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ON THE STABILIZATION OF NONLINEAR SYSTEMS VIA INPUT-DEPENDENT SLIDING SURFACES

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SUMMARY

This article proposes the use of sliding regimes, defined on a suitable input-dependent sliding surface, as a means of making more robust any model-based smooth feedback control scheme designed for the stabilization of a general nonlinear plant. The approach naturally produces a, robust, 'outer loop' redundant discontinuous feedback scheme, with several advantageous properties regarding insensitivity to external perturbation signals, modelling errors and sudden failure of the smooth portion of the feedback loop.

KEY WORDS: nonlinear systems; discontinuous feedback control; sliding regimes

1. INTRODUCTION

The advent of *differential algebra* into the realm of sliding mode control analysis and design for nonlinear controlled systems[1–6] has resulted in a rather significant departure from traditional discontinuous feedback design schemes utilizing only state-dependent surfaces. Continuous, rather than bang-bang, control input signals, and substantially smoothed chatter-free trajectories have been shown to be some of the several advantageous properties exhibited by the use of input-dependent sliding surfaces in dynamical sliding mode control schemes of nonlinear systems. The possibilities of using discontinuous feedback control policies, such as sliding mode control, pulse-width-modulation, and pulse-frequency-modulation based strategies, in nontraditional application areas, such as chemical processes and mechanical systems, is also one of the new developments emerging from this approach.[7,8]

In this article we propose to utilize an input-dependent sliding surface directly suggested by a model-based designed smooth feedback control law, assumed to ideally stabilize the nonlinear system according to a preselected stabilization criterion (optimal performance, pole placement, exact linearization, etc.). Such a sliding surface coordinate prescription has the interpretation of a *feedback control implementation error* in the sense that any deviation of the implemented input signal, from the required value generated by the smooth state feedback control law, yields an error which immediately triggers a discontinuous feedback control correction signal and forcefully imposes the designed feedback control law. The scheme thus

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operates as a parallel feedback scheme with a 'high gain' loop that enforces the originally designed smooth feedback control law. The activation of the resulting discontinuous feedback controller mechanism, here proposed, complements the original feedback scheme in a redundant fashion, which is capable of tolerating sudden failures in the main smooth designed feedback loop. Feedback implementation error signals are frequently caused by the presence of independent perturbation input signals to the actuator, by feedback designs carried out under unknown modelling errors and, also, by sensor failures, among many other causes. The benefits of our proposed discontinuous feedback control scheme are: (1) enhanced robustness for the actual operation of the designed smooth feedback control law, (2) feedback correction based on smoothed discontinuous feedback control actions, (3) redundancy, in the form of a dynamical discontinuous feedback law, of the designed smooth control scheme and (4) robustness, with respect to sudden failures, in the smooth portion of the proposed parallel feedback scheme.

Section 2 presents the general feedback control scheme based on utilizing the designed smooth feedback control law as an input-dependent sliding surface. In this section we derive and analyse some of the advantageous features of such an approach. Section 3 is devoted to an application example drawn from the chemical process control area. Simulation studies are included. Section 4 contains the conclusions and suggestions for further research.

2. MAIN RESULT

2.1. A dynamical sliding mode controller based on a prescribed smooth feedback control law.

Consider a nonlinear n-dimensional single input smooth system of the form:

$$\dot{x} = f(x, u) \tag{1}$$

Suppose, furthermore, that a smooth feedback controller has been designed which locally stabilizes the trajectories of the control system to a desirable constant equilibrium point $X(U)$, dependent upon a constant value of the input signal U. i.e., $f(X(U), U) = 0$. We assume, without any loss of generality that U is nonzero. The stabilizing feedback control law is assumed to be explicitly given by

$$u = -k(x) \tag{2}$$

in other words, the closed-loop system,

$$\dot{x} = f(\mathrm{x}, -k(x)) \tag{3}$$

is assumed to locally exhibit desirable asymptotic stability features towards the equilibrium point. In equilibrium, the value of the feedback signal $-k(X(U))$ is compatible with the equilibrium value for u, i.e., $U = -k(X(U)) \neq 0$.

