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# An application of sliding mode control to the determination of system orders

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In this paper an essential property of sliding surface coordinate functions is exploited for proposing an experimental verification of the orders of an assumed system model. The results are applied to verify the relative degrees of actual system models, obtained by experimental identification, of a single-link robotic manipulator equipped with artificial pneumatic muscles, and of an armature controlled DC motor driving an inverted pendulum.

## 1. Introduction

An important problem in automatic control is the validation of structural parameters of a proposed dynamical system model, such as the system order and the relative degree, or, alternatively, the dimension of the zero dynamics (see Isidori 1990 for the corresponding definitions and many interesting details). In certain discontinuous feedback control strategies, such as sliding mode control (Utkin 1978, 1992), this structural information is deemed to be essential and, to a certain degree, it is all that is needed to control efficiently the given system, provided a minimum phase assumption is known to be valid. This fact has been clearly established in the literature (Utkin 1978, 1992, Sira-Ramírez 1990) thanks to the insensitivity of sliding mode control to matched parametric variations and bounded external perturbations.

A sliding mode control scheme thus allows one to avoid time-consuming and sometimes expensive identification experiments aimed at precise determination of system parameters. Once the relative degree of a minimum-phase system has been established, a robust stabilizing sliding surface may then be immediately proposed as a suitable linear combination of the phase variables. Initial steps in this direction have been already proposed by Lopez *et al.* (1994) and by Nouri (1994).

The objective of this paper is to show, in rather general terms, that the consideration of the time evolution of a stabilizing sliding surface coordinate function candidate, defined in the phase space of the system, allows for the unequivocal validation of the structural characteristics of the given system. Such validation refers only to the local dimensions of the model. This possibility is immediately translated into an experimental procedure which is based on the well-known necessary and sufficient condition for the local existence of a sliding regime on the zero level set of a given scalar output function. Such a condition states that a sliding surface candidate must, necessarily, exhibit relative degree equal to one.

The procedure proposed here entitles the use of a discontinuous feedback control input to test systematically the existence of a sliding regime on successively higher

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dimensional sliding surface candidates defined in the phase space of the system. The sliding surface candidates are represented by the zero level set of suitable linear (Hurwitz) combinations of the phase variables of the system. To further validate a sliding surface candidate as an actual sliding surface, which has been already experimentally concluded to exhibit relative degree one, the method further proposes to test for the behaviour of one of the next possible candidates. This leads to the concept of the existence of a local 'sliding hypervolume'. In such a case the sliding surface can only be implicitly defined by means of a discontinuous expression. The presence of such an anomalous case is readily detectable, and hence the output relative degree is firmly established. We only treat here the single-input single-output system (SISO) case. In special circumstances, namely differential flatness, or exact linearizability, of the original system (Fliess *et al.* 1992 a, b, 1993) the method yields the order of the system.

In § 2 we provide the theoretical basis that justifies the proposed experimental procedure. In this instance the adopted mathematical framework utilizes the formalism of the linear differential algebraic approach, for system analysis. The reader is referred, for interesting details and extensive developments/within this important area to the works carried out by DiBenedetto *et al.* (1989), Moog *et al.* (1991), Glumineau and Moog (1989), Grizzle (1993) and more recently by Pomet *et al.* (1992) and Aranda *et al.* (1993). It should be pointed out, however, that the results are also readily derivable through the more conventional differential geometric approach of Isidori (1990) (see Sira-Ramírez 1990); § 3 is devoted to describing the experimental procedure carried out for the determination of the relative degree and the order of two experimental set-ups consisting of a DC motor controlling an inverted pendulum and of a single-link robotic manipulator equipped with pneumatic artificial muscles 4 contains the conclusions and suggestions for further research.

## 2. Theoretical basis

In this section we re-establish, from the framework of linear differential algebra, a well-known fact concerning the necessary and sufficient condition for a candidate sliding surface coordinate function to actually sustain a sliding regime on its zero level set. The fact that the relative degree of an actual sliding surface coordinate function must be equal to one was already implicitly assumed, and used in early works about sliding mode control (Utkin 1978), but it was properly formalized through modern nonlinear systems theory using the differential geometric viewpoint by Sira-Ramírez (1988) (see also Sira-Ramírez 1990 for further developments from the differential geometric viewpoint). Here, and basically for tutorial purposes, we re-establish this well-known fact using the formalism of linear differential algebra.

### 2.1. *Some elements from the linear differential algebraic approach to system analysis*

The definitions below are taken from Pomet *et al.* (1992) with very few modifications. Consider a nonlinear $n$-dimensional system of the form

$$\left.\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}\right\} \tag{1}$$

where the scalar function $h(x)$ is a meromorphic function of the state components. The components of the vector fields $f(x)$ and $g(x)$ are also constituted by meromorphic functions of the state coordinates $x$. The input $u$ is a scalar quantity taken to be totally unrestricted. Let $\mathscr{K}$ stand for the differential field of meromorphic functions of the components of the infinite set of variables $x, u, \dot{u}, \ddot{u}, \ldots$ Let E denote the formal vector space, defined over the field $\mathscr{K}$, spanned by the *differentials* (also called 1-forms) $\mathrm{d}x, \mathrm{d}u, \mathrm{d}\dot{u}, \mathrm{d}\ddot{u}, \ldots$, where $\mathrm{d}x$ stands for $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$.

We denote by $\omega$ a 1-form in $\Sigma$ given by

$$\omega = \sum_j \alpha_j \,\mathrm{d}v_j \tag{2}$$

where $\alpha_j \in \mathscr{K}$, $\forall j$ and $\mathrm{d}v_j$ represents either $\mathrm{d}x_i$ or $\mathrm{d}u^{(k)}$ for some suitable indices $i$ and $k$.

The time derivative $\dot{\omega}$ of a 1-form $\omega$, given in (2), is given by

$$\dot{\omega} = \sum_j \dot{\alpha}_j \,\mathrm{d}v_j + \alpha_j \,\mathrm{d}\dot{v}_j \tag{3}$$

Consider the differential $\mathrm{d}y$ of the output function $y$. Such a differential is computed in local coordinates by

$$\mathrm{d}y = \left(\frac{\partial h}{\partial x}\right)\mathrm{d}x \tag{4}$$

The relative degree (Isidori 1990) of the output function $y = h(x)$ is defined as the minimum number of successive time derivations, required on the output function $y$, such that the control input $u$ explicitly appears in the obtained expression. This definition can also be equivalently formalized in terms of the number of time derivatives, required on the differential $\mathrm{d}y$ of such a scalar output function $y$, so that the differential of the control input $\mathrm{d}u$ appears explicitly in the computed derivative.

