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Design of pulse width modulation controllers for stabilization and tracking in derived DC-to-DC power converters

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Feedback controllers, based on pulse-width-modulation (PWM), are derived for the average input current stabilization and tracking problems in derived DC-to-DC power supplies of the buck and boost types. The stabilization problems are solved on the basis of steady-state considerations about the current 'ripple' and exactly discretized nonlinear models describing the sampled PWM regulated input current trajectories. In the boost-derived converter, the stabilization problem leads to an implicit static nonlinear feedback controller, or duty ratio synthesizer, which requires online solutions of a transcendental equation at each sampling instant. The signal tracking problems are solved on the basis of discrete-time, non-kalmanian state representation models describing the average PWM regulated input current. These generalized state models naturally allow for explicit dynamical, rather than static, feedback regulators. Computer simulations, including unmodelled load variations and external stochastic perturbation inputs, are presented which test the robustness of the proposed PWM controller performances,

1. Introduction

Simplified versions of DC-to-DC power converters may be obtained by removing the storing capacitors on the output circuits. Corresponding to the buck, the boost, and the buck–boost converters (see Severns and Bloom 1983), the obtained converters are known, respectively, as 'choppers', step-up and step up-down converters. Generically, they may also be addressed as 'derived' converters (see Rashid 1993, for details).

The feedback regulation of DC-to-DC power supplies is customarily accomplished, through pulse-width-modulation (PWM) feedback strategies (see Kassakian *et al.* 1991, Rashid 1993). Typically, control objectives include input current stabilization, around a given constant value, or time-varying reference input signal tracking. In this context, infinite frequency average models (see Middlebrook and Cúk 1976) or equivalent control models are frequently invoked, at the controller design stage, in order to obtain a smooth feedback specification of the computed duty ratio

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function (see also Sira-Ramírez 1989, 1991, Sira-Ramírez and Lischinsky-Arenas 1991, and Sira-Ramírez *et al.* 1993). The performance features of the actual PWM controlled circuit responses, with respect to those predicted by the average PWM model, depend on the magnitude of the sampling frequency associated with the pulse width modulator. For low sampling frequencies, the closed-loop precision deteriorates allowing substantial errors in the stabilization and tracking tasks.

The use of average models, however, may not be entirely justified for derived DC-to-DC power supplies. First, the appealing simplicity of the dynamic models does not seem to require further simplification through a questionable smooth approximation and, secondly, the possibilities for exact discretization, which is certainly a much more involved process in the traditional version of the converters, make it reasonable to attempt a direct PWM controller design based on an exact discrete-time model of the derived converter (see Kassakian *et al.* 1991). The exact discretization circumvents all problems related to the approximation involved in the finite magnitude of the sampling frequency used for the pulse width modulator.

It is the purpose of this paper to explore, in detail, the feasibility of PWM stabilizing controller designs for stabilization and tracking in derived DC-to-DC power converters. The approach is based on exact discretization of the sampled input current. Discrete-time regulation policies based on approximate discretization and approximate linearization were explored by Ehsani *et al.* (1983). The outline of an approximate discretization approach for the stabilization of more complex DC-to-DC power supplies can also be found in Kassakian *et al.* (1991). Related developments, from a viewpoint different to that of feedback control, are found in Rashid (1993).

In this article we present the fundamentals of an exact discretization approach for the input current stabilization and tracking problems in the derived versions of the buck and the boost DC-to-DC power supplies. The results, however, can also be extended to include the buck–boost derived converter. The linearity in the input, associated with the traditional infinite frequency average models of the converters, is effectively destroyed by the exact discretization procedure. Nevertheless, the obtained models still remain linear in the state. The proposed approach offers no special difficulties for the stabilization problem in converters of the buck-derived type. For such a class of derived converters, it is also possible to obtain an explicit expression relating steady-state average input current values to steady-state sampled input current values. This key fact allows us to solve explicitly the average current stabilization problem in terms of an equivalent sampled current stabilization problem. However, in the stabilization problem for the boost-derived converter, the resulting nonlinear discrete-time duty ratio synthesizers (controllers) are of the implicit type, i.e. at each sampling instant, the feedback duty ratio function is given by the numerical solution of a transcendental equation. Similar transcendental equations allow for the offline computation of the desired steady-state average input current in terms of the steady-state sampled input current. The signal tracking problems are addressed by introducing discrete-time average models for the PWM regulated input current trajectories. The discrete-time models naturally result in generalized, i.e. non-kalmanian, state representations of the systems (see Fliess 1992). In both cases, the proposed average models naturally lead to nonlinear explicit dynamical feedback duty ratio synthesizers.

Section 2 presents an exact discretization approach for PWM feedback regulator designs solving stabilization problems defined on derived DC-to-DC power supplies of the buck and boost types. In this section we also present simulations of the proposed feedback control schemes for the derived converters. Section 3 presents the nonlinear discrete-time average models, and the corresponding solutions to the signal tracking problems, for the two types of derived converters. The conclusions and suggestions for further research in this area are presented in the final section.

2. Feedback stabilization of derived DC-to-DC power converters via an exact discretization scheme

2.1. *The buck-derived converter*

Consider the buck-derived converter circuit shown in Fig. 1 (Rashid 1993). The switch regulated model describing the behaviour of the input current, denoted by x, is given by

$$\begin{aligned} \dot{x} &= -\frac{R}{L}x + \frac{E}{L}u \\ y &= Rx \end{aligned} \tag{2.1}$$

where y is the output load voltage and the parameters R, L and E stand, respectively, by the load resistance, the inductance of the input circuit, and the constant input source voltage. The variable u denotes the switch position function taking values on the discrete set $\{0, 1\}$.