Suppose now that an auxiliary input-dependent function of the form:

$$s(x, u) = u + k(x) \tag{4}$$

is synthesized, and proposed as a *sliding surface candidate* on which the following discontinuous dynamics is imposed:

$$\dot{s}(x, u) = -W \operatorname{sign}[s(x, u)] \tag{5}$$

with W being a sufficiently large, strictly positive, constant quantity. Notice that the trajectories of (5) reach the condition $s(x, u) = 0$ in *finite time* T given by: $T = |s(x(0), u(0))|/W$.

Upon reaching of the condition $s(x, u) = 0$ by the scalar value $s(x, u)$ of the sliding surface coordinate, a *sliding motion*[9] is created on such an input-dependent manifold. The sliding mode condition is then sustained in an indefinite manner. The *ideal sliding motions* associated with the sliding regime, thus created, imply that the control input variable u precisely complies with the designed feedback control law. In other words, ideally speaking, under the sliding mode condition one has: $u = -k(x)$.

Replacing (4) into (5) leads to the following dynamical discontinuous sliding mode controller for the nonlinear system:

$$\dot{u} = -\left[\frac{\partial k(x)}{\partial x}\right] f(x, u) - W \operatorname{sign}[u + k(x)] \tag{6}$$

Equation (6) represents a time-varying nonlinear first-order differential equation for the control input signal u, with a discontinuous right-hand side. The additional complication incurred in building such a dynamical discontinuous feedback controller is superseded by the many advantages it bestows on the closed-loop features of the controlled system.

A block diagram of the feedback controller (6) is shown in Figure 1. A straightforward integration of the above expression (6) allows for the reinterpretation of the controller in terms of a redundant 'hybrid' controller comprising the original feedback law (3) implemented in parallel to an integrated (i.e. smoothed) discontinuous feedback signal, triggered by the sign of the feedback error $u + k(x)$. Indeed, integration of (6) yields

$$\begin{aligned} u &= -\int_0^t \left\{\left[\frac{\partial k(x(\sigma))}{\partial x}\right] f(x(\sigma), u(\sigma)) + W \operatorname{sign}[u(\sigma) + k(x(\sigma))]\right\} d\sigma + u(0) \\ &= -k(x(t)) - \int_0^t W \operatorname{sign}[u(\sigma) + k(x(\sigma))]\, d\sigma + [u(0) + k(x(0))] \\ &= -k(x(t)) - W \int_0^t \operatorname{sign}[u(\sigma) + k(x(\sigma))]\, d\sigma + s(x(0), u(0) \end{aligned}$$

A block diagram depicting this reinterpretation of the controller (6) is shown in Figure 2.

From the previous expression one also immediately obtains, upon rearrangement,

$$s(x(t), u(t)) = s(x(0), u(0)) - W \int_0^t \operatorname{sign}[u(\sigma) + k(x(\sigma))]\, d\sigma \tag{7}$$

from where it easily follows that, regardless of the initial value of $s(x(0), u(0))$ of the sliding surface coordinate function, $s(x, u)$, the condition $s(x(T), u(T)) = 0$ is indeed reached in the previously given finite time T.

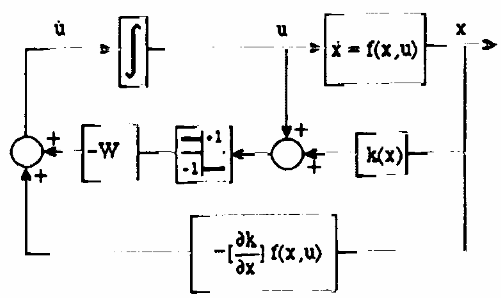


Figure 1. Dynamical sliding mode controller enforcing a designed smooth feedback control law

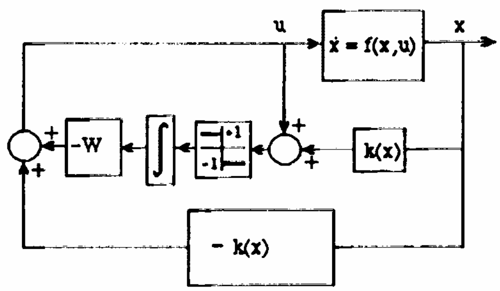


Figure 2. Reinterpretation of dynamical sliding mode controller enforcing a designed smooth feedback control law

By virtue of the above developments, we finally rewrite the dynamical controller (6) as

$$\begin{aligned} u &= v - k(x) \\ \dot{v} &= -W \operatorname{sign}\,[u + k(x)] \end{aligned} \tag{8}$$

2.2. *Some properties of the proposed dynamical discontinuous controller*

The dynamical controller (6) exhibits several advantageous properties which are summarized below.