**Definition 1:** We say that the 1-form $\mathrm{d}y$ has a finite relative degree $k$ whenever

$$\left.\begin{array}{l} \mathrm{d}y^{(j)} \in \operatorname{span}_{\mathscr{K}}\{\mathrm{d}x\} \\ \mathrm{d}y^{(k)} \notin \operatorname{span}_{\mathscr{K}}\{\mathrm{d}x\} \end{array}, \quad 0 \le j \le k-1 \right\} \tag{5}$$

Otherwise $\mathrm{d}y$ has infinite relative degree.

Define the following sequence of subspaces

$$\left.\begin{array}{l} \mathscr{E}_0 = \operatorname{span}_{\mathscr{K}}\{\mathrm{d}x, \mathrm{d}u\} \\ \mathscr{E}_j = \{\omega \in \mathrm{E} \mid \omega \in \mathrm{E}_{j-1}, \dot{\omega} \in \mathrm{E}_{j-1}\} \end{array}\right\} \tag{6}$$

It is easy to see that $\mathrm{E}_1 = \operatorname{span}_{\mathscr{K}}\{\mathrm{d}x\}$ and that the defined sequence is a descending chain, i.e.

$$\mathscr{E} \supset \mathscr{E}_0 \supset \mathscr{E}_1 \supset \mathscr{E}_2 \cdots \tag{7}$$

It has been shown by Pomet *et al.* (1992) that if the system satisfies the strong accessibility property then the descending chain is strictly decreasing and becomes stationary at zero for some finite integer $k^*$, i.e. $\mathscr{E}_{k^*} = \mathscr{E}_{k^*+1} = \cdots = \{0\}$. It is easy to see that the subspace $\mathscr{E}^k$ of $\mathscr{E}$ is constituted by all 1-forms which have relative degree greater than or equal to $k$.

**Definition 2:** Let $\mathscr{M}$ and $\mathscr{N}$ be two subspaces of E such that $\mathscr{M} \supset \mathscr{N}$. Define the complement of $\mathscr{N}$ in $\mathscr{M}$ as the subspace $\mathscr{N}^\perp$ defined by

$$\mathscr{N}^\perp = \{\omega \in \mathscr{M} \mid \omega \notin \mathscr{N}\} \tag{8}$$

Note that $\mathscr{N}^\perp$ is a subspace of $\mathscr{M}$ that satisfies $\mathscr{N} \oplus \mathscr{N}^\perp = \mathscr{M}$, where the symbol $\oplus$ stands for direct sum.

Let the subspace $E_{j+1}^{\perp}$ denote the complement of $E_{j+1}$ in $E_j$, i.e.

$$\mathscr{E}_{j+1} \oplus \mathscr{E}_{j+1}^{\perp} = \mathscr{E}_j, \quad j = 0, 1, \ldots \tag{9}$$

It is easy to see that the set of differentials in $E_{j+1}^{\perp}$ is constituted by those differentials with relative degree exactly equal to $j$. In other words

$$\mathscr{E}_{j+1}^{\perp} = \{\omega | \omega^{(j-1)} \in E_2^{\perp}\} \tag{10}$$

Note also that

$$\mathscr{E}_{k^*}^{\perp} = \mathscr{E}_{k^*+1}^{\perp} = \mathscr{E}_{k^*+2}^{\perp} = \cdots = \mathscr{E}_{k^*-1}$$

The $n$-dimensional SISO system (1) is said to be differentially flat, or input–output linearizable by static state feedback, with linearizing output given by the function $y$, if and only if $k^* = n$ and $dy^{n-1} \in E_2^{\perp}$. In this case it may be verified that $dy^{(j)} \in E_{n-j+1}^{\perp}$, for $j = 0, \ldots, n-2$. An equivalent form of characterizing differentially flat systems is as follows.

**Definition 3:** A SISO system of the form (1) is differentially flat if there exists a scalar output function $y = h(x)$ such that

$$\dim_{\mathscr{K}}(\mathscr{E}_1 \cap \text{span}_{\mathscr{K}}\{dy, d\dot{y}, \ldots, \}) = n \tag{11}$$

### 2.2. *A Linear algebraic characterization of sliding surfaces*

Consider an $n$-dimensional nonlinear single-input single-output system of the form (1).

Suppose that only two fixed feedback control functions of the form

$$u = u^+(x), \quad u = u^-(x) \tag{12}$$

are available to regulate the system. Moreover, assume that if the system evolution is forcefully restricted to the zero level set of the output function, given by

$$S = \{x \in \mathbb{R}^n | h(x) = 0\} \tag{13}$$

then a desired behaviour of the corresponding autonomous system is obtained. For instance, the system trajectories are asymptotically stable towards an equilibrium point.

Variable structure control strategies resulting in sliding regimes are based on the possibilities of using the fixed available feedback control actions $u^+(x)$ and $u^-(x)$, respectively, on the regions $y > 0$ and $y < 0$ to obtain a forced evolution of the system trajectories leading to and staying on the surface $S$. The discontinuous feedback control actions specified by

$$u = \begin{cases} u^+(x), & \text{for } y > 0 \\ u^-(x), & \text{for } y < 0 \end{cases} \tag{14}$$

are assumed to lead the state trajectories to (ideally) satisfy the conditions $y = 0$ in finite time.

