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On the sliding mode control of multivariable nonlinear systems

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Sliding mode control of multivariable nonlinear systems is addressed, from the perspective of linear differential algebra, for a special but large class of linearizable systems known as *differentially flat systems*. Essential orders and differential flatness are shown to be, a relevant concept and a sensible requirement respectively, associated with the possibilities of designing static or dynamical sliding mode multivariable feedback regulators for nonlinear systems.

1. Introduction

Discontinuous feedback regulation of nonlinear multivariable systems, such as sliding mode control, pulse width modulation or pulse frequency modulation strategies, pose special questions related to the underlying structure at infinity of the inputs-to-sliding-surfaces relation and the associated feedback decoupling problem. Most importantly, the feasibility of a well-defined sliding mode strategy for systems which are not statically decouplable requires suitable dynamical extensions obtained by invoking the essential structure at infinity. From this viewpoint, the differential flatness of the system seems to be a most natural restriction for the class of systems which enjoy feasible dynamical or static sliding mode control strategies. The following example illustrates the issues in some detail.

Example 1.1: Consider the following example, taken from Charlet *et al.* (1990):

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u_1 \\ \dot{x}_3 &= u_2 \\ \dot{x}_4 &= x_3 - x_3 u_1 \\ s_1 &= x_1 \\ s_2 &= x_4 \end{aligned}\right\} \tag{1.1}$$

where s_1 and s_2 represent proposed (outputs) sliding surface coordinates which must be robustly zeroed by means of a discontinuous multivariable feedback control strategy.

The structure at infinity of the inputs to the sliding surface system is obtained from the following relation:

$$\begin{pmatrix} \ddot{s}_1 \\ \dot{s}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ x_3 & 0 \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{1.2}$$

The system is clearly not *statically decouplable* and, hence, a *static* sliding mode feedback control policy cannot be directly enforced. Note moreover that, if further

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sliding surface coordinate derivatives are computed for s_2, in order to obtain an explicit dependence on the control input u_2, then one obtains

$$\begin{pmatrix} \ddot{s}_1 \\ \ddot{s}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -x_3\dot{u}_1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1-u_1 \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{1.3}$$

Clearly, if a sliding regime is devised for s_1 by imposing on it the discontinuous dynamics

$$\ddot{s}_1 = -\lambda\dot{s}_1 - W_1 \operatorname{sign}(\dot{s}_1 + \lambda s_1), \quad \lambda > 0 \tag{1.4}$$

then the first control input u_1 is of the form

$$u_1 = -\lambda\dot{s}_1 - W_1 \operatorname{sign}(\dot{s}_1 + \lambda s_1) \tag{1.5}$$

As a result, $\dot{u}_1$ is no longer defined after sliding is achieved and the second derivative of s_2 is also no longer defined. No sliding mode policy can be devised for s_2 in terms of u_2.

Clearly, the solution of the sliding mode design problem, for this example, rests on the possibilities of introducing *dynamical decoupling* of the given system. Indeed, extending the given system according to

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u_1 \\ \dot{u}_1 &= v_1 \\ \dot{x}_3 &= u_2 \\ \dot{x}_4 &= x_3 - x_3 u_1 \\ s_1 &= x_1 \\ s_2 &= x_4 \end{aligned}\right\} \tag{1.6}$$

where v_1 is a new auxillary input and u_1 is just an additional state variable, one now has the following input to sliding surface relations:

$$\begin{pmatrix} s_1^{(3)} \\ \ddot{s}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -x_3 & 1-u_1 \end{pmatrix} \begin{bmatrix} v_1 \\ u_2 \end{bmatrix} \tag{1.7}$$

The extended system is now decouplable and both v_1 and u_2 may be simultaneously defined as discontinuous feedback policies without further complications. Suitable sliding dynamics to be imposed on s_1 and s_2 may be proposed, for appropriate constants ζ, ω_n and λ, as

$$\left.\begin{aligned} s_1^{(3)} &= -2\zeta\omega_n \ddot{s}_1 - \omega_n^2 \dot{s}_1 - W_1 \operatorname{sign}(\ddot{s}_1 + 2\zeta\omega_n \dot{s}_1 + \omega_n^2 s_1) \\ \ddot{s}_2 &= -\lambda \dot{s}_2 - W_2 \operatorname{sign}(\dot{s}_2 + \lambda s_2) \end{aligned}\right\} \tag{1.8}$$

This choice clearly leads to the following well-defined dynamical multivariable sliding mode feedback controller with input-dependent sliding surfaces:

$$\left.\begin{aligned} \dot{u}_1 &= -2\zeta\omega_n u_1 - \omega_n^2 x_2 - W_1 \operatorname{sign}(u_1 + 2\zeta\omega_n x_2 + \omega_n^2 x_1) \\ u_2 &= \frac{1}{1-u_1}\{x_3[-2\zeta\omega_n u_1 - \omega_n^2 x_2 - W_1 \operatorname{sign}(u_1 + 2\zeta\omega_n x_2 + \omega_n^2 x_1)] \\ &\quad - \lambda(1-u_1)x_3 - W_2 \operatorname{sign}[(1-u_1)x_3 + \lambda x_4]\} \end{aligned}\right\} \tag{1.9}$$

From a different viewpoint, if we now seek to express all the variables in the original system (1.1) in terms of the sliding surface coordinates s_1 and s_2 we obtain

$$\left.\begin{aligned} x_1 &= s_1 \\ x_2 &= \dot{s}_1 \\ u_1 &= \ddot{s}_1 \\ x_4 &= s_2 \\ x_3 &= \frac{\dot{s}_2}{1-\ddot{s}_1} \\ u_2 &= \frac{\ddot{s}_2(1-\ddot{s}_1)+\dot{s}_2 s_1^{(3)}}{(1-\ddot{s}_1)^2} \end{aligned}\right\} \qquad (1.10)$$

It is also easy to see from the above calculations that the system is not linearizable by means of static discontinuous state feedback since time differentiation of the input u_1 is needed for the synthesis of u_2.

Indeed, if $s_1^{(3)}$ and $\ddot{s}_2$ are specified as discontinuous autonomous trajectories, as in (1.8), then the fact that the control input variables are expressible as *differential functions* of s_1 and s_2 leads immediately to the definition of a discontinuous feedback control law for u_2. The same fact allows one to conclude that $\dot{u}_1$ must be, necessarily, taken as an auxiliary control input for which the corresponding discontinuous feedback policy of (1.8) is readily assignable. Again, the extended system (1.6) had to be invoked and the dynamical, rather than static, character of the feasible robustly linearizing sliding mode control policy is apparent.

