) by $v_{eq}(t)$. The obtained average adaptation law is now a perturbation–dependent feedback adaptation law given by

$$\begin{aligned}\dot{\tilde{\omega}}_{eq}(t) &= -\left(\frac{\tilde{x}(t) + \tilde{\xi}(t)}{[\tilde{x}(t) + \tilde{\xi}(t)]^T[\tilde{x}(t) + \tilde{\xi}(t)]}\right) \cdot \left[\tilde{\omega}_{eq}^T(t)\ (\dot{\tilde{x}}(t) + \dot{\tilde{\xi}}(t)) - \dot{y}_d(t)\right] \\ &= -\left(\frac{[\tilde{x}(t) + \tilde{\xi}(t))][\dot{\tilde{x}}(t) + \dot{\tilde{\xi}}(t)]^T}{[\tilde{x}(t) + \tilde{\xi}(t)]^T\ [\tilde{x}(t) + \tilde{\xi}(t)]}\right)\tilde{\omega}_{eq}(t) + \left(\frac{\tilde{x}(t) + \tilde{\xi}(t)}{[\tilde{x}(t) + \tilde{\xi}(t)]^T\ [\tilde{x}(t) + \tilde{\xi}(t)]}\right)\dot{y}_d(t).\end{aligned} \quad (41)$$

Enforcing the invariance condition, corresponding to zero perturbed learning error $\dot{\hat{e}}(t) = 0$ on (42), one obtains, after some algebraic manipulations, the following expression

$$\left(I - \frac{[\tilde{x}(t) + \tilde{\xi}(t)][\dot{\tilde{x}}(t) + \dot{\tilde{\xi}}(t)]^T}{[\tilde{x}(t) + \tilde{\xi}(t)]^T[\tilde{x}(t) + \tilde{\xi}(t)]}\right)\dot{\tilde{\omega}}_{eq}(t) = 0. \quad (42)$$

According to the results of proposition 1, the matrix multiplying $\dot{\tilde{\omega}}_{eq}(t)$ in (42) is a time–varying projection operator. Hence, (42) implies that $\dot{\tilde{\omega}}(t)$ lies in the range space of the perturbed input vector, $\tilde{x}(t)+\tilde{\xi}(t)$ and has zero projection onto the zero perturbed learning error hyperplane. The adaptation mechanism for the perturbed input case has similar geometric features as the unperturbed case.

Noisy Measurements of the Unperturbed Inputs and Neuron Output

Consider now the case in which the measurements of the unperturbed input vector $x(t)$ are corrupted by some unknown but bounded noise signal, which we still denote by $\xi(t)$, satisfying the assumptions of the previous subsection. The measured vector is denoted by $x_m(t) = x(t) + \xi(t)$. We also assume that the bounded measurement noise component $\xi_{n+1}(t)$ distorts the measured value of the constant bias input to the threshold element, assumed to be nominally equals to B. However, the noise component $\xi_{n+1}(t)$ should satisfy the restriction $| \omega_{n+1}(t) | \leq V_{\xi_{n+1}} < B$. The measured input vector function $\tilde{x}_m(t) = \tilde{x}(t) + \tilde{\xi}(t)$ has the obvious meaning. Its constitutive parts are defined as before and they satisfy the same assumptions. In particular The assumption $V_\xi < V_x$ clearly implies that the vector function $\tilde{x}(t)$ is never orthogonal to the extended measured input vector function $\tilde{x}_m(t)$, i.e.,

$$\tilde{x}_m^T(t)\,\tilde{x}(t) = [\tilde{x}(t) + \tilde{\xi}(t)]^T\,\tilde{x}(t) \neq 0. \tag{43}$$

Additionally, the measured perceptron output $y_m(t) = y(t) + \zeta(t)$ is assumed to be corrupted by some additive noise signal, $\zeta(t)$, which has a bounded time derivative, i.e., $| \dot{\zeta}(t) | \leq V_{\dot{\zeta}}\ \forall\, t$.

It is assumed that the time derivative of the perturbed measured input vector $\dot{\tilde{x}}_m(t) = \tilde{x}(t)+\xi(t)$ is not synthesizable in practice. The perturbed adaptation law, proposed for this case, is of the same form as that in (36). The perturbed learning error dynamics is now obtained as

$$\dot{\hat{e}}(t) = -\left(\frac{[\tilde{x}(t) + \tilde{\xi}(t)]^T\tilde{x}(t)}{[\tilde{x}(t) + \tilde{\xi}(t)]^T[\tilde{x}(t) + \tilde{\xi}(t)]}\right) k \text{ sign } \hat{e}(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t) + \dot{\zeta}(t) - \dot{y}_d(t). \tag{44}$$

To guarantee the existence of a sliding regime on the hyperplane $\hat{e}(t) = 0$, the smallest possible value of the product of the switch gain factor k, and the time varying scalar quantity modulating its value, has to be sufficiently large as to overcome the unknown but bounded values of the term $\tilde{\omega}^T(t)\dot{\tilde{x}}(t) + \dot{\zeta}(t) - \dot{y}_d(t)$ in the error dynamics (44).