A sliding regime on $S$ is feasible only when the independently controlled system trajectories reach the manifold $S$, in finite time, from arbitrary initial points located in any arbitrary $n$-dimensional vicinity $N$ of $S$. The necessary and sufficient condition which locally guarantees the existence of a sliding regime on $S$ is given by

$$y \frac{dy}{dt} < 0, \quad x \in N \tag{15}$$

Consider the intersection $M$ of $N$ and $S$; the existence condition of (15) in turn holds if and only if the output function $y = h(x)$ is locally relative degree equal to one on the open set $M$ of the sliding surface $S$. An ideal sliding regime smoothly constrains the system trajectories to the subset $M$ of the manifold $S$. The corresponding constrained dynamics are locally defined as if they were due to a smooth controller, called the equivalent control, and denoted by $u_{EQ}(x)$. Such a virtual feedback controller is uniquely defined by the conditions

$$y = h(x) = 0, \quad \dot{y} = \left(\frac{\partial h}{\partial x}\right)[f(x) + g(x)u_{EQ}(x)] = 0 \tag{16}$$

The local existence of such an equivalent control is tantamount to the existence of a sliding regime. It is easy to show the following theorem Sira-Ramírez 1988).

**Theorem 1:** *The system* (1) *with output function $y = h(x)$ is said to locally exhibit a sliding regime on a subset $M$ of the zero level set*

$$A = \{x \in \mathbb{R}^n | h(x) = 0\}$$

*if and only if the equivalent control locally exists and satisfies, for all $x \in M$*

$$\min\{u^-(x), u^+(x)\} < u_{EQ}(x) < \max\{u^+(x), u^-(x)\} \tag{17}$$

If one assumes that locally $u^+(x) > u^-(x)$ then, under the assumptions of the above theorem, the switching logic (14) guarantees the local existence of a sliding regime on $S$ for trajectories starting from any initial point located sufficiently close to the sliding manifold $S$.

All definitions associated with sliding regimes can be given in a slightly more abstract form by exploiting the linear differential algebraic features of the underlying problem. This brings simplicity and the possibilities for further generalization, especially to the multivariable case (Glumineau and Moog 1989, Sira-Ramírez 1994).

**Definition 4:** A sliding surface coordinate function $y = h(x)$ is any meromorphic function of the components of $x$ such that $\mathrm{d}y$ is locally relative degree one on $S$.

A sliding surface coordinate function is then any scalar function $y = h(x)$ such that its first order time derivative depends explicitly upon the control input $u$; i.e. $y = h(x)$ is a scalar function of the state components $x$ which exhibits relative degree equal to one.

One formalizes such a statement in terms of 1-forms, as follows.

**Proposition 1:** *The function $y = h(x)$ is a sliding surface coordinate function if and only if*

$$\mathrm{d}y \in \mathscr{E}_2^{\perp} \tag{18}$$

**Proof:** The proof is immediate from the definitions.

**Definition 5:** A finite set of coefficients $\gamma_i$, $i = 0, 1, \ldots, k - 1$, are said to be Hurwitz if the associated monic polynomial $p_k(\lambda)$, in the complex variable $\lambda$, given by

$$p_k(\lambda) = \sum_{i=0}^{k} \gamma_i \lambda^i, \quad \gamma_k = 1 \tag{19}$$

is a Hurwitz polynomial, i.e. all roots of the equation $p_k(\lambda) = 0$ have strictly negative real parts.

A Hurwitz differential polynomial is a polynomial in the symbol d/d$t$, which has Hurwitz coefficients.

We let $P_k(\mathrm{d}/\mathrm{d}t)$ denote a monic Hurwitz differential polynomial of order $k$, with constant coefficients $\gamma_j$, $j = 0, 1, \ldots, k-1$, of the form

$$P_k\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = \sum_{j=0}^{k} \gamma_j \frac{\mathrm{d}^j}{\mathrm{d}t^j}, \quad \gamma_k = 1 \tag{20}$$

When using $P_k(\mathrm{d}/\mathrm{d}t)$, the specific form of the Hurwitz differential polynomial is left to be quite arbitrary.

The meanings of expressions such as $[P_k(\mathrm{d}/\mathrm{d}t)]\mathrm{d}y$, *or* $[P_k(\mathrm{d}/\mathrm{d}t)]y$ are clearly given by

$$\left.\begin{aligned} \left[P_k\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right]\mathrm{d}y &= \sum_{j=0}^{k} \gamma_j\, \mathrm{d}y^{(j)} \\ \left[P_k\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right]y &= \sum_{j=0}^{k} \gamma_j \frac{\mathrm{d}^j y}{\mathrm{d}t^j} \end{aligned}\right\} \tag{21}$$

Let $y = h(x)$ be a scalar output function of the system (1) and consider the following sequence of 1-forms:

$$\left\{\left[P_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right]\mathrm{d}y, \left[P_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right]\mathrm{d}y, \ldots, \left[P_k\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right]\mathrm{d}y, \ldots\right\} \tag{22}$$

with $P_0 = 1$, $P_j$, $j = 1, \ldots,$ being differential Hurwitz polynomials as defined above. The following result constitutes the basis of the proposed experimental approach described in the next section.

**Proposition 2:** *Given the system* (1), *there exists a finite integer $r$ satisfying $n \geq r \geq 1$ such that*

$$\mathrm{d}s_r = \left[P_{r-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right]\mathrm{d}y \in \mathscr{E}_2^{\perp} \tag{23}$$

for some Hurwitz differential polynomial $P_{r-1}(\mathrm{d}/\mathrm{d}t)$. In other words, a sliding regime exists on the zero level set of the scalar function $s_r = [P_{r-1}(\mathrm{d}/\mathrm{d}t)]y$ and no sliding regime can exist on any other scalar function of the form $s_j = [P_{j-1}(\mathrm{d}/\mathrm{d}t)]y$ for $j < r$.

**Proof:** The proof is immediate by construction of the sliding surface candidates and the relative degree one property of an actual sliding surface coordinate function.

As a corollary, if the output function $y$ is a linearizing coordinate, i.e. if d$y$ is relative degree $n$ and hence $\mathrm{d}y^{(n-1)} \in \mathscr{E}_2^{\perp}$, a sliding regime can then be made to exist, for the first time within the outlined procedure, only for a candidate sliding surface coordinate function of the form $s_n = [P_{n-1}(\mathrm{d}/\mathrm{d}t)]y$. The relative degree of $y$ is then equal to $n$, the order of the system.