The above considerations show that decoupled sliding regimes can be synthesized by means of *endogenous feedback*, that is one that does not require external variables to the system in order to be defined. The feasibility of the design rests on the existence of decouplable linearizing outputs s_1 and s_2 in equal number to the control inputs. These properties identify a class of multivariable systems for which discontinuous decoupled feedback linearization is indeed achievable. The preceding properties of the given system are summarized by saying that the system is *differentially flat*. □

In this article we study the sliding mode controller design problem for multivariable differentially flat systems. Differentially flat systems constitute a most important class of systems fully linearizable, by means of endogenous feedback, to controllable linear systems. Differentially flat systems have been introduced in recent works by Fliess *et al.* (1991, 1992a, 1992b, 1993a, 1993b). Practical examples of some mechanical systems, such as the truck and the trailer, the jumping robot, and the crane were presented by Fliess *et al.* (1991, 1992a, 1992b). Uncontrollable systems and systems *without* the strong accessibility property constitute typical examples of non-flat systems. In the paper by Fliess *et al.* (1993a), the *Kapitsa pendulum* and the double inverted pendulum were shown to be non-differentially flat systems which can, nevertheless, be 'flattened', in an average sense, by means of high-frequency vibratory control and the application of averaging techniques to the resulting system. Further developments and examples can be found in the paper by Martin (1992) and relations with *quasistatic* feedback have been given by Rudolph (1993). In a recent article by Pomet *et al.* (1992) connections of flat systems with non-exact Brunovsky canonical forms is established using equivalent systems of exact 1-forms. Further interesting relations with Fliess' differential algebraic approach have been given by Aranda-Bricare *et al.* (1995).

Because of the existence of linearizing outputs, differentially flat systems are naturally decouplable linearizable systems. For differentially flat systems where static decoupling is possible, sliding mode strategies may be readily prescribed. If decoupled linearization requires, however, dynamical feedback, then the notion of essential orders becomes particularly relevant in the specification of well-defined dynamical discontinuous feedback control strategies. Our presentation uses the recently developed *linear differential algebraic approach* to system analysis (see the work of Di Benedetto *et al.* (1989), Glumineau and Moog (1989), Glumineau (1992), Moog *et al.* (1991) and Grizzle (1993)). This technique is reviewed, in a tutorial fashion, in § 2 of this article. In § 3 we outline a design procedure for specifying sliding mode control laws for differentially flat systems and present some illustrative examples. The last section is devoted to the conclusions and some suggestions for further work.

2. Mathematical preliminaries

2.1. *Basics of linear differential algebra in control systems*

In this section we summarize, in a tutorial fashion, the main results, which are applicable to multivariable sliding mode control design, of the linear differential algebraic approach to control system analysis. The reader is referred to the work of Di Benedetto *et al.* (1989), for further interesting details. The presentation here closely follows that of Glumineau (1992).

We consider finite-dimensional nonlinear control systems of the form

$$\left.\begin{aligned} \dot{x} &= f(x) + G(x)u \\ s &= h(x) \end{aligned}\right\} \tag{2.1}$$

where $x \in \Re^n$ represents the state vector, $u \in \Re^m$ is the control input vector and $s \in \Re^m$ represents the set of sliding surface coordinate functions (or system outputs to be zeroed). The components of the vector function $f(x)$, the columns $g_i(x)$, $i = 1, \dots, m$ of the matrix $G(x)$ and the components of the vector function $h(x)$ are supposed to be *meromorphic functions* of the components of x, that is they are elements of the fraction field of the ring of analytic functions on a certain domain $\mathcal{D}$ of $\Re^n$.

Consider the field $\mathcal{K}$ of meromorphic functions of the finite collection of components of the set $\{x, u, \dot{u}, \dots, u^{(n-1)}\}$.

Example 2.1: A typical element η of $\mathcal{K}$ may be given by

$$H = \frac{e^{x_1} u_5^{(3)} - x_3^4}{u_2 \sin x_1 + (\ddot{u}_3)^2} \tag{2.2}$$

□

One may consider the differentials of the components of $x, u, \dots, u^{(n-1)}$ as independent 'coordinate directions' of a *formal vector space* defined over the field $\mathcal{K}$. This not only provides us with precise 'tags' for identifying the dependence of given system functions over the components of the set of vectors $x, u, \dots, u^{(n-1)}$ but also gives us the possibilities of formally working over an underlying linear vector space where we may express known properties of the given system as 'geometric properties' over such a space. This simple observation results in a suitable combination of the simplicity and power of the algebraic reasoning with the intuitive value of geometry.

Consider then the vector space $\mathscr{E}$ spanned over the field $\mathscr{K}$ by the set of *differentials* (also called *1-forms*, *exact differentials*, *covectors*, etc.), $\{dx, du, d\dot{u}, \ldots, du^{(n-1)}\}$. Notationally,

$$\mathscr{E} = \text{span}_{\mathscr{K}}\{dx, du, d\dot{u}, \ldots, du^{(n-1)}\} \tag{2.3}$$

Here dx stands for $dx_1, \ldots, dx_n$, while du denotes $du_1, \ldots, du_m$, etc.

Example 2.2: The differential $d\eta$ of a scalar function $\eta = \eta(x, u, \dot{u}, \ldots, u^{(k)})$, $k \leqslant n-1$, exhibiting an explicit dependent on the control input and its time derivatives, up to order k, is given by

$$d\eta = \sum_{i=1}^{n} \frac{\partial \eta}{\partial x_i} dx_i + \sum_{j=1}^{m} \sum_{l=0}^{k} \frac{\partial \eta}{\partial u_j^{(l)}} du_j^{(l)} \tag{2.4}$$

Evidently $d\eta$ computed in (2.4) satisfies $d\eta \in \mathscr{E}$. We say $d\eta$ is a $\mathscr{K}$-linear combination of the components of the differentials $dx, du, \ldots, du^{(k)}$. □

Example 2.3: Consider the single-input single-sliding surface case of (2.1). Then the jth time derivative of the function $s = h(x)$ depends on u and, at most, on $j-1$ of its time derivatives if and only if

$$ds^{(j)} \notin \text{span}_{\mathscr{K}}\{dx\}$$

that is $ds^{(j)}$ is not in the *subspace* of $\mathscr{E}$ spanned over the field $\mathscr{K}$ by the n possible coordinate 'directions' corresponding to dx. □

Since all differentials of time derivatives of the components of s may be, ultimately, expressed as $\mathscr{K}$-linear combinations of dx and suitable differentials of derivatives of u, that is du^j, $j = 0, 1, 2, \ldots$, one may introduce a sequence of subspaces in $\mathscr{E}$ in terms of differentials of the components of x and the differentials of derivatives of the components of s. Consider then

$$\left.\begin{aligned} \mathscr{E}_0 &= \text{span}_{\mathscr{K}}\{dx\} \\ \mathscr{E}_1 &= \text{span}_{\mathscr{K}}\{dx, d\dot{s}\} \\ &\vdots \\ \mathscr{E}_n &= \text{span}_{\mathscr{K}}\{dx, d\dot{s}, \ldots, ds^{(n)}\} \end{aligned}\right\} \tag{2.5}$$

Evidently such sequence of subspaces over $\mathscr{K}$ is a growing sequence. They are said to constitute a *chain* or, more properly, an *ascending chain*, satisfying by $\mathscr{E}_0 \subset \mathscr{E}_1 \subset \ldots \subset \mathscr{E}_n$.