Note that

$$\begin{aligned}&\left| \frac{[\tilde{x}(t) + \tilde{\xi}(t)]^T\tilde{x}(t)}{[\tilde{x}(t) + \tilde{\xi}(t)]^T[\tilde{x}(t) + \tilde{\xi}(t)]} \right| \\ &= \left| \frac{[x(t) + \xi(t)]^T x(t) + B\,(B + \xi_{n+1}(t))}{[x(t) + \xi(t)]^T[x(t) + \xi(t)] + (B + \xi_{n+1}(t))^2} \right| \\ &\geq \frac{B^2 - (V_x + V_\xi)\,V_x - B\,V_{\xi_{n+1}}}{(V_x + V_\xi)^2 + (B + V_{\xi_{n+1}})^2} =: \eta. \end{aligned} \tag{45}$$

We assume also that $\eta > 0$ The preceeding developments, and arguments similar to those already used in the proof of theorem 1, establish the basis for the proof of the following theorem which summarizes the robustness result for this case.

Theorem 2.19: Consider the problem of a sliding mode creation in a neuron with noisy measurements of the unperturbed input vector $\tilde{x}(t)$ and output signal $y(t)$. If the adaptation law for the augmented weight vector $\tilde{\omega}(t)$ is chosen as in (36) with k being a positive constant satisfying

$$k > \frac{\tilde{W}V_{\dot{x}} + V_{\dot{\zeta}} + V_{\dot{y}}}{B^2 - (V_x + V_\xi)\,V_x - BV_{\xi_{n+1}}} \cdot [(V_x + V_\xi)^2 + (B + V_{\xi_{n+1}})^2] \tag{46}$$

then, given an arbitrary initial condition $\hat{e}(0)$, the perturbed learning error $\hat{e}(t)$ converges to zero in finite time $\hat{t}_h$, estimated by

$$\hat{t}_h \leq \frac{| e(0) |}{k\,\eta - \tilde{W}(V_{\dot{x}} + V_{\dot{\xi}}) - V_{\dot{\zeta}} - V_{\dot{y}}} \tag{47}$$

in spite of all possible assumed (bounded) values of the input measurement noise. Moreover a sliding motion is sustained on $\hat{e}(t) = 0$ for all $t > \hat{t}_h$.

We use again the method of the equivalent control on the basis of the error dynamics equation (44).

$$-\frac{\tilde{x}_m^T(t)\tilde{x}(t)}{\tilde{x}_m^T(t)\tilde{x}_m(t)}\,v_{eq}(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t) + \dot{\zeta}(t) - \dot{y}_d(t) = 0.$$

Note that no singularity is present in the definition of the equivalent input $v_{eq}(t)$ since the product $\tilde{x}_m^T(t)\tilde{x}(t)$ is never zero, as remarked before.

The ideal sliding dynamics, obtained from the invariance condition $\dot{\hat{e}}(t) = 0$, yields, in this case, the following description of the equivalent adaptation law for the vector of variable weights

$$\dot{\tilde{\omega}}_{eq}(t) = -\left(\frac{\tilde{x}_m(t)\dot{\tilde{x}}^T(t)}{\tilde{x}_m^T(t)\tilde{x}(t)}\right)\tilde{\omega}_{eq}(t) + \left(\frac{\tilde{x}_m(t)}{\tilde{x}_m^T(t)\tilde{x}(t)}\right)\left[\dot{y}_d(t) - \dot{\zeta}(t)\right]. \tag{48}$$

In accordance with the invariance condition $\dot{\hat{e}}(t) = 0$ one substitutes $\dot{y}_d(t)$ by $\dot{y}(t)$ in the preceeding equation. After some simple algebraic manipulations the following expression is obtained

$$\left(I - \frac{\tilde{x}_m(t)\tilde{x}^T(t)}{\tilde{x}_m^T(t)\tilde{x}(t)}\right)\dot{\tilde{\omega}}_{eq}(t) = 0 \tag{49}$$

which clearly indicates that the velocity vector for the weight evolution belongs to the range space of the vector $\tilde{x}_m(t)$ and has zero projection onto its normal hyperplane. Once sliding occurs, the vector of variable weights is attached to a fixed point of a hyperplane normal to $\tilde{x}_m(t)$. However, the zero error learning hyperplane is skewed with respect to this hyperplane and the weight vector evolution is no longer attached to a fixed, but a variable, point on the zero learning error hyperplane. We say that an unmatched evolution is obtained for the vector of adaptive weights.

The projection onto the normal hyperplane to $\tilde{x}(t)$, of the velocity vector of the weight adaptation trajectory, $\dot{\tilde{\omega}}_{eq}(t)$, now exhibits a nonzero component. This means that the projection of the vector of weights does move relative to the error hyperplane $\hat{e}(t) = 0$ in a sliding fashion. Moreover, since the velocity vector of $\tilde{\omega}_{eq}(t)$ is no longer orthogonal to the zero learning error hyperplane, the adopted weight evolution law does not guarantee the fastest approach to the zero learning error condition and boundedness of $\tilde{\omega}_{eq}(t)$ becomes highly dependent upon the nature of the noise signal $\tilde{\xi}(t)$. The following result establishes structural condition which guarantees the fastest rate of approach of the vector of adaptive weights to satisfy the zero learning error condition.