The practical implications of the above results rest on the fact that by proposing successive sliding surface coordinate functions candidates of the form $s_k = [P_{k-1}(\mathrm{d}/\mathrm{d}t)]y$, $k = 1, 2, \ldots,$ a sliding regime can only be created, by the first time, on the zero level set of $s_r = [P_{k-1}(\mathrm{d}/\mathrm{d}t)]y$ if and only if $y$ has relative degree equal to $r$. Thus, by successively extending the dimension of the sliding surface candidate, through Hurwitz combinations of an augmented number of phase variables, the proposed sliding surface coordinate function will actually be driven to zero, in a sliding mode manner. This will only happen when the sliding surface candidate function $s_r$ depends explicitly upon the $(r-1)$th time derivative of the output function $y$. The time derivative of such a function will necessarily contain the influence of the control input function $u$ and thus a sliding regime can be locally created on the manifold $s_r = 0$ by means of a simple control input switching law.

### 2.3. *Input-dependent sliding manifolds*

According to the (experimental) testing process considered in the preceding section, it is possible that one may inadvertently, or even purposely, test a sliding surface candidate $s_r$, whose dimension is equal to the relative degree of the given system by incorporating precisely $r$ time derivatives of the output into the corresponding linear expression. Such a sliding surface candidate is seen to directly depend on the control input. The sliding surface measurement, however, does not entitle actual use of the input signal, since its synthesis is usually obtained by numerical differentiation of the output. One should remark that the topic of efficient numerical differentiation techniques in the presence of input noise is the subject of ongoing current theoretical research. The reader is invited to read the work of Diop *et al.* (1994). In such work, within the context of observer theory, numerical differentiation, in suitable combination with efficient interpolation techniques, is shown to overcome the ill-conditioned nature of numerical differentiation. Their interesting results further seem to challenge the construction of present-day state observers.

To illustrate that this situation of input dependent manifolds may be readily identified, consider the case of an $n$-dimensional system of the form (1) with relative degree equal to $r$. Consider also, to simplify the treatment further, that the linearizable part of the normal canonical form is already in Brunovsky's canonical form with (possibly auxiliary) control input $v$:

$$\left.\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= v \\ \dot{\eta} &= q(\xi_1, \ldots, \xi_r, \eta) \\ y &= \xi_1 \end{aligned}\right\} \tag{24}$$

where $\eta$ is an $(n-r)$-dimensional vector describing the zero dynamics of the system by means of

$$\dot{\eta} = q(0, \ldots, 0, \eta) \tag{25}$$

Consider then the sliding surface candidate $s_r = P_r(\mathrm{d}/\mathrm{d}t)y$. In phase coordinates the expression for such a sliding surface is an input-dependent expression written as

$$s_r = \gamma_0 \xi_1 + \gamma_1 \xi_2 + \cdots + \gamma_{r-1} \xi_r + v \tag{26}$$

Evidently, the creation of a local sliding regime on the zero level set $s_r = 0$ of the sliding surface candidate function $s_r$ is possible whenever one takes as an auxiliary control input the time derivative of the original input $v$, i.e. one sets $\dot{v} = -W \operatorname{sign} s_r$ (Sira-Ramírez 1992). Note that this alternative actually corresponds to placing an integrator in front of the input channel for $v$ and using the input to the integrator as a new input to the system. In such a case the time derivative of the sliding surface candidate $s_r$ satisfies

$$\dot{s}_r = \gamma_0 \xi_2 + \gamma_1 \xi_3 + \cdots + \gamma_{r-1} v - W \operatorname{sign} s_r \tag{27}$$

For sufficiently large values of $W$, it is clear that the condition $s_r \dot{s}_r < 0$ is locally satisfied.

However, if as suggested by the testing procedure described above, no integrators are placed in front of the input channel for $v$, then after synthesis of the quantity $s_r$ by means of (say) numerical differentiation of the output signal, or suitable indirect measurements, one actually uses the discontinuous control input $v = -W \operatorname{sign} s_r$ rather than its integral. The resulting sliding surface candidate then satisfies the following implicit equation

$$s_r = \gamma_0 \xi_1 + \gamma_1 \xi_2 + \cdots + \gamma_{r-1} \xi_r - W \operatorname{sign} s_r \tag{28}$$

One rewrites the above equation in the following manner:

$$\beta(s_r) = s_r + W \operatorname{sign} s_r = \gamma_0 \xi_1 + \gamma_1 \xi_2 + \cdots + \gamma_{r-1} \xi_r \tag{29}$$

where one may also assume, without loss of generality, that the right-hand side of (29) is still a Hurwitz linear combination of the phase variables. The function $\beta(s_r)$ is everywhere defined except at $s_r = 0$ (Fig. 1). The domain of definition of such a function is then $\mathbb{R}\backslash 0$. The function $\beta(s_r)$ can now be inverted and an explicit piecewise linear description of the sliding surface candidate is obtained as follows (see also Drakunov and Utkin (1990) where a similar technique is used in connection with discrete-time sliding modes)

$$s_r = \begin{cases} \sum_{i=0}^{r-1} \gamma_i \xi_{i+1} - W, & \text{for } \sum_{i=0}^{r-1} \gamma_i \xi_{i+1} > W \\ 0 & \text{for } \left|\sum_{i=0}^{r-1} \gamma_i \xi_{i+1}\right| \leq W \\ \sum_{i=0}^{r-1} \gamma_i \xi_{i+1} + W, & \text{for } \sum_{i=0}^{r-1} \gamma_i \xi_{i+1} < -W \end{cases} \tag{30}$$

Figure 2 depicts the nature of the above sliding surface description. Note that $s_r = 0$ no longer corresponds to an hyperplane but to an entire region, bounded by the following two parallel hyperplanes defined in the $r$-dimensional phase space of

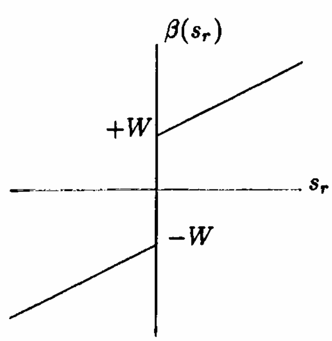


Figure 1. Implicit, relative degree $r$, sliding surface candidate function.