A regulation strategy, based on a PWM specification of the switch position function, may be, generally speaking, specified by (see Sira-Ramírez *et al.* 1993)

$$u(t) = \begin{cases} 1 & \text{for } t_k \leqslant t < t_k + \mu(t_k)T \\ 0 & \text{for } t_k + \mu(t_k)T < t < t_k + T \end{cases} \tag{2.2}$$

$$t_{k+1} = t_k + T; \quad k = 0, 1, 2, \ldots$$

where $\mu(\cdot)$ is known as the actual duty ratio function, taking values in the interval $[0, 1]$ of the real line. T is the sampling period and t_k is the sampling instant. A typical example of a PWM commanded switch position function trajectory is depicted in Fig. 2.

Since the duty ratio μ is specified online in a feedback manner, i.e. computed as a function explicitly depending on the sampled value of the input current $x(t_k)$ at each instant t_k, one may obtain values of μ which lie outside the closed interval $[0, 1]$. We must, therefore, make a distinction between the computed duty ratio function, denoted by $\mu_c(\cdot)$ and the actual duty ratio function, denoted by $\mu(\cdot)$. The relation between these variables is simply given by

$$\mu(t) = \begin{cases} 1 & \text{for } \mu_c(t) > 1 \\ \mu_c(t) & \text{for } 0 \leqslant \mu_c(t) \leqslant 1 \\ 0 & \text{for } \mu_c(t) < 0 \end{cases} \tag{2.3}$$

The actual duty ratio function is thus the forceful limitation of the computed duty ratio function to the closed interval $[0, 1]$.

The buck-derived converter owes its popular name 'chopper' to the fact that the input current is limited to taking values on the interval $[0, E/R]$, as can easily be verified from the circuit equations. The corresponding (positive) output load voltages delivered by the converter cannot, therefore, exceed the value E of the external source voltage.

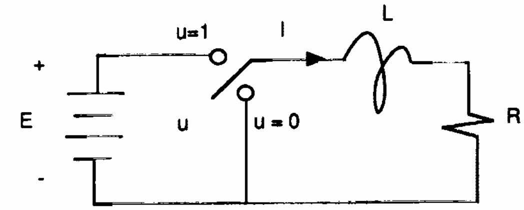

Figure 1. The buck-derived converter.

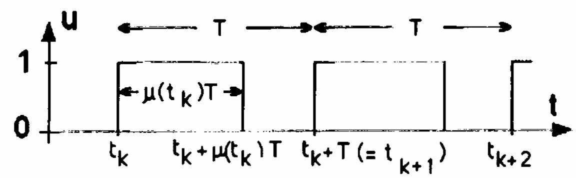

Figure 2. PWM commanded switch position function.

2.2. *An exact discretization of the PWM regulated buck-derived converter*

The linear nature of the two possible topologies of the converter circuit facilitate the derivation of an exact discrete-time model for the evolution of the sampled values of the input current in the buck-derived converter (2.1), when subject to a switching policy of the form (2.2). Indeed, given the value of x at time t_k, denoted by $x(t_k)$, the value of the input current at the end of the 'pulse', of width $\mu(t_k)\,T$, is obtained as

$$x(t_k+\mu(t_k)\,T) = \exp\{-\theta_1\,\mu(t_k)\,T\}\,x(t_k)+\frac{\theta_2}{\theta_1}[1-\exp\{-\theta_1\,\mu(t_k)\,T\}] \tag{2.4}$$

where we have let the parameter θ_1 denote the quotient R/L and θ_2 denote E/L.

The sampled value of the input current at the end of the sampling interval is obtained, after some further computations, as

$$x(t_k+T) = \exp\{-\theta_1\,T\}\,x(t_k)+\frac{\theta_2}{\theta_1}\exp\{-\theta_1\,T\}\,[\exp\{\theta_1\,\mu(t_k)\,T\}-1] \tag{2.5}$$

If we denote $\Psi_1 = e^{-\theta_1 T}$ and $\Psi_2 = \theta_2/\theta_1$, the discrete-time model for the evolution of the input current, depicted at the sampling instants, is given by the following model

$$x(t_{k+1}) = \Psi_1\,x(t_k)+\Psi_1\,\Psi_2[\Psi_1^{-\mu(t_k)}-1] \tag{2.6}$$

where the value of the duty ratio function at time $t_k, \mu(t_k)$, must now be effectively regarded as the 'control input' variable, to be specified at the beginning of each sampling period. The discrete-time model for the sampled input current is, therefore, nonlinear in the new control input, $\mu(t_k)$.

The only eigenvalue associated with the linear sampled state dynamics, given by Ψ_1, is evidently positive and strictly smaller than unity. The steady-state value of the sampled input current, denoted by $\bar{x}_\infty$, corresponding to constant duty ratio function of value μ_∞, is then readily obtained from (2.6) as

$$\bar{x}_\infty = \frac{\Psi_1\,\Psi_2}{1-\Psi_1}(\Psi_1^{-\mu_\infty}-1) \tag{2.7}$$

The overbar in (2.7) refers to the 'lower' portion of the actual zigzagged trajectory

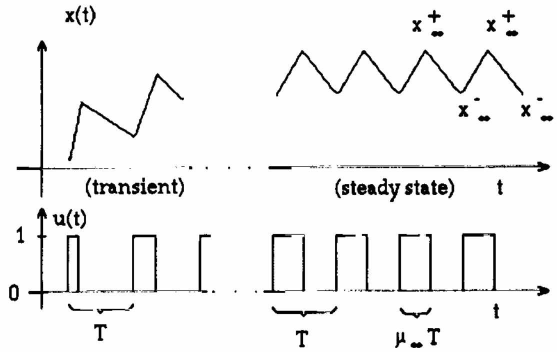

Figure 3. Transient and steady-state PWM controlled state trajectory.

partially described by (2.6) (see Fig. 3). Evidently, a feedback regulation policy, which specifies the duty ratio function $\mu(t_k)$ solely on the basis of the sampled state $x(t_k)$, is by no means satisfactory. The reason for such a statement stems from the fact that the 'ripple', unavoidably associated with the switch-regulated evolution of $x(t_k)$, is not taken into account by the model (2.6) alone. One must also take into account the values of $x(t)$ at the end of each width-modulated control input pulse occurring within the sampling period of length T. In other words, one must take into account the values of $x(t)$ at the instants $t = t_k + \mu(t_k)T$; $k = 0, 1, 2, \ldots$ (see Fig. 3).