Theorem 2.20: Let $\tilde{\mathcal{X}}(t)$ denote the 1-D range space, at time t, of the vector $\tilde{x}(t)$, then the equivalent feedback adaptation laws, given by (48), satisfies

$$\dot{\tilde{\omega}}_{eq}(t) \in \tilde{\mathcal{X}}(t) \forall t \tag{50}$$

if, and only if,

$$\tilde{\xi}(t) \in \tilde{\mathcal{X}}(t). \tag{51}$$

Proof: See the Appendix.

Condition (51) is the well-known matching condition which is to be satisfied by the structure of the input measurement perturbation noise. This matching condition means that all effects of the measurement perturbations will be confined to the time varying subspace $\tilde{\mathcal{X}}(t)$ where the discontinuous feedback actions will overcome them. The boundedness of the perturbation signals implies furthermore that the regulated motions of the adaptive weights vector will be robustly brought to the zero learning error hyperplane.

III. Applications to Inverse and Direct Dynamics Identification

In discrete-time feedforward neural networks, the basic building block unit connecting physically available input variables to the neurons, or Adalines, is constituted by a transversal filter consisting of an ideal sampler and a string of pure delay elements in a "ladder" array (see [4]). This unit is usually addressed as the IS/F-module (for Ideal Sampler-Filter). The output of each pure delay unit constitutes a component of the discrete-time state vector of the ladder filter. These states are provided, as input signals, to the neuron module.

In continuous time (i.e., analog) neuron units, the *IS/F*-module must be replaced by a string of integrators, which is the dynamical continuous-time "equivalent" of the pure delay element. However, such an arrangement is inherently unstable and some internal feedback must be devised so that the resulting unit provides stable, i.e., bounded, signals as inputs to the neuron unit. We thus propose the use of a stable filter, i.e., a stable time-invariant linear dynamical system

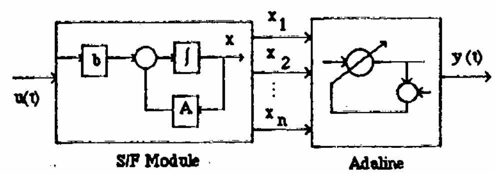

Fig. 3. Schematic representation of the stable filter module and Adaline.

whose (available) states will be used as inputs to the neuron unit. The scalar input function $u(t)$ is the input to the filter and represents the physically available signal to be processed by the neuron (usually a plant input, or output). Fig. 3 depicts a schematic representation of the Stable Filter (SF) module connected to the Adaline module.

Let A denote the constant matrix representing the internal, time invariant, feedback connections of the SF-module. Let b be the vector representing the input channel structure to the stable filter. The pair (A, b), with state vector $x(t)$ represents the SF-module.

Consider the augmented version of the input pair (A, b), as follows

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}; \; \tilde{b} = \begin{bmatrix} b \\ 0 \end{bmatrix}. \tag{52}$$

The SF-module state equations are therefore given by

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{b}\, u(t) \tag{53}$$

where $\tilde{x}(t)$ is the state vector of the SF-module, constituting also the vector of augmented inputs to the neuron unit, considered in the previous section.

Note that since the vector function $x(t)$ is implicitly assumed to be available for measurement, the vector $\tilde{x}(t)$, and the vector $\dot{\tilde{x}}$, are, indeed, available for measurement from the particular topology of the *SF* module (note that each state variable component describing the filter is physically measurable, and so are all their first order time derivatives, which are just the inputs to the several integrators present in the constructed filter). Any sliding mode control strategy to be used on the basis of the SF-Adaline combination can, therefore, assume that these two signals are actually available for measurement. Note also that any noise affecting the signal $u(t)$ influences the filter state vector $x(t)$ in such a manner that the filtered noise components present on each input $x_i(t)$ to the neuron is already of the "matched" type.

The composite discontinuous weight adaptation dynamics takes then the form

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{b}u(t) \\ \dot{\tilde{\omega}}(t) &= -\left(\frac{\tilde{x}(t)\tilde{x}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\, \tilde{A}^T + u(t)\, \frac{\tilde{x}(t)\, \tilde{b}^T}{\tilde{x}^T(t)\tilde{x}(t)} \right) \tilde{w}(t) \\ &\quad - \left(\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)} \right) k \operatorname{sign} e(t) \\ y(t) &= \tilde{x}^T(t)\tilde{\omega}(t) \end{aligned} \tag{54}$$

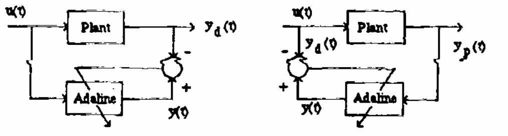

Fig. 4. Forward and inverse dynamics identification schemes using SF-Adalines.