Example 2.4: Consider the n-dimensional single-input system $\dot{x} = f(x) + g(x)u$. Let $\mathscr{X} = \text{span}_{\mathscr{K}}\{dx\} = \mathscr{E}_0$ and let $\mathscr{Y} = \text{span}_{\mathscr{K}}\{dy, d\dot{y}, \ldots, dy^{(n-1)}\}$, for some scalar function $y = h(x)$. The system is transformable to a linear controllable system of the form $y^{(n)} = v$, with v being a new control input, if and only if there exists such a scalar function y, for which the following conditions are satisfied:

$$dy^{(j)} \in \mathscr{X}, \quad j = 0, 1, \ldots, n-1$$

$$dy^{(n)} \notin \mathscr{X}$$

$$\dim_{\mathscr{K}}(\mathscr{Y} \cap \mathscr{X}) = n$$

The first and the last condition imply that $y, \dot{y}, \ldots, y^{(n-1)}$ define a local diffeomorphism $z_i = y^{(i-1)}(x)$, while the second condition simply says that $y^{(n)} = \dot{z}_n = \alpha(x) + \beta(x)\,u$ with, necessarily, $\beta(x) \neq 0$. Evidently, $\dot{z}_i = z_{i+1}$, $i = 1, 2, \ldots, n-1$. The new input v is immediately defined as $v = \alpha(x) + \beta(x)\,u$ and, hence, $y^{(n)} = \dot{z}_n = v$. □

Example 2.5: Consider a multivariable linear time-invariant system $\dot{x} = Ax$, $s = Cx$ and let $\mathscr{K} = \mathfrak{R}$. If we let $\mathscr{X} = \operatorname{span}_{\mathfrak{R}}\{\mathrm{d}x\}$ and $\mathscr{S} = \operatorname{span}_{\mathfrak{R}}\{\mathrm{d}s, \mathrm{d}\dot{s}, \ldots, \mathrm{d}s^{(n-1)}\}$, then

$$\left.\begin{aligned} \mathrm{d}s &= C\,\mathrm{d}x \in \mathscr{E}_0 \\ \mathrm{d}\dot{s} &= CA\,\mathrm{d}x \in \mathscr{E}_0 \\ &\vdots \\ \mathrm{d}s^{(n-1)} &= CA^{n-1}\,\mathrm{d}x \in \mathscr{E}_0 \end{aligned}\right\} \tag{2.6}$$

The system is *observable* if, and only if,

$$\dim_{\mathfrak{R}}(\mathscr{X} \cap \mathscr{S}) = n$$

If we now let the system be controlled by u, as in $\dot{x} = Ax + Bu$, we have

$$\left.\begin{aligned} \mathrm{d}s &= C\,\mathrm{d}x \in \mathscr{E}_0 \\ \mathrm{d}\dot{s} &= CA\,\mathrm{d}x + CB\,\mathrm{d}u \in \mathscr{E}_1 \subset \operatorname{span}_{\mathfrak{R}}\{\mathrm{d}x, \mathrm{d}u\} \\ &\vdots \end{aligned}\right\} \tag{2.7}$$

Note that $\mathrm{d}\dot{s} \notin \mathscr{X}$ if, and only if, the matrix B is not identically zero, similarly $\mathrm{d}\ddot{s} \notin \operatorname{span}_{\mathfrak{R}}\{\mathrm{d}x, \mathrm{d}u\}$ if, and only if, the matrix CAB is not identically zero, etc. □

2.2. *Structure at infinity, invertibility and decouplability*

As hinted by the previous examples, it becomes relatively simple, in the adopted framework, to define the (row) *relative degree* of a particular sliding surface coordinate function s_i in the vector s (the term *vector relative degree* has also been used by Isidori (1990)).

The *row relative degree* of the sliding surface $s_i = h_i(x)$, $i = 1, \ldots, m$ is defined as the integer $n_i \geqslant 1$ such that

$$\left.\begin{aligned} &\mathrm{d}s_i^{(j)} \in \mathscr{E}_0, \quad \text{for } 0 \leqslant j \leqslant n_i - 1 \\ &\mathrm{d}s_i^{(n_i)} \notin \mathscr{E}_0 \end{aligned}\right\} \tag{2.8}$$

An ill-defined sliding surface s_i is constituted by a function which has infinite relative degree, that is $\mathrm{d}s_i^{(k)} \in \mathscr{E}_0\, \forall\, k \geqslant 1$ (the time derivatives of such a sliding surface never depend, explicitly, on the components of the control input vector).

Consider now the quotient space $\mathscr{E}_1/\mathscr{E}_0$. According to the definition of the involved subspaces, this quotient space is, evidently, also a vector space, over $\mathscr{K}$, spanned now by the 'residual basis' directions, abusively denoted here by $\mathrm{d}\dot{s}$. Note that, if the dimension of this quotient vector space is zero, then necessarily, $\dot{s}$ depends only on x.

This is clear since $\mathrm{d}\dot{s}$ in $\mathscr{E}_1$ can be written as a $\mathscr{K}$-linear combination of the basis covectors $\mathrm{d}x$ spanning $\mathscr{E}_0$. On the other hand, if the dimension of such quotient space is, say σ_1 (necessarily inferior to m), then precisely σ_1 components of $\mathrm{d}\dot{s}$ cannot be expressed as a $\mathscr{K}$-linear combination of the components of $\mathrm{d}x$ alone. As a consequence, some of the components of $\mathrm{d}u$ have to be used. This clearly indicates that σ_1 components of $\dot{s}$ indeed depend on some of the components of u. The system has then σ_1 zeros at infinity of order equal to 1.

Suppose then that $\dim_{\mathscr{K}} \mathscr{E}_1/\mathscr{E}_0 = \sigma_1$, and consider next the quotient space $\mathscr{E}_2/\mathscr{E}_1$. If $\dim_{\mathscr{K}} \mathscr{E}_2/\mathscr{E}_1 = \sigma_2 \neq 0$, then σ_2 components of $\mathrm{d}\ddot{s}$ cannot be written as $\mathscr{K}$-linear combinations of the components of both $\mathrm{d}x$ and $\mathrm{d}\dot{s}$, that is as a $\mathscr{K}$-linear combination of $\mathrm{d}x$ and those components of $\mathrm{d}u$ already present in $\mathrm{d}\dot{s}$. If $\sigma_2 = \sigma_1$, the new dimensions present in $\mathscr{E}_2$ correspond to the differentials of the derivatives of the components of u already present in $\mathrm{d}s$ and no new components of $\mathrm{d}u$ would appear. If, on the other hand, $\sigma_2 > \sigma_1$, it follows that additional components in $\mathrm{d}u$ have to be invoked. The increase in dimension of $\mathscr{E}_2$ over that of $\mathscr{E}_1$ is clearly conformed by the σ_1 elements in $\mathrm{d}\dot{u}$, which are inherited from the dependence of $\mathrm{d}\dot{s}$ on $\mathrm{d}u$, and the additions of new components of $\mathrm{d}u$. Clearly then σ_2 represents the number of zeros at infinity of order *less than or equal to* 2. Moreover $\sigma_2 - \sigma_1$ precisely represents the number of zeros at infinity of order 2.

The above simple explanation allows one to generalize the situation as follows (Moog *et al.* 1991).

The ascending chain $\mathscr{E}_0 \subset \mathscr{E}_1 \subset \dots \subset \mathscr{E}_n$ allows one to define the following sequence of non-decreasing integers: $\sigma_1 \leqslant \sigma_2 \leqslant \dots \leqslant \sigma_n$ given by

$$\sigma_k = \dim_{\mathscr{K}} \frac{\mathscr{E}_k}{\mathscr{E}_{k-1}} \tag{2.9}$$

One now easily concludes that (Glumnieau 1992)

$$\left.\begin{aligned} p_1 &= \sup\{\sigma_k, k \geqslant 1\}, \text{ equals the total number of zeros at infinity} \\ p_1 &= p_1 - \sigma_{i-1}, \text{ equals the number of zeros at infinity of order larger} \\ &\qquad\qquad \text{or equal to } i, \text{ for } i \leqslant 2 \end{aligned}\right\} \tag{2.10}$$

If one lets $n_j = \operatorname{card}\{p_i; \geqslant j\}$ then the list $\{n_j; j \geqslant 1\}$ represents the list of orders of zeros at infinity.