(1) The discontinuities associated with the underlying sliding motion, taking place along the input-dependent manifold: $s(x, u) = 0$, are relegated to the first-order time derivative of the control input signal u. Hence, the resulting controller is, indeed, *continuous*. Bang-bang input signals, otherwise characteristic of sliding mode control schemes,[9] are thus effectively suppressed by the dynamic nature of the proposed controller.[4-6]

(2) Suppose that at certain time $t = T_f$, the smooth portion of the feedback loop, feeding the signal component $-k(x)$ to the control input signal u, fails for an indefinite period of time (see Figure 3). Assume, furthermore, that at the failure time T_f the discontinuous portion of the controller was currently exhibiting a sliding mode behaviour (i.e., ideally $s(x(T_f), u(T_f)) = 0$). Suppose also that the system's state was already stabilized at its equilibrium value $x = X(U)$ and, hence, $u(T_f) = -k(X(U)) = U$. The feedback control law being enforced at any time $t > T_f$, after the failure of the smooth portion of the feedback loop, satisfies

$$u = -W \int_{T_f}^{t} \operatorname{sign}[u(\sigma) + k(x(\sigma))]\, d\sigma \tag{9}$$

i.e.,

$$\dot{u}(t) = -W \operatorname{sign}\,[u(t) + k(x(t))] \tag{10}$$

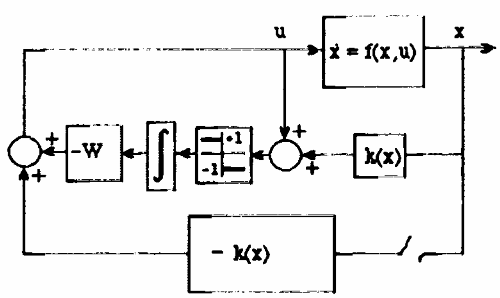


Figure 3. A feedback signal failure in the smooth portion of the redundant controller

It follows that the value of $u(t)$ is instantaneously set to zero at T_f^+. It easily follows from (9) and (10) that u evolves, for $t > T_f$, in such a fashion that the feedback error signal: $s(x, u) = u + k(x)$ is being constantly diminished in absolute value. The discontinuous part of the controller locally sustains the motions of the system in a sliding regime around the manifold: $u + k(x) = 0$, thus recovering, on the average, the original feedback control law $u = -k(x)$. Indeed, consider the value of the product $s(x(t), u(t))\, \mathrm{d}s(x(t), u(t))/\mathrm{d}t$ for $t > T_f$, when (10), rather than (6), is valid

$$\begin{aligned} s(x(t), u(t))\dot{s}(x(t), u(t)) &= s(x(t), u(t))\left[\dot{u} + \frac{\partial k}{\partial x} f(x, u)\right] \\ &= s(x(t), u(t))\left[-W \operatorname{sign}[s(x(t), u(t)] + \frac{\partial k}{\partial x} f(x, u)\right] \\ &= -W\,|s(x(t), u(t))| + s(x(t), u(t))\left[\frac{\partial k}{\partial x} f(x, u)\right] \\ &= \leqslant -|s(x(t), u(t))|\left[W + \frac{\partial k}{\partial x} f(x, u)\operatorname{sign}(s(x(t), u(t)))\right] \end{aligned} \tag{11}$$

Thus, for a sufficiently large value of the constant W, the sign of the above expression can always be made negative on an open neighbourhood defined around $s(x, u) = 0$, and, hence, a sliding regime is seen to locally exist on $s(x, u) = 0$.