The corresponding (average) equivalent adaptation law is simply obtained now as,

$$\begin{aligned}\dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{b}u(t) \qquad (55)\\ \dot{\tilde{\omega}}_{eq}(t) &= -\left(\frac{\tilde{x}(t)\tilde{x}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\,\tilde{A}^T + u(t)\,\frac{\tilde{x}(t)\,\tilde{b}^T}{\tilde{x}^{(}t)\tilde{x}(t)}\right)\tilde{w}_{eq}(t)\\ &\quad + \left(\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\dot{y}_d(t)\\ y(t) &= \tilde{x}^T(t)\tilde{\omega}_{eq}(t) = y_d(t)\end{aligned}$$

The particular form adopted for $\dot{\tilde{x}}(t)$ does not have any bearing on the geometric features associated with the sliding mode adaptation algorithm. As it can be easily verified, (69), is independent of $\dot{\tilde{x}}(t)$ and hence, it is independent of the particular values of the pair $(\tilde{A}, \tilde{b})$. The stability features of $\tilde{\omega}(t)$, or of its average value $\tilde{\omega}_{eq}(t)$ do depend, however, on the values adopted by the pair $(\tilde{A}, \tilde{b})$, and the value given to the input function $u(t)$.

Proposition 3.1: The actual and the average sliding mode controlled dynamics for $y(t)$ (54), (55) are independent of the matrices $\tilde{A}$ and $\tilde{b}$ i.e., the convergence of the output function $y(t)$ to the desired output $y_d(t)$ by means of the sliding mode adaptation algorithm is insensitive with respect to the SF-module parameters

Proof: See the Appendix.

For the forward and inverse dynamics identification tasks, we use standard definitons, which may be found in [4]. Fig. 4 depicts the forward and inverse dynamics identification schemes used for the example in the next section.

Identification of Forward and Inverse Dynamics for the Kapitsa Pendulum

Here we consider a truly nonlinear system of the nonflat type, studied by Fliess and coworkers in [3], consisting of a unit mass rod with a suspension point which freely moves only on a vertical direction. The Kapitsa pendulum is, thus, an inverted pendulum where the control actions are constrained to move the suspension point only along a vertical axis (see Fig. 5).

We considered a nonstabilizing open loop control $u(t)$, applied to the plant, and obtained the corresponding output $y_p(t)$ of the nonlinear system, represented by the angular position of the rod with respect to the vertical axis. In the forward dynamics identification problem $y_p(t)$ is regarded as the desired signal, $y_d(t)$, to be followed by the neuron output $y(t)$. In that case, the input function $u(t)$ to the system, is also the input to the SF unit. For the inverse dynamics identification

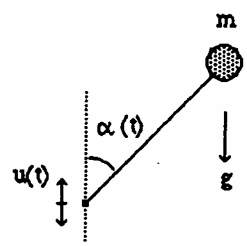

Fig. 5. Kapitsa pendulum.

the roles of $u(t)$ and $y_p(t)$ were reversed, with respect to the neuron system.

The open loop control function $u(t)$ was chosen, according to ([3]), of the form

$$u(t) = A_1 + A_2\cos(\frac{t}{2\,\epsilon}) + A_3\sin(\frac{t}{\epsilon}) \qquad (56)$$

where A_1, A_2 and A_3 are constant parameters. The nonlinear system is assumed to be unknown and only its input and output signals are assumed to be measurable for the adaptation process. For simulation purposes, however, the following model was used

$$\begin{aligned}\dot{\alpha}(t) &= p(t) + \frac{u(t)}{l}\,\sin\,\alpha(t)\\ \dot{p}(t) &= \left(\frac{g}{l} - \frac{u^2(t)}{l^2}\,\cos\alpha(t)\right)\,\sin\alpha(t)\\ &\quad - \frac{u(t)}{l}\,p(t)\,\cos\alpha(t)\\ \dot{z}(t) &= u(t)\\ y_p(t) &= \alpha(t) \qquad (57)\end{aligned}$$

where $\alpha(t)$ is the angle of the rod with the vertical axes, $p(t)$ is proportional to the generalized impulsion. The constants g and l represent, respectively, the gravity acceleration and the length of the rod. The velocity of the suspension point acts as the control variable $u(t)$. The variable $z(t)$ is then the vertical position of the suspension point.

Numerical values for the parameters of the Kapitsa pendulum model were set to be $g = 9.81[\frac{m}{s^2}]$ and $l = 0.7$[mts]. An open loop control input signal $u(t)$ of the form given in (56) with the following constant parameters

$$A_1 = 0.4; A_2 = 2\,; A_3 = 3; \epsilon = 0.05$$

was used, for both tasks.