We now relate the values of x at times $t_{k+1} + \mu(t_{k+1})T$ and $t_k + \mu(t_k)T$ so as to obtain the values of the 'upper' corners of the zigzagged input current trajectory. One obtains, after some algebraic manipulations

$$x(t_{k+1} + \mu(t_{k+1})T) = \Psi_1^{\mu(t_{k+1})}\Psi_1^{1-\mu(t_k)}x(t_k + \mu(t_k)T) + \Psi_2(1 - \Psi_1^{\mu(t_{k+1})}) \tag{2.8}$$

The eigenvalue associated with the above linear state dynamics is clearly given by the product $\Psi_1^{\mu(t_{k+1})}\Psi_1^{1-\mu(t_k)}$. This quantity is strictly positive and smaller than unity for values of μ bounded by the unit interval $[0, 1]$. The steady-state value of the 'upper' corners of the state trajectory, described by (2.8), corresponding to a constant value μ_∞ of the duty ratio function, is given by (see Fig. 3)

$$x_\infty^+ = \frac{\Psi_2}{1 - \Psi_1}(1 - \Psi_1^{\mu_\infty}) \tag{2.9}$$

The relation between the steady-state values x_∞^+ and x_∞^- can be obtained from (2.7) and (2.9) as

$$x_\infty^- = x_\infty^+ \Psi_1^{1-\mu_\infty} \tag{2.10}$$

Since Ψ_1 is a positive number, which is strictly less than 1, one can conclude that $x_\infty^- < x_\infty^+$ for $\mu_\infty \in [0, 1]$.

The steady state 'ripple', denoted by r_∞, may then be described as the following difference

$$r_\infty = x_\infty^+ - x_\infty^- = \frac{\Psi_2}{1 - \Psi_1}(1 - \Psi_1^{\mu_\infty})(1 - \Psi_1^{1-\mu_\infty}) \tag{2.11}$$

We define a steady-state average value for the input current trajectory as

$$x_{\mathrm{av}}(\infty) = x_\infty^- + \tfrac{1}{2}r_\infty = \tfrac{1}{2}(x_\infty^- + x_\infty^+) \tag{2.12}$$

Using the expressions (2.7) and (2.9) in (2.12) one obtains

$$x_{av}(\infty) = \frac{1}{2}\left(\frac{\Psi_2}{1-\Psi_1}\right)(1-\Psi_1^{\mu_\infty})(1+\Psi_1^{1-\mu_\infty}) \tag{2.13}$$

We proceed to express the steady-state value of the sampled input current trajectory x_∞^-, in terms of the average steady-state input current $x_{av}(\infty)$. This relation allows us to define a suitable stabilizing feedback duty ratio (control) policy on the basis of the sampled states of the discrete-time model (2.6). The feedback policy properly takes into account the ripple associated with the controller trajectory, and asymptotically achieves a pre-specified desired steady-state value for the average input current. To achieve this goal one simply eliminates the steady-state value of the duty ratio, μ_∞ from the expressions (2.7) and (2.13). One then obtains,

$$x_\infty^- = -\Psi_2\left[\left(\frac{1}{2}\left(1-\frac{2x_{av}(\infty)}{\Psi_2}\right)+\frac{\Psi_1}{1-\Psi_1}\right)-\left(\frac{1}{4}\left(1-\frac{2x_{av}(\infty)}{\Psi_2}\right)^2+\frac{\Psi_1}{(1-\Psi_1)^2}\right)^{1/2}\right] \tag{2.14}$$

2.3. *A stabilizing PWM control policy for the buck-derived converter*

The stabilization problem for the buck-derived converter consists of specifying a PWM feedback regulation policy of the form (2.2) such that the steady-state average value of the controlled input current trajectory $x(t)$ reaches a desired constant value $x_{av}(\infty) = X$.

A stabilizing feedback regulation policy $\mu(t_k)$ can then be explicitly obtained on the basis of the sampled states of the discrete-time model (2.6) by forcing $x(t_k)$ to stabilize asymptotically around the value x_∞^-, corresponding to X, which we denote by $x_\infty^-(X)$ and rewrite as

$$x_\infty^-(X) = -\Psi_2\left[\left(\frac{1}{2}\left(1-\frac{2X}{\Psi_2}\right)+\frac{\Psi_1}{1-\Psi_1}\right)-\left(\frac{1}{4}\left(1-\frac{2X}{\Psi_2}\right)^2+\frac{\Psi_1}{(1-\Psi_1)^2}\right)^{1/2}\right] \tag{2.15}$$

We impose on the sampled controlled system the following linear asymptotically stable closed-loop behaviour

$$x(t_{k+1}) = \alpha(x(t_k)-x_\infty^-(X))+x_\infty^-(X); \quad |\alpha|<1 \tag{2.16}$$

Substituting the right-hand side of expression (2.6) on (2.16) and solving for the duty ratio function $\mu(t_k)$ one obtains the following nonlinear computed duty ratio feedback control policy

$$\mu_c(t_k) = -\frac{1}{\ln\Psi_1}\ln\left[1+\frac{(\alpha-\Psi_1)x(t_k)+(1-\alpha)x_\infty^-(X)}{\Psi_1\Psi_2}\right]; \quad k=0,1,2,\ldots \tag{2.17}$$

The actual duty ratio function $\mu(t_k)$ may be readily obtained from expression (2.3). Figure 4 depicts the PWM feedback regulation scheme based on the exact discrete-time dynamics model of the sampled input current.