Remark. It should be pointed out that the region of attraction of the sliding manifold $s(x, u) = 0$, for control laws satisfying (10) must be precisely determined in each case. It may very well happen that after a feedback loop failure, such as the one described above, the surviving discontinuous portion of the controller is incapable of achieving a sliding motion on the zero error manifold: $s(x, u) = 0$. Such possibility is highly dependent upon the possibility of complying with the negativity of the final expression in (11).

(3) Sliding mode controllers are known to be highly insensitive to external perturbation signals and to modelling errors. Thus the above scheme always imposes, in a robust fashion, the 'right feedback control law'. Changes in state, due to external perturbation inputs to the system, result in corresponding changes in the feedback control law $-k(x)$, both, at the smooth and discontinuous portions of the proposed controller. If the designed smooth control law is known to enjoy robustness properties, with respect to a certain class of perturbation input signals, the proposed controller simply inherits those properties and results in a forceful imposition, on the average, of the required smooth control law. Owing to the abundance of results in the area of robustness of sliding mode control schemes, the reader is referred to the literature and is kindly invited to become convinced of this important feature of discontinuous feedback control schemes.

3. AN APPLICATION EXAMPLE

3.1. A continuously stirred tank reaction system

Consider the following simple nonlinear dynamical model of a controlled CSTR in which an isothermal, liquid-phase, multicomponent chemical reaction takes place:[10,11]

$$\begin{aligned} \dot{x}_1 &= -(1 + D_{a1})x_1 + u \\ \dot{x}_2 &= D_{a1}x_1 - x_2 - D_{a2}x_2^2 \\ y &= x_1 + x_2 - Y \end{aligned} \tag{12}$$

where x_1 represents the normalized (dimensionless) concentration C_p/C_{P0} of a certain species P in the reactor, with $Y = C_{P0}$ being the desired concentration of the species P and Q measured in moles per cubic metre. The state variable x_2 represents the normalized concentration C_Q/C_{P0} of the species Q. The control variable u is defined as the ratio of the per-unit volumetric molar feed rate of species P, denoted by N_{PF}, and the desired concentration C_{P0}, i.e., $u = N_{PF}/(FCP0)$ where F is the volumetric feed rate in cubic metres per second. The constants D_{a1} and D_{a2} are respectively defined as k_1V/F and k_2VC_{P0}/F with V being the volume of the reactor, in cubic metres, and k_1 and k_2 the first-order rate constants (in s^{-1}).

It is assumed that the species Q is highly acidic while the reactant species R is neutral. In order to avoid corrosion problems in the downstream equipment, it is desired to regulate the total concentration y to a prescribed set-point value specified by the constant Y. It is assumed that the control variable u is naturally bounded in the closed interval $[0, U_{max}]$ reflecting the bounded (physical) limits of molar feed rate of the species P.

A stable constant equilibrium point for this system is given by

$$u = U; x_1(U) = \frac{U}{(1 + D_{a1})}; x_2(U) = \frac{1}{2D_{a2}}\left[-1 + \sqrt{1 + \frac{4D_{a1}D_{a2}U}{(1 + D_{a1})}}\right] \tag{13}$$

3.2. *A smooth linearizing controller design for the CSTR system*

It is easy to verify that the following smooth state feedback controller results in an exact input–output linearization of the given system (12):

$$u = (1 - \lambda)\ (x_1 + x_2) + D_{a2}X_2^2 + \lambda Y \tag{14}$$

where λ is a positive quantity regulating the exponential decay in the imposed linear asymptotically stable dynamics for the output y:

$$\dot{y} = \lambda y \tag{15}$$

It may be verified, after some tedious but straightforward computations, that the closed-loop system (12), (15) exhibits a locally asymptotically stable zero dynamics around the equilibrium point (13).[10]

3.3. *Redundant dynamical sliding mode controller design for the CSTR system*

According to the results of Section 2 we choose as a sliding surface the input dependent sliding surface:

$$s(x, u) = u - (1 - \lambda)\ (x_1 + x_2) - D_{a2}x_2^2 - \lambda Y \tag{16}$$

and by imposing the dynamics (2.5) on $s(x, u)$ one obtains the following dynamical sliding mode controller:

$$\begin{aligned} u &= v + (1 - \lambda)\ (x_1 + x_2) + D_{a2}x_2^2 + \lambda Y \\ \dot{v} &= -W \operatorname{sign}\left[u - (1 - \lambda)\ (x_1 + x_2) - D_{a2}x_2^2 - \lambda Y\right] \end{aligned} \tag{17}$$