The *SF* module was designed as a stable low pass filter with the following state representation

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \qquad (58)\\ \dot{x}_2(t) &= x_3(t)\\ \dot{x}_3(t) &= -x_1(t) - 3x_2(t) - 3x_3(t) + u(t)\\ \dot{x}_4(t) &= 0\end{aligned}$$

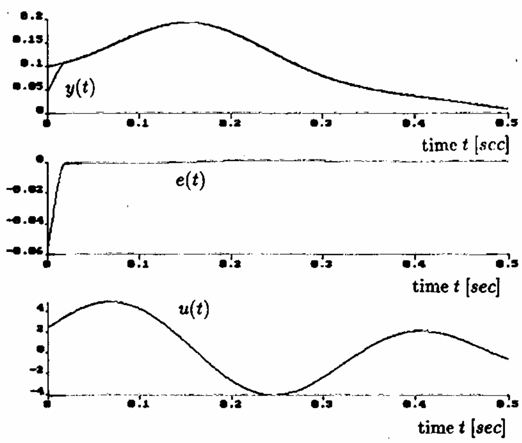

Fig. 6. Noise-free forward dynamics identification of the Kapitsa pendulum.

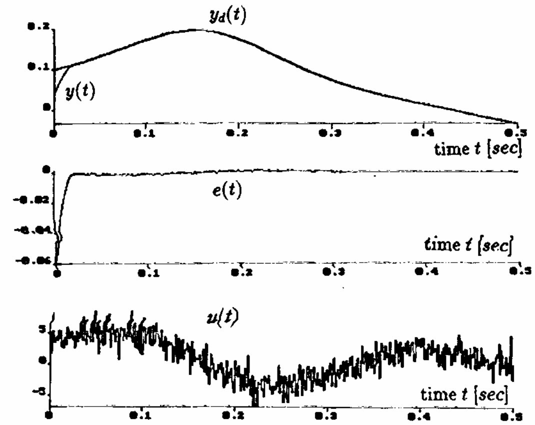

Fig. 7. Robust Forward dynamics identification with bounded noise input for the Kapitsa pendulum.

where the state variable $x_4(t)$ represents the bias component with initial condition equals to B. Such a constant parameter is taken, for this example, as $B = 1$.

The results of a simulated forward dynamics identification tasks, for an Adaline with a total of 4 weights (including one bias variable weight) are shown in Fig. 6. In this figure, the desired output trajectory $y_d(t)$ is constituted by the nonlinear pendulum system output, i.e., $y_p(t) = \alpha(t) = y_d(t)$ and the input $u(t)$ to the SF–module is the same input given to the nonlinear system. The learning (tracking) error response $e(t)$ is shown to converge to zero in approximately 0.02 s. To alleviate the "chattering" phenomena, present in the neuron output and learning error responses, as well as to speed up the simulation time for the SIMNON package, the following standard substitution was adopted for the ideal switch function

$$k \text{ sign } e(t) \approx k \frac{e(t)}{|e(t)| + \delta} \tag{59}$$

with $\delta = 0.05$.

Highly accurate following is seen to be achieved without chattering around the desired output signal. The open loop unperturbed input signal trajectory $u(t)$, afecting both the pendulum and the SF–neuron arrangement, is also shown in this figure. In the simulation no additive noise affecting the input signal $u(t)$ was assumed. The value used for the variable structure gain k was set to $k = 5$.

For comparison and qualitative neuron performance evaluation, simulations were also carried out for the same forward dynamics identification task with an input signal $u(t)$ subject now to a computer generated additive bounded noise $\xi(t)$. The generated noise signal is a discrete–time stochastic process normally distributed at each instant of time with zero mean and standard deviation equals to 1. The value of k was set to be the same as for the previous simulation and the same switch substitution was carried out. The perturbed input $u(t) + \xi(t)$ affects, as before, both the input signal to the pendulum and to the SF–adaline system. The measured filter state is now a perturbed vector function. The noisy states are used to conform the sliding mode adaptive strategy in accordance with (36). The perturbed filter state constituting the input to the adaline is, thus, of the "matched" type. The corresponding simulation results are shown in Fig. 7.

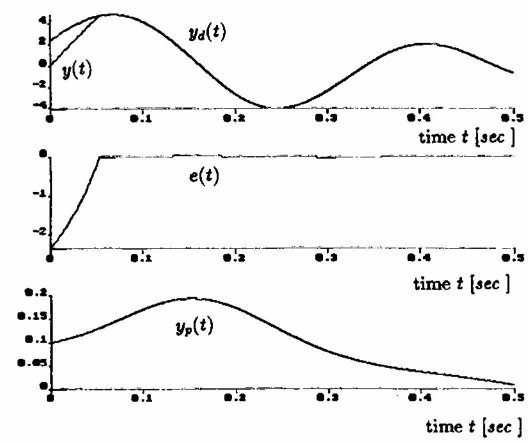

Fig. 8. Noise-free inverse dynamics identification of the Kapitsa pendulum.

The inverse dynamics identification task was also implemented using the same SF–module described above. The variable structure control gain used in this case was $k = 90$. The simulation results, without additive noise for the measured output signal $y_p(t)$ of the nonlinear system, are presented, for a 4 weights Adaline, in Fig. 8. Fig. 9 presents the corresponding results for an additive noise input signal, of the same characteristics as before, affecting the measured signal $y_p(t)$ given as an input to the filter-neuron combination. In this case the value of k was substantially increased to $k = 1000$ due to the large values of the first order time derivative of the desired output signal $y_d(t)$, represented now by the noisy signal $u(t) + \xi(t)$, with $u(t)$ as given in (56).