A system is said to be *right invertible* if the number of independent outputs (sliding surfaces) equals the number of outputs (Fliess 1989). A system is *left invertible* if the number of independent outputs equals the number of inputs. For *square systems*, that is systems with the same number of inputs as outputs, left and right invertibility are then equivalent. The input-to-sliding-surface system is said to be *decouplable* (i.e. each sliding surface vector component depends on only one input vector component) if and only if it is right invertible (Fliess 1989).

The previous result has also been established, in linear differential algebraic terms, by Glumineau (1992). In our square input-to-sliding surface system case, such a result is as follows.

The system (2.1) is input-to-sliding surface *decouplable* if and only if

$$\dim_{\mathscr{K}} \left(\frac{\mathscr{E}_n}{\mathscr{E}_{n-1}}\right) = m \tag{2.11}$$

Alternatively, by introducing $\mathscr{X} = \text{span}_{\mathscr{K}}\{dx\}$ and $\mathscr{S} = \text{span}_{\mathscr{K}}\{ds, d\dot{s}, \ldots, ds^{(n)}\}$, the square system (2.1) will be decouplable if, and only if,

$$\dim_{\mathscr{K}}(\mathscr{X} \cap \mathscr{S}) = n \tag{2.12}$$

Example 2.6: Consider the simplified model of the angular velocities evolution of a rigid spacecraft provided only with two actuators (jets) acting on two of the principal axes, while the uncontrolled axis is not an axis of symmetry (see, for instance, Aeyels (1985))

$$\left.\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 x_2 \\ s_1 &= x_1 \\ s_2 &= x_3 \end{aligned}\right\} \tag{2.13}$$

Clearly

$$\left.\begin{aligned} dx &= [dx_1\, dx_2\, dx_3]', \quad ds_1 = dx_1, \quad ds_2 = dx_3 \\ d\dot{s}_1 &= du_1, \quad d\dot{s}_2 = x_2\, dx_1 + x_1\, dx_2 \\ d\ddot{s}_1 &= d\dot{u}_1, d\ddot{s}_2 = u_1\, dx_2 + u_2\, dx + x_2\, du_1 + x_1\, du_2 \\ ds_1^{(3)} &= d\ddot{u}_1, ds_2^{(3)} = \dot{u}_2\, dx_1 + \dot{u}_1\, dx_2 + x_2\, d\dot{u}_1 + x_1\, d\dot{u}_2 \end{aligned}\right\} \tag{2.14}$$

From this simple calculation one obtains

$$\left.\begin{aligned} \mathscr{E}_0 &= \text{span}_{\mathscr{K}}\{dx_1\, dx_2\, dx_3\} \\ \mathscr{E}_1 &= \text{span}_{\mathscr{K}}\{dx_1\, dx_2\, dx_3\, du_1\} \\ \mathscr{E}_2 &= \text{span}_{\mathscr{K}}\{dx_1\, dx_2\, dx_3\, du_1\, du_2\, d\dot{u}_1\} \\ \mathscr{E}_3 &= \text{span}_{\mathscr{K}}\{dx_1\, dx_2\, dx_3\, du_1\, du_2\, d\dot{u}_1\, d\dot{u}_2\, d\ddot{u}_1\} \end{aligned}\right\} \tag{2.15}$$

and then

$$\sigma_1 = 1, \quad \sigma_2 = 2, \quad \sigma_3 = 2 \tag{2.16}$$

The relative degree of s_1 is 1, that of s_2 is 2, the system has one zero at infinity of order 1 and one zero at infinity of order 2. The system is also right invertible and hence decouplable by means of static feedback. □

2.3. *Essential orders*

An important definition in nonlinear multivariable control is that of *essential orders*. The essential order fundamentally represent the obstructions to decoupled linearization by means of static feedback. In the context of sliding mode control, the essential orders represent the structure of the obstructions to defining discontinuous dynamics for certain sliding surface coordinates. We present here the definition given by Glumineau and Moog (1989), in a slightly different form.

Define, associated with the ith component of the sliding surface vector s_i, the sequence $\mathscr{E}^{\{i,(k)\}}$ of subspaces in $\mathscr{E}$ characterized by

$$\mathscr{E}_n = \mathscr{E}^{\{i,(k)\}} \oplus \text{span}_{\mathscr{K}}\{ds_i^{(k)}\} \tag{2.17}$$

where $\oplus$ stands for direct subspace addition.

The essential order of the sliding surface coordinate s_i is said to be n_{ie} whenever

$$\left.\begin{array}{ll} s^{(j)} \notin \mathscr{E}^{\{i,(j)\}} & \text{for } 0 \leqslant j \leqslant n_{ie}-1 \\ s_i^{(n_{ie})} \notin \mathscr{E}^{\{i,(n_{ie})\}} & \end{array}\right\} \tag{2.18}$$

The essential order of a particular output function represents the smallest order of the time derivatives of the function for which such a derivative becomes dependent upon a new input coordinate, irrespective of the fact that such an input may be present in some other outputs, in their derivatives of any order, or even if it is present in any of the derivatives of higher order of the considered output function itself. Additionally, this dependence is such that it does not cause the considered output derivative to be functionally related to its higher-order derivative expressions, or to other output functions nor to any of their possible time derivatives.

In the context of dynamical sliding mode control, essential orders precisely indicate the number of necessary dynamic extensions that must be performed, for certain input variables, in order to have well-defined required discontinuities in some of the time derivatives of the system outputs (or system sliding surfaces).

It is easy to see that the essential orders of output functions are higher than or equal to but never smaller than their corresponding row relative degrees. This explains the important obstruction that may be present in the definition of discontinuous feedback control policies in multivariable systems which are not statically input-to-sliding-surface decouplable.

Example 2.7: In Example 2.6, the essential orders can be easily computed by arranging the relations, between the differentials $ds_i^{(k)}$, $k = 0, 1, 2, 3$ and the state and input differentials $dx, du, d\dot{u}, d\ddot{u}$, as a matrix array (also called a *jacobian matrix*)

$$\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \\ d\dot{s}_1 \\ d\dot{s}_2 \\ d\ddot{s}_1 \\ d\ddot{s}_2 \\ ds_1^{(3)} \\ ds_2^{(3)} \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ x_2 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \mathbf{u_2} & \mathbf{u_1} & \mathbf{0} & \mathbf{x_2} & \mathbf{x_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \dot{u}_2 & \dot{u}_1 & 0 & 0 & 0 & x_2 & x_1 & 0 & 0 \end{pmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \\ du_1 \\ du_2 \\ d\dot{u}_1 \\ d\dot{u}_2 \\ d\ddot{u}_1 \\ d\ddot{u}_2 \end{bmatrix} \tag{2.19}$$

The lines in bold type correspond to the first differentials of the derivatives of s_1 and s_2 which cannot be placed as non-trivial $\mathscr{K}$-linear combinations of the state and output components differentials. The essential orders are then 1 for s_1 and 2 for s_2 (everywhere except at $x_1 = 0$). The essential orders thus coincide with the relative degrees. The differentials $d\dot{s}_1$ and $d\ddot{s}_2$ are called *essential differentials* and the corresponding rows in the jacobian matrix are addressed as *essential rows* (Glumineau 1992). □

The importance of the essential orders in the decoupling problem was thoroughly addressed in Glumineau and Moog (1989) and Glumineau (1992) and it is summarized by the following statement.