Expression (2.17) allows for the determination of the region of non-saturation of the actual duty ratio function. Indeed, the double inequality: $0<\mu_c<1$, yields the following corresponding region for the sampled state

$$0 < (\alpha-\Psi_1)x(t_k)+(1-\alpha)x_\infty(X) < \Psi_2(1-\Psi_1) \tag{2.18}$$

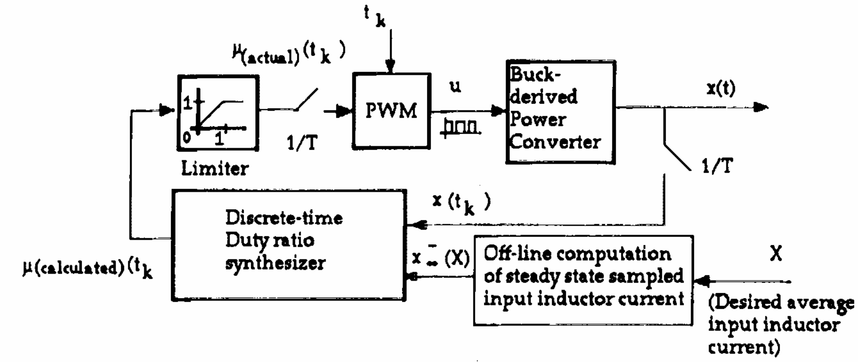

Figure 4. A PWM feedback regulation scheme for the buck-derived converter based on exact discretization.

2.4. *Simulation results*

In order to test the robustness of the previously proposed PWM feedback regulation policy we carried out simulations on the following noise perturbed model of the buck-derived converter

$$\dot{x} = -\frac{R}{L}x + \left(\frac{E + v(t)}{L}\right)u \tag{2.19}$$

where $v(t)$ is a (computer generated) stochastic perturbation signal representing an unmodelled additive noisy voltage source affecting the behaviour of the circuit. The values for the parameters defining the converter were taken to be

$$R = 2{\cdot}8 \times 10^{-2}\,\Omega; \quad L = 1{\cdot}0 \times 10^{-2}\,\text{mH}; \quad E = 126\,\text{V}$$

The sampling period was chosen to be $T = 0{\cdot}125$ ms ($1/T = 8$ kHz) and the desired steady-state value of the average dynamics was set to be $X = 1237$ A. The eigenvalue for the closed loop linear dynamics, α, was set to be 0·3. The corresponding value of the steady-state input current was found to be $x^-_\infty(1237) = 1080{\cdot}7$ A. The required steady-state average value of the input current as well as the steady-state values x^+_∞ and x^-_∞ are well within the allowable range which guarantees non-saturation of the actual duty ratio function.

Figure 5 depicts a typical simulated PWM feedback controlled trajectory for the input current arising from the perturbed model (2.19). This figure also shows the actual duty ratio function $\mu(t)$ and the corresponding switch position function $u(t)$. At the end of the figure we show the perturbation signal $v(t)$. As shown, in spite of the influence of the unmodelled perturbation signal, $v(t)$ the derived nonlinear discrete-time duty ratio controller performs remarkably well.

The robustness of the proposed feedback control scheme was also tested with respect to a class of modelling errors, represented by significant, but temporary, circuit parameter variations. We performed several simulations, which included a sudden, unmodelled, load resistance variation. Figure 6 depicts the responses of the closed-loop regulated inductor current to variations of 0 %, 20 %, 40 % and 80 %, above the nominal load resistance value R, while the plant was still being affected by the external stochastic perturbation noise $v(t)$. Such variations were left to occur in the time

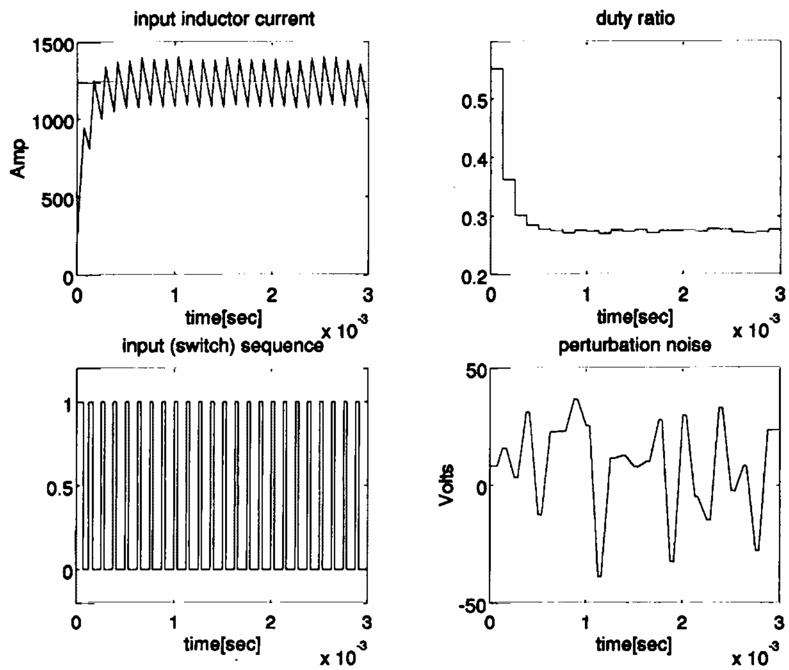

Figure 5. Simulation results of PWM regulation of perturbed buck-derived converter.