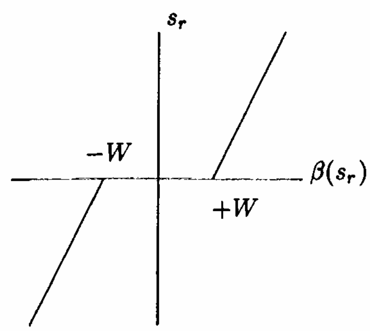


Figure 2. Explicit, relative degree $r$, piecewise linear sliding surface candidate function.

the system

$$\left.\begin{aligned} \sum_{i=0}^{r-1} \gamma_i \xi_{i+1} &= W \\ \sum_{i=0}^{r-1} \gamma_i \xi_{i+1} &= -W \end{aligned}\right\} \tag{31}$$

We shall call such a region a sliding hypervolume. The corresponding time derivative of $s_r$ is formally obtained as

$$\dot{s}_r = \begin{cases} \sum_{i=0}^{r-2} \gamma_i \xi_{i+2} - W\gamma_{r-1}, & \text{for } s_r = \sum_{i=0}^{r-1} \gamma_i \xi_{i+1} - W > 0 \\ 0, & \text{for } |\sum_{i=0}^{r-1} \gamma_i \xi_{i+1}| \leq W \\ \sum_{i=0}^{r-2} \gamma_i \xi_{i+2} + W\gamma_{r-1} s_r, & \text{for } s_r = \sum_{i=0}^{r-1} \gamma_i \xi_{i+1} + W < 0 \end{cases} \tag{32}$$

Note that from the assumption of having a Hurwitz linear combination of the phase variables, the coefficient $\gamma_{r-1}$ is necessarily positive. It then follows that for sufficiently large values of $W$, the inequality $s_r \dot{s}_r < 0$ is locally satisfied outside the sliding hypervolume. It follows immediately that the sliding hypervolume is locally attractive for the regulated trajectories and, at least formally, one may conclude that a sliding regime does exist on that region of the space where $s_r = 0$. The controlled phase trajectories locally approach the hyperplanes, described in (31), bounding the sliding hypervolume $s_r = 0$ in the phase space.

Within the sliding hypervolume, where the sign of $s_r$ is undefined, the values of the function $\beta$ abruptly change from either $+W$ to $-W$ or from $-W$ to $+W$, depending on the region from which the sliding hypervolume is approached by the system trajectories ($s_r > 0$ or $s_r < 0$, respectively). Such abrupt changes entitle trajectories with infinite speed jumping back and forth from one delimiting hyperplane of the region described by $s_r = 0$ to the other.

From the experimental viewpoint the observed function, say $\psi$, consists of an appropriate Hurwitz linear combination of the output function $y$ and its first $r$ time derivatives; i.e. it consists of an expression of the following form:

$$\psi = \dot{s}_r = \sum_{i=0}^{r-1} \gamma_i \xi_{i+1} - W \operatorname{sign} s_r \tag{33}$$

This measured quantity $\hat{s}_r$, which is not numerically equal to $s_r$, is actually

$$\psi = \hat{s}_r = \beta(s_r) - W \operatorname{sign} s_r$$

Thus, once $\hat{s}_r$ reaches zero, i.e. when $\beta(s_r)$ reaches either $+W$ or $-W$, a large bounded chattering should be immediately observed on the values of $\psi = \hat{s}_r$ with (ideally) corresponding infinitely fast switchings of the control input $v$.

## 3. Experimental verification of the relative orders of a DC motor inverted pendulum arrangement and of a one-link manipulator with artificial muscles

### 3.1. *DC motor example*

We considered an actual laboratory set-up of a DC motor whose rotor axis is coupled to a rigid link with a significant mass load at the free extreme. The output of the system was taken to be the angular position $y = \theta$ of the motor shaft, with respect to the vertical axis, and the input $u$ to the system was taken as the armature voltage.

The proposed testing procedure entitled the use of a growing sequence of sliding surface candidates $s_i = P_{i-1}(\mathrm{d}/\mathrm{d}t)y$, $i = 1, 2, \ldots$, and use at the input of a corresponding discontinuous feedback control action based on the sign of such a sliding surface candidate function.

$$u = -W \operatorname{sign} s_i = -W \operatorname{sign}\left[P_{i-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)y\right], \quad W > 0 \tag{34}$$

A zeroth-order sliding surface candidate $s_1 = P_0(\mathrm{d}/\mathrm{d}t)y$ was first taken, using the measured motor angular position $y = \theta$ of the DC motor system. The output $y = \theta$ was considered to simply represent the angular stabilization error with respect to the vertical equilibrium position, which was arbitrarily assigned the value $\theta = 0$.

$$s_1 = P_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)y = 0 \tag{35}$$

The experimental results shown in Fig. 3 clearly show that there exists no sliding regime on $s_1 = 0$. An indication of this assessment is the small oscillations of the angular position error trajectory around zero, along with the corresponding low frequency switchings undergone by the input voltage trajectory $u(t)$. The control input voltage amplitude $W$ was increased to its maximum tolerable margin of 10 V. Even for this maximum amplitude value, the discontinuous-based feedback control signal was seen to be not able to induce the desired sliding regime. It was therefore concluded that the system was of relative degree higher than one.

A first-order sliding surface candidate $s_2$ expressed in terms of the angular stabilization error $y$ and the angular velocity $\dot{y} = \dot{\theta}$ was next taken as a sliding surface candidate for the system.

$$s_2 = P_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)y = \left(\gamma_0 + \frac{\mathrm{d}}{\mathrm{d}t}\right)y = \gamma_0 y + \dot{y}, \quad \gamma_0 > 0 \tag{36}$$

The positive constant $\gamma_0$ was arbitrarily chosen to be $\gamma_0 = 25$, and the position error derivative $\dot{y}$ was directly measured from the DC motor system using a sufficiently