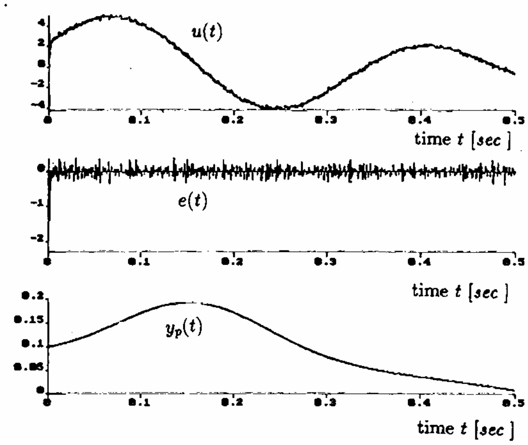

Fig. 9. Robust inverse dynamics identification with bounded noise input for the Kapitsa pendulum.

IV. Conclusion

In this paper a new dynamical discontinous feedback adaptative learning algorithm has been proposed, for linear adaptive combiners, which robustly drives the learning error to zero in finite time. The components of the vector of variable weights are assumed to be provided with continuous time adaptation possibilities. The dynamical adaptive learning scheme is based on sliding mode control ideas and it represents a simple, yet robust, mechanism for guaranteeing finite time reachability of a zero learning error condition. The approach is also highly insensitive to bounded external perturbation inputs, measurement noises and designed input filter parameters.

Bounded average weight evolution is guaranteed under several conditions relative to the underlying linear time–varying system describing the average evolution of the vector of adaptive weights. Some of these conditions are closely related to those of persistency of excitation and thus links our approach with standard adaptive control results.

The matching condition, with respect to bounded input signal and neuron output measurment noises, guarantees a minimum norm solution for the velocity of weight adaptation and fastest convergence to the zero error hyperplane. Measurable inputs, already containing external perturbation components, result in a "matched" input channel structure which always guarantees orthogonal velocity of convergence to the sliding hyperplane. The matched structure appears to be trivially satisfied in most automatic control oriented applications.

Chattering–free dynamical sliding mode controllers for nonlinear systems have been recently proposed by Sira-Ramírez ([12]) using input-dependent sliding surfaces. The adaline case study presented here represents an instance in which the sliding surface (zero learning error condition) is actually an "input" dependent manifold. The obtained sliding "controller" is thus continuous rather than bang–bang.

Extensions of the results to more general classes of multilayer neuron arrangements is being pursued at the present time, with encouraging results.

Appendix A

Proof of Theorem 2.3

Consider a Lyapunov function candidate given by

$$V(e(t)) = \frac{1}{2}e^2(t). \tag{60}$$

The time derivative of $V(e(t))$ is given by

$$\begin{aligned}\dot{V}(e(t)) &= e(t)\ (\dot{\tilde{\omega}}^T(t)\tilde{x}(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t))\\ &= -k \mid e(t) \mid + e(t)\ (\tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t))\\ &\leq -k \mid e(t) \mid + (\tilde{W}V_{\dot{x}} + V_{\dot{y}}) \mid e(t) \mid\\ &= (-k + \tilde{W}V_{\dot{x}} + V_{\dot{y}}) \mid e(t) \mid <\ 0\ \forall\, e \neq 0.\end{aligned} \tag{61}$$

Thus, the controlled trajectories of the learning error converge to zero in a stable manner. We may actually show that such a convergence takes place in finite time.

Indeed, the differential equation satisfied by the regulated error trajectories $e(t)$ is simply given by

$$\dot{e}(t) = -k \text{ sign } e(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t). \tag{62}$$

For all times $t \leq t_h$, the solution, $e(t)$, of such a differential equation, with initial condition $e(0)$ at $t = 0$, satisfies

$$\begin{aligned}e(t) - e(0) &= -kt \text{ sign } e(0)\\ &\quad + \int_0^t (\tilde{\omega}^T(\sigma)\dot{\tilde{x}}(\sigma) - \dot{y}_d(\sigma))d\sigma\end{aligned} \tag{63}$$

at time $t = t_h$ the solution takes the value zero and, hence,

$$\begin{aligned}-e(0) &= -k\ t_h \text{ sign } e(0)\\ &\quad + \int_0^{t_h} (\tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t))\ dt.\end{aligned} \tag{64}$$

Multiplying both sides of the equality by $-$sign $e(0)$ one immediately obtains the estimate in (12) of t_h by means of the following inequality

$$\begin{aligned}\mid e(0) \mid &= k\ t_h - (\int_0^{t_h} (\tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t))\ dt) \text{ sign } e(0)\\ &\geq [k - (\tilde{W}V_{\dot{x}} + V_{\dot{y}})]t_h.\end{aligned} \tag{65}$$