Theorem 2.1 (Glumineau and Moog 1989): *If there exists a static or dynamic state feedback solving the decoupling problem then, for each one of the outputs of the system,* *it is verified that its (decoupled) relative degree is not inferior to its essential order. Moreover, there exists a (possibly extended) decoupled system, deduced from the original system, for which the essential orders coincide with the decoupled relative degrees.*

The *structure at infinity* of the system is just the list of row relative degrees of the outputs in a fixed but arbitrary order. Similarly, the *essential structure* is the list of essential orders of the outputs arranged in the same order.

By suitable extension, the structure at infinity of the decoupled system can be made to coincide with the essential structure of the original system. This precisely identifies a class of non-statically decouplable multivariable nonlinear systems for which dynamical sliding mode control policies can be feasibly defined. The additional restriction of differential flatness focus attention on those systems which are also linearizable to decoupled *controllable* systems.

For systems decouplable by means of *regular static state feedback* the essential structure coincides with the structure at infinity (Glumineau 1992) and the prescription of sliding mode policies possess no special problem.

Example 2.8: In Example 2.7, the system has the same essential structure as the structure at infinity. The system is hence decouplable by means of static feedback. Indeed, the system decoupling matrix is not everywhere singular as it may be easily inferred from

$$\begin{bmatrix} \dot{s}_1 \\ \ddot{s}_2 \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ x_2 & x_1 \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2.20}$$

□

2.4. *Differentially flat systems*

Differentially flat systems constitute a widespread class of dynamical systems which represent the simplest possible extension of controllable linear systems to the nonlinear systems domain. Flat systems (in short) enjoy the property of possessing a finite set of *differentially independent outputs* (i.e. coordinates which do not satisfy, by themselves, nonlinear differential equations), called *linearizing outputs*, such that all variables in the system may be expressed as special functions, termed *differential functions*, of such coordinates (they are functions of the linearizing coordinates and of a finite number of their time derivatives). Flat systems are systems linearizable to decoupled controllable systems. The fact that inputs and state components are functions of the linearizing (flat) outputs imply that linearization can be carried out within the class of *endogenous* feedback policies (the static or dynamical nature of the required feedback will depend, fundamentally, on the relation between the essential structure and the structure at infinity of the system with respect to such flat outputs).

The customary statements that identify such linearizing outputs are then the following.

(1) The number of independent linearizing outputs is identical with the number of inputs of the system.

(2) All variables of the system, including the control input variables, can be written exclusively in terms of *differential functions* of the linearizing outputs (i.e. functions of the linearizing outputs and of a finite number of their time derivatives).

(3) Generally speaking, the linearizing outputs can, in turn, be expressed as differential functions of the system state vector components. This includes the possibility of having explicit dependences of the linearizing outputs on the states, the control inputs and a finite number of their time derivatives.

Since sliding surface coordinates might not all coincide with the linearizing output coordinates, sliding surfaces and their required evolution, characterized by differential equations with discontinuous right-hand sides, will also be expressible, in general, as a set of coupled discontinuous nonlinear differential equations involving only the linearizing coordinates. This immediately allows one to define the control inputs as *endogenous discontinuous feedback* policies. Since control inputs may be constituted by 'extended inputs' (i.e. derivatives of original input variables), then flat systems can be properly said to be robustly linearly decoupled by means of dynamical endogenous discontinuous feedback control.

Let $\{y_1 \dots, y_m\}$ be a set of independent outputs of the system, not necessarily coincident with the set of sliding surfaces. Denote by y the vector of such scalar outputs. The components of the vector y satisfy $y_i \in \mathscr{K}$, $i = 1, \dots, m$. Since the outputs may depend upon a finite number of control input derivatives, one may avoid counting time derivatives by considering the *infinite-dimensional* version of the formal vector space $\mathscr{E}$ which we denote here as

$$\mathscr{E}_\infty = \text{span}_{\mathscr{K}}\{\text{d}x, \text{d}u, \text{d}\dot{u}, \text{d}\ddot{u}, \dots\} \tag{2.21}$$

This not only bestows further generality but also is in the same spirit of recent developments in control theory of using an *infinite-dimensional space* of control input *jets*, and the associated infinite-dimensional vector fields, for the analysis of nonlinear systems (Fliess *et al.* 1993b, 1993c). Denote by $\mathscr{F}$ the quotient field of the ring of analytic functions of the components of the infinite set $\{y, \dot{y}, \ddot{y}, \dots\}$. We may now introduce a formal infinite-dimensional vector space of differentials $\mathscr{Y}_{\mathscr{F}}$, defined over the field $\mathscr{F}$, and given by

$$\mathscr{Y}_{\mathscr{F}} = \text{span}_{\mathscr{F}}\{\text{d}y^{(k)}; k \geqslant 0\} \tag{2.22}$$

Consider also the formal vector space

$$\mathscr{Y} = \text{span}_{\mathscr{K}}\{\text{d}y^{(k)}; k \geqslant 0\} \tag{2.23}$$

We denote, as before, by $\mathscr{X}$ the subspace of $\mathscr{E}$ given by

$$\mathscr{X} = \text{span}_{\mathscr{K}}\{\text{d}x\} \tag{2.24}$$

Flat systems are thus characterized by the existence of an m-dimensional vector y of scalar outputs satisfying the following three conditions:

$$\text{d}y \in \mathscr{E}_\infty$$
$$\dim(\mathscr{X} \cap \mathscr{Y}) = n$$
$$\{\text{d}x, \text{d}u\} \in \mathscr{Y}_{\mathscr{F}}$$

A trivial but important property of flat systems is the fact that flatness is *invariant* with respect to arbitrary dynamical extensions of the system.

Example 2.9: Consider the kinematics model of a bicycle moving on a horizontal plane. Let the state variables be defined as the position coordinates (x, y) of the contact point of the plane with the rear wheel, the angular direction θ of the bicycle's longitudinal axis, with respect to the horizontal coordinate axis x, and the angular orientation ϕ of the front wheel plane with respect to x. The horizontal distance between the centres of the wheels is l. The describing equations are given by

$$\left.\begin{aligned} \dot{x} &= u_1 \cos\theta \\ \dot{y} &= u_1 \sin\theta \\ \dot{\theta} &= u_1 \frac{\tan\phi}{l} \\ \dot{\phi} &= u_2 \end{aligned}\right\} \tag{2.25}$$

The state coordinate transformation (Murray and Sastry 1993)

$$z_1 = x, \quad z_2 = \frac{\tan\phi}{l\cos^3\theta}, \quad z_3 = \tan\theta, \quad z_4 = y \tag{2.26}$$

and the redefinition of the input variables

$$\left.\begin{aligned} v_1 &= u_1 \cos\theta \\ v_2 &= u_1 \frac{3\tan^2\phi\sin\theta}{l^2\cos^4\theta} + u_2 \frac{1}{l\cos^2\phi\cos^3\theta} \end{aligned}\right\} \tag{2.27}$$

takes the system (2.25) into a *chained system* of the form (Murray and Sastry 1993)

$$\left.\begin{aligned} \dot{z}_1 &= v_1, \quad \dot{z}_2 = v_2 \\ \dot{z}_3 &= z_2 v_1 \\ \dot{z}_4 &= z_3 v_1 \end{aligned}\right\} \tag{2.28}$$