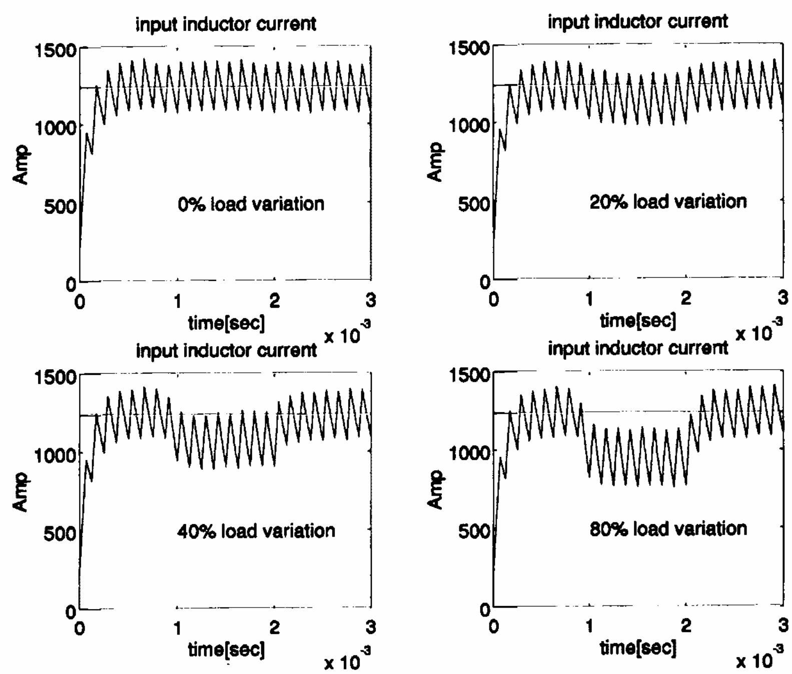

Figure 6. Simulation results of PWM regulation of perturbed buck-derived converter subject to several load variations.

interval between 0·001 s and 0·002 s. It can be seen that, up to a 20% load variation, the performance of the controller is quite robust. The controller is seen to drive the input current response to its original preassigned value, right after the load perturbation is over.

2.5. *The boost-derived converter*

In this section we briefly summarize the derivation of an implicit nonlinear feedback regulator for the indirect output voltage stabilization of the boost derived converter, shown in Fig. 7 (see Rashid 1993). The feedback loop synthesizing the required duty ratio function is based on a desired steady-state average input current value X.

The boost-derived switch regulated model

$$\begin{aligned}\dot{x} &= -\frac{R}{L}x(1-u)+\frac{E}{L}\\ &= -\theta_1(1-u)x+\theta_2\end{aligned} \tag{2.20}$$

PWM feedback regulation strategy for the switched position

$$u(t) = \begin{cases} 1 & \text{for } t_k \leqslant t < t_k+\mu(t_k)T \\ 0 & \text{for } \; t_k+\mu(t_k)T \leqslant t < t_k+T \end{cases} \tag{2.21}$$

$$t_{k+1} = t_k + T; \quad k = 0, 1, 2, \ldots$$

An exact discretization of the PWM regulated boost-derived dynamics

$$x(t_k+\mu(t_k)T) = \theta_2\mu(t_k)T + x(t_k) \tag{2.22}$$

$$\begin{aligned} x(t_k+T) &= \exp\{-\theta_1 T(1-\mu(t_k))\}[\theta_2\mu(t_k)T+x(t_k)]+\frac{\theta_2}{\theta_1}[1-\exp\{\theta_1 T(1-\mu(t_k))] \\ &= \Psi_1^{(1-\mu(t_k))}x(t_k)+\Psi_1^{(1-\mu(t_k))}[\mu(t_k)\Psi_3-\Psi_2]+\Psi_2 \end{aligned} \tag{2.23}$$

with $\Psi_1 = \exp\{-\theta_1 T\}$, $\Psi_2 = \theta_2/\theta_1$ and $\Psi_3 = \theta_2 T$.

Steady-state value of the sampled input current

$$x_\infty^- = \frac{\Psi_1^{(1-\mu_\infty)}[\mu_\infty\Psi_3-\Psi_2]+\Psi_2}{1-\Psi_1^{(1-\mu_\infty)}} \tag{2.24}$$

Discrete-time dynamics of the 'upper corners' of the PWM regulated input current trajectory

$$X(t_{k+1}+\mu(t_{k+1})T) = \Psi_1^{(1-\mu(t_k))}x(t_k+\mu(t_k)T)+\Psi_3\mu(t_{k+1})+\Psi_2(1-\Psi_1^{1-\mu(t_k)}) \tag{2.25}$$

Steady-state value of the 'intersampling' peaks of the input current

$$x_\infty^+ = \frac{\mu_\infty\Psi_3+\Psi_2(1-\Psi_1^{(1-\mu_\infty)})}{1-\Psi_1^{(1-\mu_\infty)}} \tag{2.26}$$

Steady-state 'ripple'

$$r_\infty = x_\infty^+ - x_\infty^- = \Psi_3\mu_\infty \tag{2.27}$$

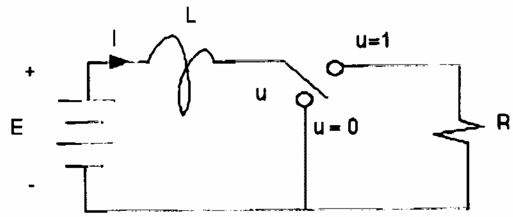

Figure 7. The boost-derived converter.