3.4. *Simulations*

Simulations were performed for the system (12) with the dynamical controller (17). The numerical values adopted for the system parameters[10] and for the dynamical sliding mode controller parameters W and λ were

$$D_{a1} = 1;\ D_{a2} = 1;\ Y = 3$$
$$W = 10;\ \lambda = 4$$

The equilibrium points for the control input and the product concentrations, corresponding to these parameters, are computed from (13). These result in

$$U = 4;\ x_1(U) = 2;\ x_2(U) = 1$$

Figure 4 shows the state responses of the dynamically sliding mode controlled system asymptotically converging to their corresponding equilibrium points. Figure 5 depicts the continuous trajectory of the control input, while Figure 6 shows the evolution of the input dependent sliding surface coordinate function $s(x, u)$, converging to zero in finite time.

In order to check the robustness of the proposed dynamical sliding mode control scheme with respect to sudden failures in the originally designed smooth feedback loop we also simulated the performance of the system with the following dynamical discontinuous feedback controller:

$$\begin{aligned} u &= v + \kappa[(1-\lambda)(x_1 + x_2) + D_{a2}x_2^2 + \lambda Y] \\ \dot{v} &= -W \operatorname{sign}[u - (1-\lambda)(x_1 + x_2) - D_{a2}x_2^2 - \lambda Y] \end{aligned} \tag{18}$$

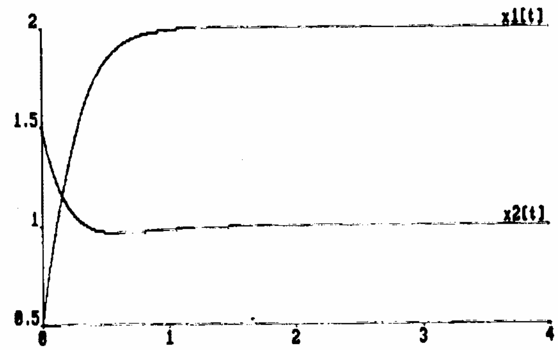


Figure 4. State trajectory responses of dynamically sliding mode controlled system

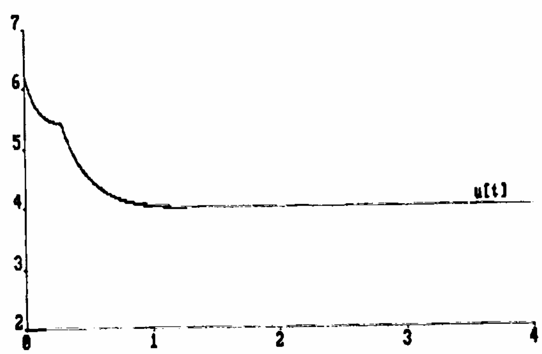


Figure 5. Continuous trajectory of the control input signal

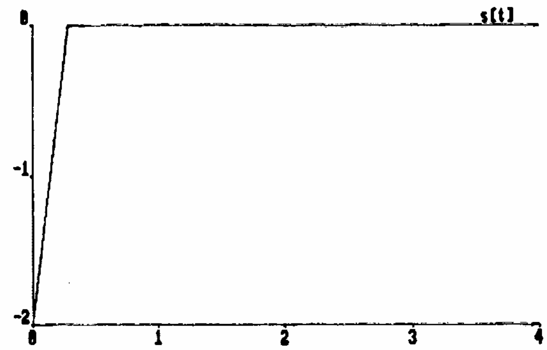


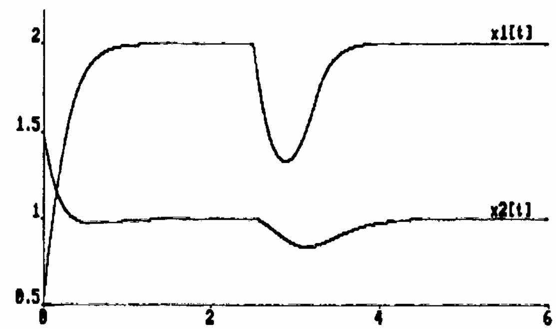
Figure 6. Evolution of the input-dependent sliding surface coordinate function

where the variable κ, simulating the feedback loop failure, was allowed to be

$$\kappa = \begin{cases} 1 \text{ for } t \leq T_f \\ 0 \text{ for } t > T_f \end{cases} \tag{19}$$

with $T_f = 2{\cdot}5$.