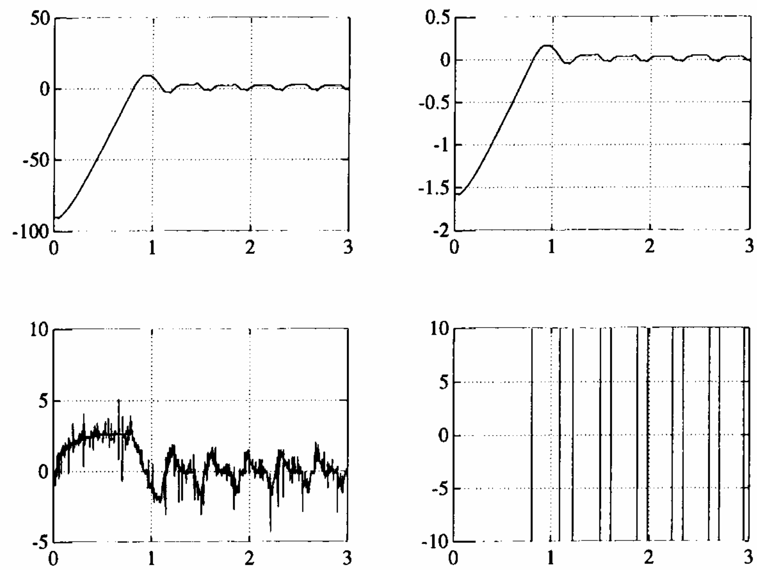


Figure 3. Zeroth order sliding surface behaviour, discontinuous control input and corresponding phase trajectories for DC motor experiments.

accurate tachometer. As before, the control input was synthesized as

$$u = -W \operatorname{sign}(\gamma_0 y + \dot{y}) \tag{37}$$

The obtained experimental results are shown in Fig. 4. A substantially improved behaviour of the stabilization error signal was obtained with very small, or entirely negligible, oscillations around zero. The control input exhibited rather fast bang–bang switchings, indicating that high-frequency oscillations were taking place around the zero level set of the controlled sliding surface candidate function. One concludes that the position error is then, for all practical purposes, of relative degree equal to two. Increase of the control amplitude $W$ to its maximum allowable value ($W = 10\,\mathrm{V}$) was seen not to change substantially the already observed sliding mode behaviour. It was then concluded that the system was not of relative degree higher than two.

To test further the validity of the previous result, a second-order sliding surface candidate of the form

$$\psi = \hat{s}_3 = P_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)y = \gamma_0 y + \gamma_1 \dot{y} + \gamma_2 \ddot{y}, \quad \gamma_0 = 25, \quad \gamma_1 = \gamma_2 = 1 \tag{38}$$

with a control input voltage $u$ of the form

$$u = -W \operatorname{sign}(\gamma_0 y + \gamma_1 \dot{y} + \gamma_2 \ddot{y}) \tag{39}$$

was actually tested. The second-order time derivative of the angular position signal represents the angular acceleration of the rotor axis. Since it is well known that the rotor current is proportional to the angular acceleration, the rotor current was directly measured, instead of the acceleration, for the synthesis of the function $\psi$. This explains the presence of the constant $\gamma_2$ in (38) and (39). It is clearly seen from the physics of the problem that actually an input-dependent sliding surface candidate was being tested.

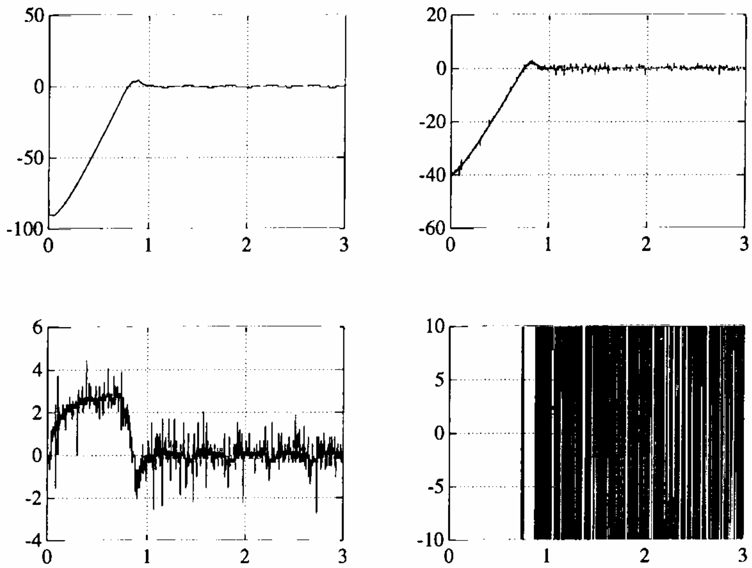


Figure 4. First order sliding surface behaviour, discontinuous control input and corresponding phase trajectories for DC motor experiments.

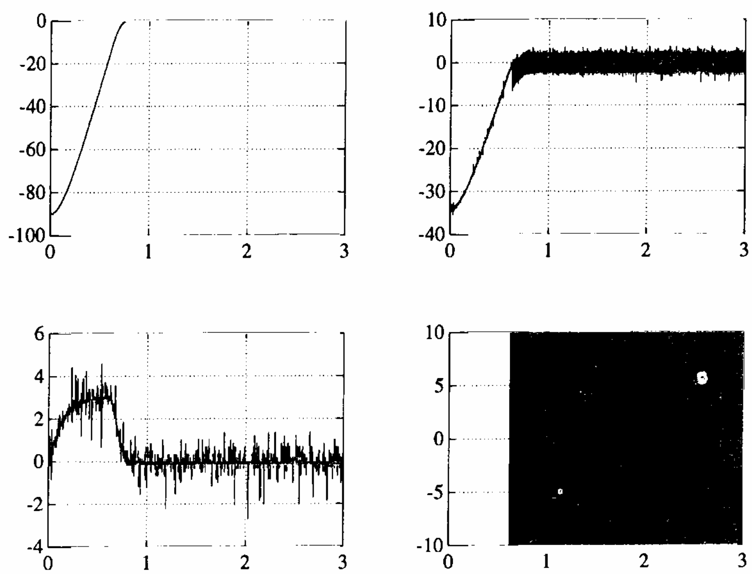


Figure 5. Second order sliding surface behaviour, discontinuous control input and corresponding phase trajectories for DC motor experiments.

The corresponding experimental results are shown in Fig. 5. The existence of a sliding volume implicitly defined by $s_3 = \hat{s}_3$ is clearly assessed from the fact that very high frequency switchings of the corresponding control input were observed, and a bounded chattering motion of the measured quantity $\psi = \hat{s}_3$ occurred. The existence of a sliding volume was concluded and the relative degree of the system was, therefore, experimentally found out to be equal to two.