All variables in the transformed system can be expressed as functions of the outputs $y_1 = z_1$ and $y_2 = z_4$ and of their time derivatives (i.e. as a differential function of the position coordinates (x, y) of the contact point of the bicycle's rear wheel with the plane). Indeed,

$$\left.\begin{aligned} z_3 &= \frac{\dot{y}_2}{\dot{y}_1} \\ z_2 &= \frac{\dot{z}_3}{\dot{z}_1} = \frac{\ddot{y}_2\dot{y}_1 - \dot{y}_2\ddot{z}_1}{\dot{y}_1{}^3} \\ v_1 &= \dot{y}_1 \\ v_2 &= \frac{y_2^{(3)}\dot{y}_1 - \dot{y}_2 y_1^{(3)}}{(\dot{y}_1)^3} + \frac{3(\ddot{y}_2\ddot{y}_1\dot{y}_1 - \dot{y}_2(\ddot{y}_1)^2)}{(\dot{y}_1)^4} \end{aligned}\right\} \tag{2.29}$$

If we take the linearizing outputs as the sliding surfaces, that is $s_1 = z_1$ and $s_2 = z_4$, then one may easily find that both s_1 and s_2 have row relative degrees equal to 1. The essential order of s_1 is equal to 3 and that of s_2 is also 3. In order to decouple the system, by means of endogenous dynamical discontinuous feedback, we only need to extend the input coordinate u_1 at least twice and leave the rest of the system untouched. □

It should be remarked that, roughly speaking, if a system is not differentially flat, then it essentially contains an uncontrollable part or else it *does not* satisfy the so-called *strong accessibility property* (see Nijmeijer and van der Schaft (1990) for the definition). Regulation of the non-flat system either may not be achieved at all or is sometimes possible in an *ad hoc* average sense (Fliess *et al.* 1993 a). In the context of discontinuous multivariable feedback control, it thus makes sense to restrict attention only to differentially flat systems, at least until a more general classification of nonlinear systems is available.

3. Sliding mode control of nonlinear multivariable systems

3.1. *A brief survey*

Sliding mode control of multivariable nonlinear systems has been addressed from different perspectives ever since the work of Utkin (1978) (see also the new book by Utkin (1992)). The main emphasis in these books was towards obtaining statically decoupled sliding regimes through the *method of the hierarchy of controls*. Applications of this technique to the regulation and tracking of nonlinear robotic systems were first made by Young (1978). Later, a more direct approach was presented by Slotine and Sastry (1983) and by Slotine (1984). A clear exposition of the basic issues in multivariable sliding mode control may also be found in a tutorial article by DeCarlo *et al.* (1988). Further developments were given in the book by Slotine and Li (1991). The differential geometric method was explored by Sira-Ramírez (1988) and by Bartolini and Zolezzi (1986). A complete picture of the geometric approach, at least from the perspective of static sliding mode control, is contained in a recent article by Kwatny and Kim (1990). A completely new approach was initiated, in the context of linear systems, by Fliess and Messager (1991). A module theoretical approach dealing simultaneously with time-varying and time-invariant linear multivariable systems has been recently proposed by Fliess and Sira-Ramírez (1993 a, 1993 b). The reader is invited to explore the recent research trends in sliding mode control in general, and for the multivariable case in particular, in two special issues of the journals *International Journal of Control* (1993) and *IEEE Transactions on Industrial Electronics* (1993). The books edited by Zinober (1990, 1994) also contain recent contributions and perspectives for the field.

3.2. *Sliding mode control of differentially flat systems*

From the preceding sections, it is apparent that the class of differentially flat systems constitutes a natural class of systems for which discontinuous endogenous feedback control strategies can be properly defined. Static or dynamical discontinuous endogenous feedback control policies can always be appropriately defined for such class of systems.

A sliding mode design procedure, for the considered class of linearizable systems, can then be proposed as follows.

(1) Establish the differential flatness of the multivariable nonlinear system by identifying a linearizing set of outputs (note that, so far, no general algorithm exists for such a task)

(2) Obtain all system variables, including the sliding surface coordinates, in terms of differential functions of the linearizing outputs.

(3) Compute the structure at infinity and the essential structure of the system with respect to the linearizing outputs.

(4) If necessary, extend the system to have the essential structure coincide with the structure at infinity of the decoupled system.

(5) Proceed to specify desirable decoupled unforced discontinuous differential equations for the given sliding surface coordinates. This set of differential equations must be prescribed in such a way that, from the corresponding set of coupled nonlinear discontinuous dynamics imposed on the linearizing outputs, the solved highest-order derivatives of such outputs coincide with those obtained in the expressions for the control inputs. The adopted unforced sliding surface discontinuous dynamics guarantee either finite time reachability to zero of particular sliding surface coordinates (namely those enjoying first-order sliding mode dynamics) or else yield asymptotically stable trajectories (for those coordinates requiring *higher-order* sliding mode dynamics).

The main result pertaining sliding mode control of nonlinear systems, restricted to the class of differentially flat systems, may be summarized as follows.

'Decoupled linearization by means of a statically discontinuous feedback control policy can always be appropriately defined for differentially flat multivariable systems which are statically decouplable to linear controllable systems (i.e. those whose essential structure coincides with the structure at infinity). Endogenous dynamical discontinuous feedback policies can also always be defined for differentially flat multivariable systems which are not statically decouplable. A suitable dynamical extension of the original system always exist which makes the extended system exhibit, for the linearizing outputs, an essential structure which is coincident with the corresponding decoupled structure at infinity'.

3.3. *Some design examples*

Example 3.1: Consider the system describing the spacecraft dynamics, treated in Example 2.6,

$$\left.\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 x_2 \\ s_1 &= x_1 \\ s_2 &= x_3 \end{aligned}\right\} \tag{3.1}$$

It is designed to regulate the angular velocity x_1 around a given constant angular velocity $x_1 = X_1 \neq 0$, while x_3 is to be regulated to zero. We thus take as sliding surfaces $s_1 = x_1 - X_1$ and $s_2 = x_3$.

The system is differentially flat, with linearizing coordinates given precisely by $y_1 = x_1$ and $y_2 = x_3$ (which are trivially related to the sliding surfaces and, in fact, exhibit the same structure at infinity and the same essential structure). Indeed, these coordinates do not satisfy any differential equation and, moreover, all variables in the system are expressible as differential functions of such output coordinates

$$\left.\begin{aligned} x_1 &= y_1 \\ x_2 &= \frac{\dot{y}_2}{y_1} \\ x_3 &= y_2 \\ u_1 &= \dot{y}_1 \\ u_2 &= \frac{\ddot{y}_2 y_1 - \dot{y}_2 \dot{y}_1}{y_1^2} \end{aligned}\right\} \tag{3.2}$$

As already verified, this system is decouplable by means of static feedback since the essential structure for the given sliding surfaces coincides with the structure at infinity. Since the system is also flat, it is linearizable by static endogenous discontinuous feedback.