Evidently, for any $t < t_h$ and for the chosen sliding mode controller gain k, in (11), one has from (62)

$$\begin{aligned}e(t)\dot{e}(t) &= -k \mid e(t) \mid + (\tilde{\omega}^T(t)\dot{\tilde{x}}(t) - \dot{y}_d(t))e(t)\\ &\leq (-k + \tilde{W}V_{\dot{x}} + V_{\dot{y}}) \mid e(t) \mid < 0\end{aligned} \tag{66}$$

and a sliding mode exists on $e(t) = 0$ for $t > t_h$. □

Proof of Theorem 2.5

The proof proceeds along similar lines of that of theorem 1 after realizing that the controlled learning error satisfies the following differential equation with discontinuous right-hand side

$$\dot{e}(t) = -k \text{ sign } e(t) - \dot{y}_d(t). \tag{67}$$

□

Proof of Proposition 2.10

Consider again (21), along with the condition $\dot{e}(t) = 0$, i.e., with $\dot{y}_d(t) = \dot{y}(t)$. We rewrite (21) as

$$\begin{aligned}\dot{\tilde{\omega}}_{eq}(t) &= -\left(\frac{\tilde{x}(t)\dot{\tilde{x}}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\tilde{\omega}_{eq}(t) + \left(\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\dot{y}(t)\\ &= \left(\frac{\tilde{x}(t)\tilde{x}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\dot{\tilde{\omega}}_{eq}(t)\end{aligned} \tag{68}$$

rearranging (68) one obtains

$$\left(I - \frac{\tilde{x}(t)\tilde{x}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\dot{\tilde{\omega}}_{eq}(t) = 0. \tag{69}$$

□

Proof of Proposition 2.13

Consider the inequalities

$$\int_{t_0}^{\infty} \frac{|\dot{y}_d(t)|}{\|\tilde{x}(t)\|}\,\mathrm{d}t \le \int_{t_0}^{\infty} |\dot{y}_d(t)|\,\mathrm{d}t = \beta \tag{70}$$

and assume that the initial states, $\tilde{\omega}_{eq}(t_0)$, of the weight adaptation trajectories are bounded by a constant $\tilde{W}_0$.

From the variation of constants formula, the solutions of the linear time-varying differential equation (21) are written as

$$\tilde{\omega}_{eq}(t) = \Phi(t,t_0)\tilde{\omega}_{eq}(t_0) + \int_{t_0}^{t}\Phi(t,\sigma)\frac{\tilde{x}(\sigma)}{\tilde{x}^T(\sigma)\tilde{x}(\sigma)}\dot{y}_d(\sigma)\mathrm{d}\sigma. \tag{71}$$

By virtue of (70), the norm of $\tilde{\omega}_{eq}(t)$ satisfies

$$\begin{aligned}\|\tilde{\omega}_{eq}(t)\| &\le \|\Phi(t,t_0)\|\,\|\tilde{\omega}_{eq}(t_0)\| + \left\|\int_{t_0}^{t}\Phi(t,\sigma)\frac{\tilde{x}(\sigma)}{\tilde{x}^T(\sigma)\tilde{x}(\sigma)}\dot{y}_d(\sigma)\,\mathrm{d}\sigma\right\|\\ &\le \|\Phi(t,t_0)\|\,\|\tilde{\omega}_{eq}(t_0)\| + \int_{t_0}^{t}\|\Phi(t,\sigma)\|\frac{|\dot{y}_d(\sigma)|}{\|\tilde{x}(\sigma)\|}\,\mathrm{d}\sigma\\ &\le \gamma(\tilde{W}_0+\beta);\forall\, t > t_0\end{aligned} \tag{72}$$

the result follows. □

Proof of Proposition 2.15

We take as definition of the matrix norm the induced norm

$$\| F(t) \| = \max_{\|z\|=1} \| F(t)z \|. \tag{73}$$

Evidently, from the definition of $F(t)$ in (26), it readily follows that

$$\| F(t) \| \le \max_{\|z\|=1} \frac{\|\dot{\tilde{x}}(t)\|\,\|z\|}{\|\tilde{x}(t)\|} \le \|\dot{\tilde{x}}(t)\|. \tag{74}$$

□

Proof of Theorem 2.17

It is easy to realize, from the definition of the augmented input vector $\tilde{x}(t)$ in (2), that the input channel matrix for the signal $\dot{y}_d(t)$, given by $\tilde{x}(t)/(\tilde{x}^T(t)\tilde{x}(t))$, is bounded for all $t \in (-\infty,+\infty)$. Moreover, according to proposition 5, the boundedness of $\dot{\tilde{x}}(t)$ implies the boundedness of $F(t)$. The proof of the theorem may now follow, quite closely, the proof found in pp. 167 of [2] □