### 3.2. *A single-link robotic manipulator with artificial muscles*

An experimental single-link manipulator, equipped with pneumatic antagonistic artificial muscles and a servovalve actuator, was used to validate an experimentally identified model obtained using the MATLAB System Identification package. Such a model turned out to be given by the following third-order linear system with a fast first-order asymptotically stable zero dynamics and negligible time delay.

$$\frac{\theta(p)}{\Delta i(p)} = \frac{18{\cdot}8 + 0{\cdot}17\, p}{p^3 + 15{\cdot}06\, p^2 + 50{\cdot}6\, p} \tag{40}$$

where $p$ designs the Laplace transform of the derivative operator and the output of the system was taken to be the angular position $\theta$ with respect to the horizontal equilibrium position. Such an equilibrium position corresponded to a constant torque developed by the antagonstic muscles arrangement. The input to the system was taken to be the servovalve incremental current $\Delta i$. The servovalve current $i$ has a constant value $i = I = 0{\cdot}5\ A$ corresponding to the horizontal equilibrium position. The incremental servovalve current $\Delta i$ around its constant nominal value was set to represent the discontinuous feedback control input action, seeking to induce a sliding regime on the several increased order proposed scalar output functions.

The experiment started with a zeroth-order sliding surface coordinate function candidate $s_1 = P_0(\mathrm{d}/\mathrm{d}t)\theta = \theta$ representing the angular position error with respect to the vertical position. The control input current was synthesized as $u = I + \Delta i = I - W \operatorname{sign} y$. The obtained experimental results are shown in Fig. 6. Evidently the

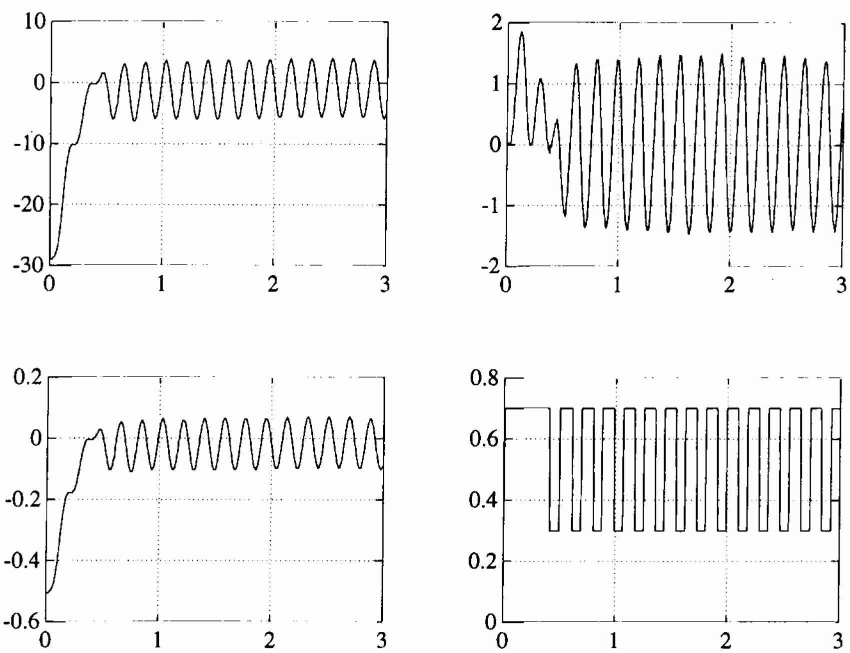

Figure 6. Zeroth order sliding surface behaviour, discontinuous control input and corresponding phase trajectories for single link manipulator with artificial muscles.

existence of a sliding regime on such a sliding surface candidate could not be concluded owing to the oscillatory behaviour of the surface coordinate and the corresponding low-frequency switchings of the control input current. The system was therefore of relative degree higher than one.

A first-order sliding surface $s_2$ was next proposed which entitled a linear (Hurwitz) combination of the angular position error and the angular velocity. The proposed sliding surface output function was then set to be

$$s_2 = P_1\left(\frac{d}{dt}\right)y = \gamma_0 y + \dot{y}, \quad \gamma_0 > 0 \tag{41}$$

where the positive constant $\gamma_0$ was chosen as $\gamma_0 = 5$. The discontinuous feedback control input $u$ was chosen as

$$u = 0{\cdot}5 - W \operatorname{sign}(\gamma_0 y + \dot{y})$$

with $W = 0{\cdot}2\ A$. This value of $W$ corresponds to an incremental pressure reasonably endured by the rubber muscles. The angular velocity was directly obtained by means of a tachometric measurement devised on the actual experimental set-up. The experimental results, depicted in Fig. 7, show an oscillatory sliding surface behaviour at the beginning and a corresponding low-frequency bang-bang incremental current input signal. After the oscillations have settled, the obtained sliding surface candidate trajectory is seen to qualify as a sliding motion, when compared with the zeroth-order surface counterpart. After this moment the control input is seen to undergo higher-frequency switchings characteristic of the local existence of a sliding regime.

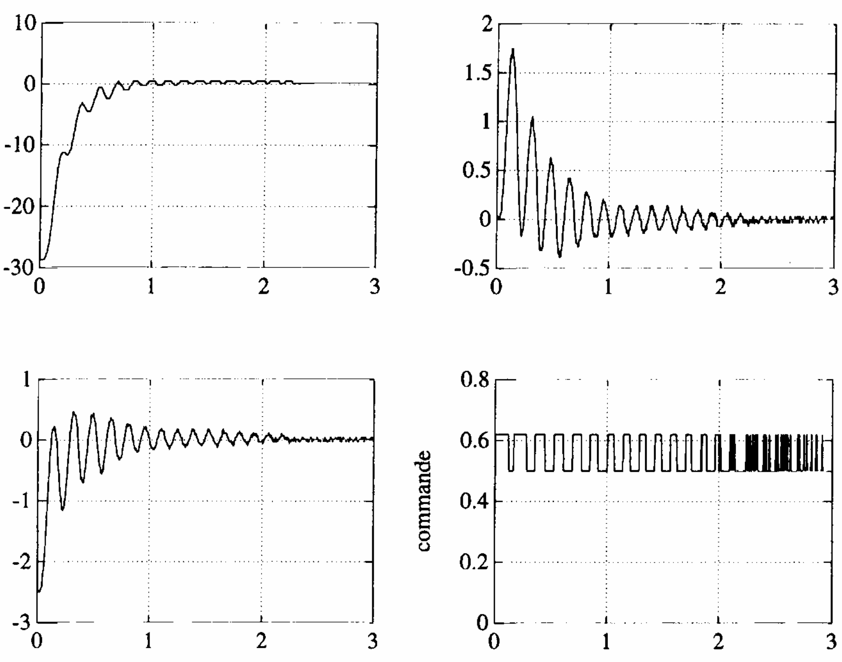


Figure 7. First order sliding surface behaviour, discontinuous control input and corresponding phase trajectories for single link manipulator with artificial muscles.