According to the design procedure previously outlined, one imposes the following slide mode dynamics on the sliding surfaces coordinates:

$$\left.\begin{aligned} \dot{s}_1 &= -W_1 \operatorname{sign} s_1 \\ \ddot{s}_2 &= -\lambda \dot{s}_2 - W_2 \operatorname{sign}(\dot{s}_2 + \lambda s_2) \end{aligned}\right\} \tag{3.3}$$

which yields the following desirable discontinuous dynamics for the linearizing coordinates y_1, y_2:

$$\left.\begin{aligned} \dot{y}_1 &= -W_1 \operatorname{sign}(y_1 - X_1) \\ \ddot{y}_2 &= -\lambda \dot{y}_2 - W_2 \operatorname{sign}(\dot{y}_2 + \lambda y_2) \end{aligned}\right\} \tag{3.4}$$

By virtue of the system flatness the required control inputs are readily obtained as

$$\left.\begin{aligned} u_1 &= -W_1 \operatorname{sign}(y_1 - X_1) \\ u_2 &= \frac{1}{y_1}[-\lambda \dot{y} \phi_2 - W_2\, sign(\dot{y}_2 + \lambda y_2)] + \frac{\dot{y}_2}{y_1^2} W_1 \operatorname{sign}(y_1 - X_1) \end{aligned}\right\} \tag{3.5}$$

In terms of the original state coordinates the following discontinuous feedback control law accomplishes the desired control objectives

$$\left.\begin{aligned} u_1 &= -W_1 \operatorname{sign}(x_1 - X_1) \\ u_2 &= \frac{1}{x_1}[-\lambda x_1 x_2 - W_2 \operatorname{sign}(x_1 x_2 + \lambda x_3)] + \frac{x_1 x_2}{x_1^2} W_1 \operatorname{sign}(x_1 - X_1) \end{aligned}\right\} \tag{3.6}$$

Clearly stabilization to $x_1 = X_1 = 0$ is unfeasible owing to the underlying singularity. □

Example 3.2: We consider the kinematic model of a powered monocycle moving on a horizontal plane:

$$\left.\begin{aligned} \dot{x} &= v_1 \cos\theta \\ \dot{y} &= v_1 \sin\theta \\ \dot{\theta} &= v_2 \end{aligned}\right\} \tag{3.7}$$

where x and y are the position coordinates of the wheel and θ is the wheel angular direction measured with respect to the x axis. The control input v_1 represents the forward velocity of the wheel and v_2 is the steering rate. (The model also represents the simplified kinematics of a car model (Nijmeijer and van der Schaft 1990.) In order to slightly simplify the treatment we perform the following state and input coordinates transformation which brings the system into *chained form* (Murray and Sastry 1993):

$$\left.\begin{aligned} & x_1 = x, \quad x_3 = y, \quad x_2 = \tan\theta \\ & u_1 = v_1 \cos\theta; \quad u_2 = \frac{v_2}{\cos^2\theta} \end{aligned}\right\} \tag{3.8}$$

that is

$$\left.\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_2 &= x_2 u_1 \end{aligned}\right\} \tag{3.9}$$

The system is easily seen to be differentially flat, with linearizing outputs y_1 and y_2 given, by the position coordinates x_1 and x_3 respectively. Indeed, these coordinates are independent since they do not satisfy any differential equation, their number equals the number of control inputs and finally all variables in the system are expressible as differential functions of the linearizing coordinates

$$\left.\begin{aligned} x_1 &= y_1 \\ x_2 &= \frac{\dot{y}_2}{\dot{y}_1} \\ u_1 &= \dot{y}_1 \\ u_2 &= \frac{\ddot{y}_2 \dot{y}_1 - \dot{y}_2 \ddot{y}_1}{\dot{y}_1^2} \end{aligned}\right\} \tag{3.10}$$

The jacobian matrix relating the differentials of the states and outputs to the differentials of the control inputs is given, in this case, by

$$\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \\ dy_1 \\ dy_2 \\ d\dot{y}_1 \\ d\dot{y}_2 \\ d\ddot{y}_1 \\ d\ddot{y}_2 \\ dy_1^{(3)} \\ dy_2^{(3)} \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_1 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\mathbf{u}}_1 & \mathbf{0} & \mathbf{u}_2 & \mathbf{u}_1 & \mathbf{x}_2 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \ddot{u}_1 & 0 & \dot{u}_2 & 2\dot{u}_1 & 2u_2 & u_1 & x_2 & 0 \end{pmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \\ du_1 \\ du_2 \\ d\dot{u}_1 \\ d\dot{u}_2 \\ d\ddot{u}_1 \\ d\ddot{u}_2 \end{bmatrix} \tag{3.11}$$

It can immediately be seen that the row relative degree of the linearizing output y_1 is 1 and that of y_2 is also 1, everywhere except for $x_2 = 0$. The system is not decouplable by static feedback. The essential orders are also readily seen to be 2 and 2. By extending u_1 just once, the system becomes decouplable with relative degrees coincident with the essential orders. Consider then

$$\left.\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{u}_1 &= v_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \end{aligned}\right\} \tag{3.12}$$

where now u_1 is just another state variable component. The system is still differentially flat, with the same linearizing coordinates $y_1 = x_1$ and $y_2 = x_3$. Indeed, the variables of the extended system are now expressed as

$$\left.\begin{aligned} x_1 &= y_1 \\ x_2 &= \frac{\dot{y}_2}{\dot{y}_1} \\ u_1 &= \dot{y}_1 \\ v_1 &= \ddot{y}_1 \\ u_2 &= \frac{\ddot{y}_2\dot{y}_1 - \dot{y}_2\ddot{y}_1}{\dot{y}_1^2} \end{aligned}\right\} \tag{3.13}$$

The new jacobian matrix relating the differentials is now given by

$$\begin{bmatrix} dx_1 \\ du_1 \\ dx_2 \\ dx_3 \\ dy_1 \\ dy_2 \\ d\dot{y}_1 \\ d\dot{y}_2 \\ d\ddot{y}_1 \\ d\ddot{y}_2 \\ dy_1^{(3)} \\ dy_2^{(3)} \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\mathbf{u}}_2 & \mathbf{v}_1 & \mathbf{0} & \mathbf{x}_2 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \ddot{u}_2 & \dot{v}_1 & 0 & 2u_2 & 2v_1 & x_2 & u_1 \end{pmatrix} \begin{bmatrix} dx_1 \\ du_1 \\ dx_2 \\ dx_3 \\ dv_1 \\ du_2 \\ d\dot{v}_1 \\ d\dot{u}_2 \end{bmatrix} \tag{3.14}$$

It is clear that the relative degrees of the outputs are now 2 and 2, which coincide with the corresponding essential orders. The extended system is then decouplable by means of static discontinuous feedback, that is the original system is decouplable by dynamical discontinuous feedback.