Steady-state average value of the input current trajectory

$$x_{av}(\infty) = x_\infty^- + \tfrac{1}{2}r_\infty \tag{2.28}$$

$$x_{av}(\infty) = \frac{[\Psi_3\mu_\infty(1+\Psi_1^{(1-\mu_\infty)})+2\Psi_2(1-\Psi_1^{(1-\mu_\infty)})]}{2(1-\Psi_1^{(1-\mu_\infty)})} \tag{2.29}$$

Existence of steady-state duty ratio function for the desired value of steady-state average input current

$$\Psi_1^{1-\mu_\infty} = \frac{2X-\Psi_3\mu_\infty-2\Psi_2}{2X+\Psi_3\mu_\infty-2\Psi_2} \tag{2.30}$$

The existence of a unique solution of (2.30), for μ_∞, follows from the fact that, as functions of μ_∞, the graph of the function on the left-hand side of (2.30) continuously increases in [0, 1], while the graph of the function on the right-hand side continuously decreases on such an interval. The graphs can be shown always to intersect each other, at most once, within the interval [0, 1].

Desired linear asymptotically stable closed loop dynamics

$$x(t_{k+1}) = \alpha(x(t_k)-x_\infty^-(X))+x_\infty^-(X); \quad |\alpha|<1 \tag{2.31}$$

Implicit nonlinear feedback duty ratio synthesizer

$$\Psi_1^{(1-\mu_c(t_k))} = \frac{\alpha x(t_k)-\Psi_2+(1-\alpha)x_\infty^-(X)}{x(t_k)-\Psi_2+\mu_c(t_k)\Psi_3} \tag{2.32}$$

A necessary and sufficient condition for the existence of a unique solution for the duty ratio

$$x_\infty^-(X) < x(t_k)+\frac{\Psi_1}{1-\alpha} \tag{2.33}$$

Remark: Implicit feedback controllers of the form (2.32) demand an online numerical solution of the corresponding transcendental equation for the computed duty ratio function $\mu_c(t_k)$, once the sampled state, $x(t_k)$, is available. The calculation time required at each sampling instant may be quite significant, thus introducing an important limitation in the implementation of the feedback control law. In the simulations carried out below a provision for such a computational time may be easily incorporated by further restricting the duty ratio function to be higher than a certain fixed positive lower bound, say of value μ_L. In other words, instead of the hard limiting condition $\mu \in [0, 1]$ on the computed duty ratio function μ, one enforces the limitation, $\mu \in [\mu_L, 1]$, with $\mu_L > 0$. At the beginning of the sampling period, and during the

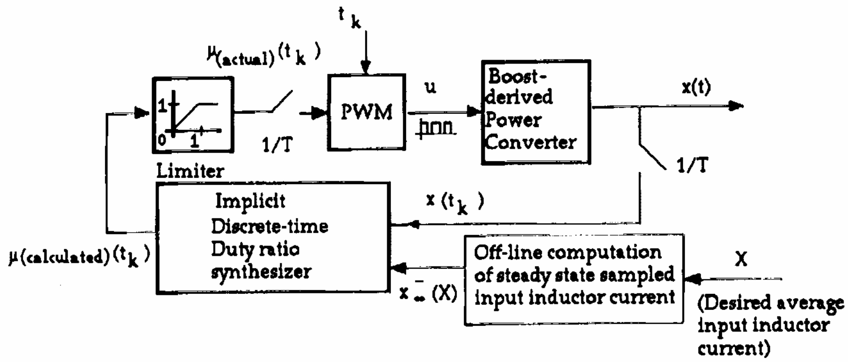

Figure 8. A PWM feedback regulation scheme for the boost-derived converter based on exact discretization.

fraction of the sampling interval, given by $\mu_L T$, control calculations must be performed. □

2.6. *Simulation results*

In order to test the robustness of the previous derived PWM feedback regulation policy, based on exact discretization, we used the following noise perturbed model of the boost-derived converter

$$\dot{x} = -\frac{R}{L}(1-u)x + \left(\frac{E+v(t)}{L}\right) \tag{2.34}$$

where $v(t)$ represented an unmodelled computer generated stochastic perturbation signal representing a noisy voltage source affecting the behaviour of the circuit. The values for the parameters defining the converter were taken to be the same as in the buck-derived case

$$R = 2{\cdot}8 \times 10^{-2}\,\Omega; \quad L = 1{\cdot}0 \times 10^{-2}\,\text{mH}; \quad E = 126\,v$$

The sampling period was chosen to be $T = 0{\cdot}125$ ms ($1/T = 8$ kHz) and the desired steady-state value of the average dynamics was set to be $X = 6000$ A. The eigenvalue for the closed loop linear dynamics, α, was set to be 0·3. The corresponding value of the steady-state input current was found to be $x_\infty^-(6000) = 5804$ A. The required steady-state average value of the input current as well as the steady-state values x_∞^+ and x_∞^- are well within the allowable range, which guarantees non-saturation of the actual duty ratio function.

Figure 9 depicts a typical simulated PWM feedback controlled trajectory for the input current arising from the perturbed model (2.34). This figure also shows the actual duty ratio function $\mu(t)$ and the corresponding switch position function $u(t)$. As can be seen from the duty ratio trajectory, in this case, the control calculation time could have been accommodated within the time interval $\mu_L T = 0{\cdot}0250$ ms, with $\mu_L = 0{\cdot}2$. At the end of the figure we show the perturbation signal $v(t)$. As shown, in spite of the unmodelled perturbation signal the derived nonlinear discrete-time duty ratio controller performs quite satisfactorily. Load variations similar to those carried out in the previous example were also performed, with similar results. The simulations are not shown in the interest of brevity.