Figure 7 shows the state responses of the dynamically sliding mode controlled system subject to the sudden failure of the form (19). The sliding mode controller is seen to



Figures 7. State trajectory responses of dynamically sliding mode controlled system subject to a sudden failure in the smooth portion of the feedback controller

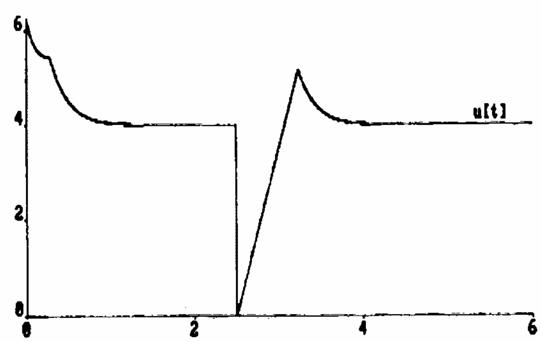


Figure 8. Trajectory of the control input signal subject to a sudden failure

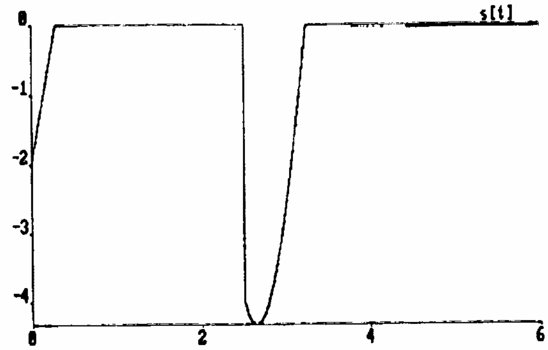


Figure 9. Behaviour of the input-dependent sliding surface coordinate function subject to a sudden failure

restabilize the state trajectories to their corresponding equilibrium points. Figure 8 depicts the corresponding trajectory of the failed control input, showing, at failure time T_f, the instantaneous resetting to the value zero of the control input signal and its subsequent recovery to a sliding mode regulation of the system. Notice that the surviving portion of the controller still generates a continuous feedback signal. Figure 9 shows the behaviour of the input dependent sliding surface coordinate function $s(x, u)$ before and after the feedback signal failure.

4. CONCLUSIONS

A robust redundant feedback control scheme, based on dynamical sliding mode control, has been proposed for nonlinear systems for which a smooth feedback control policy is already available. The proposed scheme utilizes the implementation error associated with the designed smooth feedback control policy as a sliding surface and proceeds to forcefully impose the desirable relation by means of an appropriately induced sliding regime. The resulting dynamical controller is then reinterpreted in terms of two subsystems, One being the smooth portion of the controller, represented by the originally designed stabilizing, static, feedback control law, and the other being a parallel regulator loop based on dynamically generated (i.e., smoothed) discontinuous control actions of the sliding mode (i.e., relay) type. The scheme was shown to be advantageous in several respects, among which, we found local robustness with respect to sudden failures in the static portion of the proposed feedback controller. An application example, drawn from a nontraditional application area for sliding mode control, was also presented. The basic features of the proposed redundant dynamical discontinuous feedback control scheme were illustrated by means of simulations.

Dynamical sliding mode control of nonlinear systems has been extended, in a unifying manner, to pulse-width modulation and pulse-frequency modulation based schemes.[7,12]. Such unified treatment involves a systematic use of generalized canonical forms of nonlinear systems as proposed by Fliess.[13] The redundant feedback controllers, here described, can also be extended, in a rather similar manner, to the above mentioned classes of discontinuous feedback control policies. Extension to multi-input systems should have little or no difficulties, provided decoupled sliding regimes of the form (2.5) are imposed on the several input-dependent sliding surfaces representing every component of the designed multivariable smooth feedback controller.

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