Proof of Theorem 2.20

Evidently if $\xi(t) \in \tilde{\mathcal{X}}(t)$, then, according to the assumption relating the bounds of $\tilde{x}(t)$ and $\tilde{\xi}(t)$, by which $V_x < V_\xi$, there exists a time varying scalar function $\mu(t)$ taking values in the open interval $(-1,+1)$, such that $\tilde{xi}(t) = \mu(t)\tilde{x}(t)\ \forall\, t$. It is then easy to see that the projection along $\tilde{\mathcal{X}}(t)$ onto its normal subspace, of the generated average vector velocity of the weight adaptation trajectories satisfy

$$\begin{aligned}&\left(I - \frac{\tilde{x}_m(t)\tilde{x}^T(t)}{\tilde{x}_m^T(t)\tilde{x}(t)}\right)\dot{\tilde{\omega}}_{eq}(t)\\ &= \left(I - \frac{[\tilde{x}(t)+\mu(t)\tilde{x}(t)]\tilde{x}^T(t)}{[\tilde{x}(t)+\mu(t)\tilde{\xi}(t)]^T\tilde{x}(t)}\right)\dot{\tilde{\omega}}_{eq}(t)\\ &= \left(I - \frac{\tilde{x}(t)\tilde{x}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\dot{\tilde{\omega}}_{eq}(t) = 0\end{aligned} \tag{75}$$

and, therefore $\dot{\tilde{\omega}}_{eq}(t) \in \tilde{\mathcal{X}}(t)$.

Assume now that $\dot{\tilde{\omega}}_{eq}(t) \in \tilde{\mathcal{X}}(t)\ \forall\, t$. From (49) it follows that $\dot{\tilde{\omega}}_{eq}(t)$ may be expressed, as

$$\dot{\tilde{\omega}}_{eq}(t) = \alpha(t)\,\tilde{x}_m(t) \tag{76}$$

where the scalar function $\alpha(t)$ is given by

$$\alpha(t) = \frac{\tilde{x}^T(t)\dot{\tilde{\omega}}_{eq}(t)}{\tilde{x}_m^T(t)\tilde{x}(t)} \tag{77}$$

in other words $\tilde{x}_m(t) = \tilde{x}(t) + \tilde{\xi}(t) \in \tilde{\mathcal{X}}(t)$. But this is only possible if, and only if, $\tilde{\xi}(t) \in \tilde{\mathcal{X}}(t)\ \forall\, t$ □

Proof of Proposition 3.1

We only prove the proposition for the case of (54), since the proof corresponding to (55) proceeds along similar lines.

Indeed, taking the time derivative of $y(t)$in (54) and using the expressions in the first two equations in (54), one finds, after some straightforward algebraic manipulations, that

$$\begin{aligned}\dot{y}(t) &= \dot{\tilde{\omega}}^T(t)\tilde{x}(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t) \\ &= -k \text{ sign } e(t) = -k \text{ sign}[y(t) - y_d(t)].\end{aligned} \tag{78}$$

□

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REFERENCES

[1] E. Colina-Morles, and N. Mort, "Neural network-based adaptive control design," *J. Systems Eng.* vol. 1, no. 1, pp. 9-14, 1993.

[2] R. W. Brockett, *Finite Dimensional Linear Systems.* New York: Academic, 1970.

[3] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, "Défaut d'un système non linéaire et commande haute fréquence," *C. R. Acad. Sci. Paris*, vol. I-316, 1993.

[4] J. G. Kuschewski, S. Hui, and S. H. Żak, "Application of feedforward networks to dynamical system identification and control," *IEEE Trans. Contr. Syst. Technol.*, vol. 1, pp. 37–49, Mar. 1993.

[5] J. H. Li, A. N. Michel, and W. Porod, "Analysis and synthesis of a class of neural networks: Variable structure systems with infinite gain," *IEEE Trans. Circuits Syst.*, vol. 36,, pp. 713–731, May 1989.

[6] S. Sastry and M. Bodson *Adaptive Control: Stability, Convergence, and Robustness.* Englewood Cliffs, NJ: Prentice-Hall, 1989.

[7] H. Sira-Ramírez, "Nonlinear discrete variable structure systems in quasi-sliding mode," *Int. J. Contr.*, vol. 54, pp. 1171-1187, Nov. 1991.

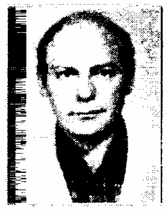
[8] H. Sira-Ramírez and S. H. Żak, "The adaptation of perceptrons with applications to inverse dynamics identification of unknown dynamic systems," *IEEE Trans. Syst., Man, Cybern.*, vol. 21, pp. 634–643, May/June 1991.

[9] V. I. Utkin, *Sliding Modes in Control Optimization.* New York: Springer-Verlag, 1992.

[10] B. Widrow and M. A. Lehr, "30 years of adaptive neural networks: Perceptron, madaline, and backpropagation," *Proc. IEEE*, vol. 78, pp. 1415-1442, Sep. 1990.

[11] S. H. Żak and H. Sira-Ramírez, "On the adaptation algorithms for feedforward neural networks," in *Theoretical Aspects of Industrial Design*, D.A. Flield and V. Komkov, Eds. New York: SIAM)1990, pp. 116-133. 1990.

[12] H. Sira-Ramírez, "On the sliding mode control of nonlinear systems," *Syst. Contr. Lett.*, vol. 19, pp. 303–312, 1992.



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