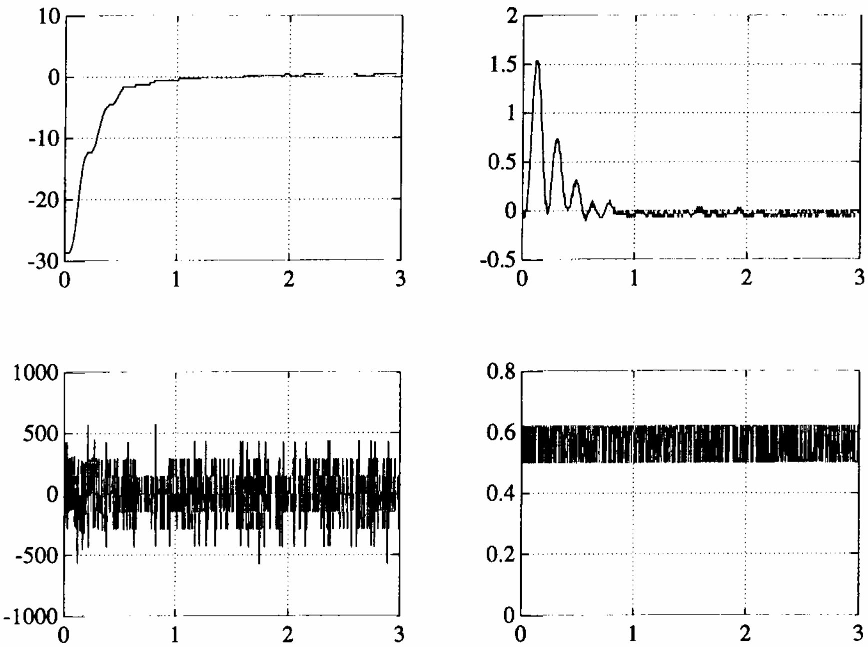


Figure 8. Second order sliding surface behaviour, discontinuous control input and corresponding phase trajectories for single link manipulator with artificial muscles.

The system is therefore of relative degree equal to 2. It was felt, however, that a still higher-order sliding surface candidate needed to be tested to confirm the result.

The experimental results corresponding to a second-order sliding surface candidate

$$\psi = \hat{s}_3 = P_2\left(\frac{d}{dt}\right)y = \gamma_0 y + \gamma_1 \dot{y} + \ddot{y}, \quad \gamma_0 > 0, \gamma_1 > 0 \tag{42}$$

with $\gamma_0 = 5$ and $\gamma_1 = 1$ and the control input

$$u = I - W \operatorname{sign}(\gamma_0 y + \gamma_1 \dot{y} + \ddot{y}) \tag{43}$$

with $W = 0{\cdot}2\,A$ are shown in Fig. 8. The second-order derivative of the angular position error was obtained through a direct numerical differentiation scheme.

The existence of a sliding volume generated by the input-dependent sliding surface candidate function $s_3$, implicitly defined by the equation $\hat{s}_3 = s_3$, is clearly concluded from the obtained experimental results depicted in 8. The angular position output of the system exhibits, therefore, a relative degree $r$ equal to 2.

It should be emphasized that it is well known that reasonable models of robotic manipulators equipped with artificial muscles are differentially flat, i.e. they are exactly linearizable by means of endogenous dynamical feedback (see, for instance, Sira-Ramírez *et al.* 1994). The given system may also be taken as a third-order system with negligible, or very fast, asymptotically stable zero dynamics. This fact is already present in the identified model above, as well as in models proposed by Inoue (1987), Hamerlain *et al.* (1991) and by Tondu and Lopez (1994).

## 4. Conclusions

In this paper we have developed a method which has also been experimentally implemented to verify the relative degree of a presumed system model. The method is based on testing the behaviour of successively higher-order scalar sliding surface coordinate function candidates. These functions define corresponding higher-order linear sliding hyperplanes in the phase space of the system. In all cases the system is assumed to be excited by a discontinuous control input which is based on the knowledge of such a sliding surface candidate. The used discontinuous feedback control inputs may be synthesized on the basis of the sign of the currently proposed sliding surface candidate, or some reasonable continuous approximation of the sign function. When a sliding regime is locally achieved on the zero level set of a particular sliding surface candidate function, the relative degree of the system is then uniquely determined from the dimensions of the corresponding sliding manifold. The relative degree of the system exceeds by one the dimensions of the obtained sliding hyperplane. An obtained sliding mode behaviour, on a given sliding surface candidate, is unequivocally assessed and it is fairly easy to identify in practice. In fact if a sliding hyperplane candidate of dimensions equal to the relative degree of the system is proposed (i.e. an input-dependent sliding surface candidate), the obtained sliding regime corresponds to a sliding motion on a sliding hypervolume which is readily identified in terms of the observed behaviour of the measured sliding function candidate. The proposed method not only enjoys a well-based theoretical foundation, but it is also straightforward to implement in a laboratory set-up. For those systems where a linearizing output is known, and it happens to be available for measurement, the method readily yields the order of the system.

The technique was applied to verify the relative degrees of experimentally derived models for a DC motor, driving an inverted pendulum, and for a robotic manipulator equipped with artificial antagonistic pneumatic muscles.

Further developments can be propsed, which include verification of the structure at infinity of multivariable systems, including the so-called essential structure or essential orders at infinity (Glumineau and Moog 1989).Generally speaking, the existence of a sliding regime for the multivariable nonlinear system case requires dynamic extension, of certain input coordinates, via the addition of integrators on some suitably chosen input channels. This is seen to be necessary to achieve an underlying required (dynamical) input-to-sliding surface decoupling property. The realization of these theoretical features in experimental environments, however, should, pose no serious practical difficulties.

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