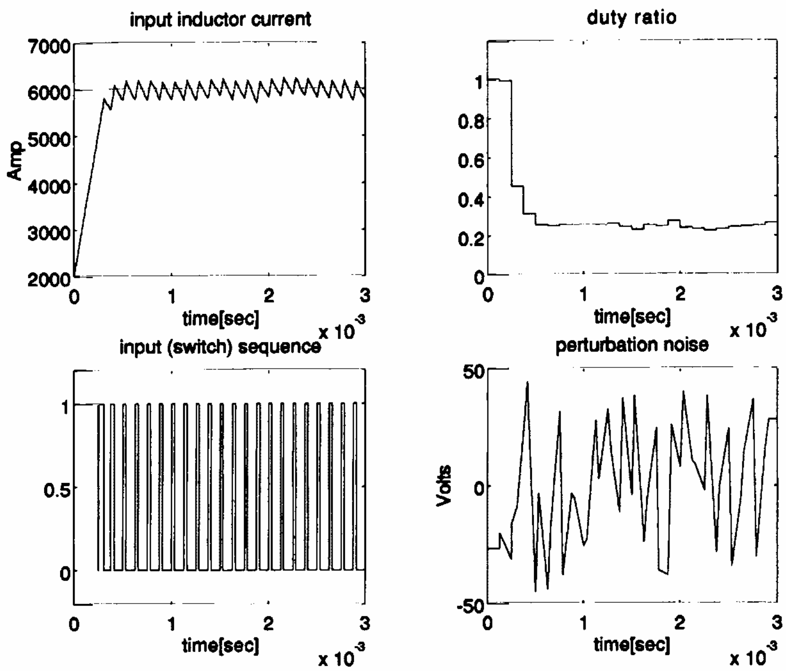

Figure 9. Simulation results of PWM stabilization of perturbed boost-derived converter.

3. Feedback signal tracking for derived DC-to-DC power converters via a discrete-time average model

In this section we present a discrete-time model for the average input currents of the derived DC-to-DC power converters presented in the previous section. The average models are obtained by a state elimination procedure on the expressions describing the mean value of the input current at the sampling times and at the end of the pulse associated with the switch position function. The state elimination leads to a non-kalmanian representation of the average input current dynamics. The models allow for a rather direct specification of the duty ratio synthesizers as dynamical feedback regulators.

We define the average value of the input current trajectory, at time t_k, as

$$z(t_k) = x(t_k) + \tfrac{1}{2}[x(t_k + \mu(t_k)\,T) - x(t_k)] = \tfrac{1}{2}[x(t_k) + x(t_k + \mu(t_k)\,T)] \qquad (3.1)$$

Remark: Note that the association of the average value $\frac{1}{2}[x(t_k) + x(t_k + \mu(t_k)\,T)]$ with the time instant t_k, as $z(t_k)$, is clearly quite arbitrary. In fact one could choose an intermediate instant in the interval, $[t_k, t_k + \mu(t_k)\,T]$, to represent the corresponding value of time. However, this convention only complicates the presentation and does not substantially differ from the one we have chosen. □

Similarly, the average of value of the input current at time $t_k + T$ is given by

$$z(t_{k+1}) = \tfrac{1}{2}[x(t_{k+1}) + x(t_{k+1} + \mu(t_{k+1})\,T)] \qquad (3.2)$$

Average PWM dynamics can be obtained from (3.1) and (3.2) by means of a state elimination procedure. Indeed, note that for each one of the treated converters, the expression (3.1) for $z(t_k)$ can be rewritten in terms of the sampled state $x(t_k)$ and the duty ratio $\mu(t_k)$ at time t_k. In a similar fashion, expression (3.2) for $z(t_{k+1})$ may also be,

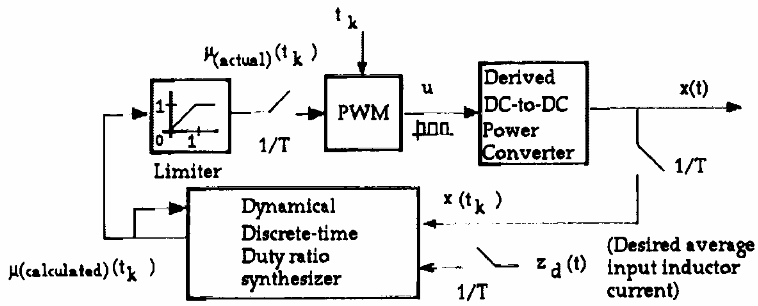

Figure 10. A discrete-time based PWM feedback regulation scheme for the signal tracking problem in derived DC-to-DC power converters.

ultimately, written in terms of $x(t_k)$ and the duty ratio function μ at times t_k and t_{k+1}, i.e. $\mu(t_k)$ and $\mu(t_{k+1})$. From the resulting equations for $z(t_k)$ and $z(t_{k+1})$ one may proceed to eliminate the state $x(t_k)$ thus obtaining $z(t_{k+1})$ as a function of $z(t_k)$ and the duty ratio function $\mu(t_k)$, $\mu(t_{k+1})$. Because of the linearity in the state associated with all of the invoked expressions, the resulting average PWM dynamics are also linear in the state $z(t_k)$. One obtains dynamics of the form

$$z(t_{k+1}) = \Phi_1(\mu(t_k), \mu(t_{k+1}))\, z(t_k) + \Phi_2(\mu(t_k), \mu(t_{k+1})) \tag{3.3}$$

which, evidently constitutes a non-kalmanian state representation, or more properly, a generalized state-space representation for the average input current dynamics (Fliess 1992).

By imposing desired linear discrete-time tracking error dynamics on the proposed average input current dynamics (3.3) a dynamical duty ratio synthesizer is readily obtained.

The dynamical feedback regulation scheme to be used for the PWM solution of the tracking problems, associated with the derived DC-to-DC power converters, as shown in Fig. 10.

In the following sections we present the average discrete-time input current model along with the duty ratio synthesizer for the solution of the corresponding tracking problem for the two derived converters studied in this article.