Consider the problem of following a circle of radius R with constant angular velocity $\dot{\psi} = \Omega$, in a counterclockwise sense, here $\psi = \operatorname{atan}(y_2/y_1)$. The sliding surfaces s_1 and s_2 are then represented by the error of the square of the radial position and by the angular velocity error $\dot{\psi} - \Omega$. In terms of the linearizing coordinates,

$$\left.\begin{aligned} s_1 &= y_1^2 + y_2^2 - R^2 \\ s_2 &= \frac{y_1\dot{y}_2 - y_2\dot{y}_1}{y_1^2 + y_2^2} - \Omega \end{aligned}\right\} \tag{3.15}$$

We impose the following discontinuous dynamics on the sliding surfaces:

$$\left.\begin{aligned} \ddot{s}_1 &= -\lambda \dot{s}_1 - W_1 \operatorname{sign}(\dot{s}_1 + \lambda s_1) \\ \ddot{s}_2 &= -W_2 \operatorname{sign} s_2 \end{aligned}\right\} \tag{3.16}$$

In terms of the linearizing coordinates, such robust discontinuous linearizing dynamics result in

$$\left.\begin{aligned} \ddot{y}_1 &= \frac{y_1}{y_1^2+y_2^2}[-\dot{y}_1^2-\dot{y}_2^2-\lambda(y_1\dot{y}_1+y_2\dot{y}_2)-\tfrac{1}{2}W_1 \operatorname{sign}\sigma_1] \\ &\quad -y_2\left(\frac{(y_1\dot{y}_2-\dot{y}_1 y_2)(2y_1\dot{y}_1+2y_2\dot{y}_2)}{y_1^2+y_2^2}-W_2 \operatorname{sign}\sigma_2\right) \\ \ddot{y}_2 &= \frac{y_2}{y_1^2+y_2^2}[-\dot{y}_1^2-\dot{y}_2^2-\lambda(y_1\dot{y}_1+y_2\dot{y}_2)-\tfrac{1}{2}W_1 \operatorname{sign}\sigma_1] \\ &\quad +y_1\left(\frac{(y_1\dot{y}_2-\dot{y}_1 y_2)(2y_1\dot{y}_1+2y_2\dot{y}_2)}{y_1^2+y_2^2}-W_2 \operatorname{sign}\sigma_2\right) \end{aligned}\right\} \tag{3.17}$$

where

$$\left.\begin{aligned} \sigma_1 &= \dot{s}_1+\lambda s_1 = 2y_1\dot{y}_1+2y_2\dot{y}_2+\lambda(y_1^2+y_2^2-R^2) \\ \sigma_2 &= s_2 = \frac{y_1\dot{y}_2-y_2 y_1}{y_1^2+y_2^2}-\Omega \end{aligned}\right\} \tag{3.18}$$

The required endogenous discontinuous feedback control inputs can be immediately computed from the relations arising from the flatness property of the system in (3.13):

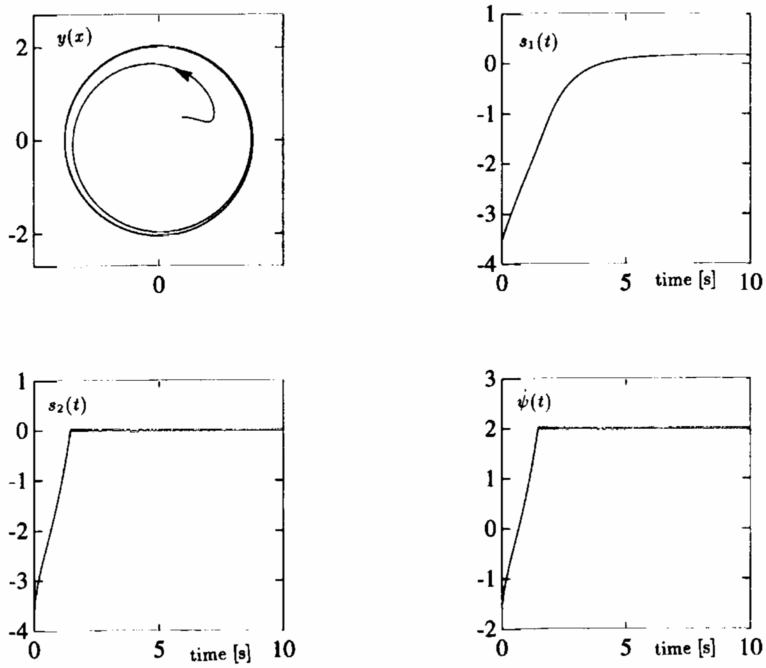
$$\left.\begin{aligned} v_1 = \dot{u}_1 &= \frac{y_1}{y_1^2+y_2^2}[-\dot{y}_1^2-\dot{y}_2^2-\lambda(y_1\dot{y}_1+y_2\dot{y}_2)-\tfrac{1}{2}W_1 \operatorname{sign}\sigma_1] \\ &\quad -y_2\left[\frac{(y_1\dot{y}_2-\dot{y}_1 y_2)(2y_1\dot{y}_1+2y_2\dot{y}_2)}{y_1^2+y_2^2}-W_2 \operatorname{sign}\sigma_2\right] \\ u_2 &= \frac{\dot{y}_1 y_2-y_1\dot{y}_2}{\dot{y}_1^2(y_1^2+y_2^2)}[-\dot{y}_1^2-\dot{y}_2^2-\lambda(y_1\dot{y}_1+y_2\dot{y}_2)-\tfrac{1}{2}W_1 \operatorname{sign}\sigma_1] \\ &\quad +\frac{y_1\dot{y}_1+y_2\dot{y}_2}{\dot{y}_1^2}\left(\frac{(y_1\dot{y}_2-\dot{y}_1 y_2)(2y_1\dot{y}_1+2y_2\dot{y}_2)}{y_1^2+y_2^2}-W_2 \operatorname{sign}\sigma_2\right) \end{aligned}\right\} \tag{3.19}$$

The obtained control laws can now be placed in terms of the original state and input coordinates.

3.4. *Simulation results*

Computer simulations, using MATLAB were carried out for the original system (3.7), regulated by the derived multivariable decoupling sliding mode controller (3.17) and (3.18). The value of the circle radius was taken to be $R = 2$ m, while the desired angular velocity was taken to be $\dot{\psi} = \Omega = 2{\cdot}0$ rad s^{-1}. The numerical values associated with the designed controllers were set to be

$$W_1 = 1, \quad \lambda = 1, \quad W_2 = 2$$



Sliding mode-controlled-trajectories for the powered unicycle example.

The Figure shows the controlled trajectory of the unicycle in the (x, y)-plane, as the function $y(x)$. This figure also shows the time evolution of the sliding surface coordinates $s_1(t)$ and $s_2(t)$ and the controlled angular velocity $\dot{\psi}(t)$ of the unicycle around the circle. The sliding surface coordinates trajectories $s_2(t)$ and $s_1(t)$ converge respectively in finite time and in asymptotic fashion to their desired value of zero. As expected, the nature of such convergence depends on the static or dynamic nature of the underlying sliding regime.

4. Conclusions

In this article we have presented, for a class of linearizable multivariable systems known as differentially flat systems, a design method which achieves robust feedback linearization by means of endogenous discontinuous feedback control of the sliding mode type. The notion of essential orders has been shown to be quite relevant in the feasibility of dynamical or static discontinuous decoupled linearization for the class of differentially flat systems. The exposition uses the language of linear differential algebra in a rather conventional manner. Several examples demonstrate the feasibility of a systematic treatment of such class of problems.

Within the same framework utilized here, an interesting study of the decoupling problem for the case of perturbed systems has been initiated by Castro-Linares and Moog (1994). The implication of such a study on dynamical sliding mode control policies seems challenging and a excellent topic for further work.

Recently, the linear differential algebraic approach has been successfully extended by Grizzle (1993) to include nonlinear discrete-time multivariable systems. The appropriate definition of discrete time sliding modes, and some of their relevant properties, may be sought within this general context as a problem that requires deserved attention.

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