3.1. *The buck-derived converter*

The dynamics of the average value of the input current $z(t_k)$ is obtained by following the state elimination procedure outlined at the beginning of this section. The average model results in

$$\begin{aligned} z(t_{k+1}) &= \Psi_1\left(\frac{1+\Psi_1^{\mu(t_{k+1})}}{1+\Psi_1^{\mu(t_k)}}\right) z(t_k) \\ &\quad + \tfrac{1}{2}\Psi_2\left\{\frac{\Psi_1^{1-\mu(t_k)}(1-\Psi_1^{\mu(t_k)})(1+\Psi_1^{\mu(t_{k+1})}) + (1-\Psi_1^{\mu(t_{k+1})})(1+\Psi_1^{\mu(t_k)})}{1+\Psi_1^{\mu(t_k)}}\right\} \end{aligned} \tag{3.4}$$

An interesting fact, which makes the average model (3.4) non-traditional, is that the resulting recursion formula, obtained for the evolution of the average current values, requires the values of the duty ratio function, i.e. of the control input variable, at two consecutive sampling instants.

Suppose $\lim_{k\to\infty}\mu(t_k)$ exists, and assume it is given by the constant value μ_∞, then, evidently, $\lim_{k\to\infty}\mu(t_k) = \lim_{k\to\infty}\mu(t_{k+1}) = \mu_\infty$ and the corresponding steady-state value, z_∞, of the average input current, as computed from (3.4), is given by

$$z_\infty = \frac{1}{2}\left(\frac{\Psi_2}{1-\Psi_1}\right)(1-\Psi_1^{\mu_\infty})(1+\Psi_1^{1-\mu_\infty}) \tag{3.5}$$

which exactly corresponds to the steady-state value $x_{av}(\infty)$ computed in (2.13) and found from slightly different considerations.

3.2. *An average input current tracking problem for the buck-derived converter*

Let $z_d(t)$ represent a desired time-varying reference input signal. It is desired to have the average input current $z(t_k)$ asymptotically track the sampled values, $z_d(t_k)$ of the reference input signal.

Let $e(t_k)$ denote the average tracking error at time t_k, given by $e(t_k) = z(t_k) - z_d(t_k)$. The following tracking error dynamics may then be imposed on the closed loop system

$$e(t_{k+1}) = \alpha e(t_k); \quad |\alpha| < 1 \tag{3.6}$$

In terms of the average input current, such dynamics result in the following expression

$$z(t_{k+1}) = \alpha(z(t_k) - z_d(t_k)) + z_d(t_{k+1}) \tag{3.7}$$

Since $z_d(t_{k+1})$ is assumed to be known beforehand, the preceding equation does not have the connotation of an acausal system.

Substituting the right-hand side of expression (3.4) into (3.7), and solving for $\mu(t_{k+1})$ one obtains the following time-varying dynamical nonlinear feedback controller, or duty ratio synthesizer, for the average input current tracking problem

$$\begin{aligned}\mu(t_{k+1}) = \frac{1}{\ln\Phi_1}\ln\Bigg\{ & \frac{2(1+\Psi_1^{\mu(t_k)})}{\Psi_2[\Psi_1^{1-\mu(t_k)}(1-\Psi_1^{\mu(t_k)}) - (1+\Psi_1^{\mu(t_k)})] + 2\Psi_1 z(t_k)} \\ & \times\Bigg[1 - \frac{1}{2}\Psi_2\left(\frac{\Psi_1^{1-\mu(t_k)}(1-\Psi_1^{\mu(t_k)}) + (1+\Psi_1^{\mu(t_k)})}{1+\Psi_1^{\mu(t_k)}}\right) \\ & - \left(\frac{\Psi_1}{1+\Psi_1^{\mu(t_k)}} - \alpha\right) z(t_k) - \alpha z_d(t_k) + z_d(t_{k+1})\Bigg]\Bigg\}\end{aligned} \tag{3.8}$$

Remark: The initialization of the above controller requires the specification of $z(t_0)$ which, in fact, involves knowledge of both $x(t_0)$ and $\mu(t_0)$. The initial input current $x(t_0)$ may be measured, or simply set by previously discharging the energy initially stored in the inductor. The initial duty ratio, $\mu(t_0)$, must then be arbitrarily chosen. This implies that the quantities $x(t_0)$ and $x(t_0+\mu_0 T)$ are assumed to be initially known. Note also that a PWM control policy based on the average model (3.3) necessarily requires online measurements of the average input current, i.e. values of $x(t)$ have to be measured both at the end of each pulse, within the sampling interval, and at the beginning of each sampling period. □

The computed duty ratio function must never exceed the natural limiting values of the actual duty ratio function, represented by the closed interval [0, 1]. Possibilities for saturation of the actual duty ratio function depend on the time derivative of the desired input reference signal, on the imposed closed-loop eigenvalue and on the circuit parameters themselves. By more conventional averaging techniques (see Sira-Ramírez *et al.* 1993) one can obtain an estimate of the tracking limitations of the circuit.

3.3. *Simulation results*

The previously derived PWM feedback tracking policy, based on exact discretization and the introduced average model was used on the same noise perturbed model (2.19) of §2.4.

It is desired to track a 'trapezoidal' reference input signal, as shown in Fig. 11, with $z_{\max} = 1237$ A. This reference signal is expressed as

$$z_d(t) = \begin{cases} 1237t & \text{for } 0 < t \leqslant 1 \text{ ms} \\ 1237 & \text{for } 1 < t \leqslant 2 \text{ ms} \\ 1237 - 1237(t-2) & \text{for } 2 < t \leqslant 3 \text{ ms} \end{cases}$$

Figure 12 depicts the simulated PWM feedback controlled trajectory for the input current arising from the controlled perturbed model (2.19). This figure also shows the actual duty ratio function $\mu(t)$ and the corresponding switch position function $u(t)$. In this figure we also show the perturbation signal $v(t)$. As shown, in spite of the unmodelled perturbation signal, the derived nonlinear discrete-time duty ratio controller tracks remarkably well the desired reference signal, as long as the computed duty ratio function does not take values outside the interval [0, 1]. Note that close to the end of the tracking horizon the actual duty ratio function μ is seen to 'saturate' to the value of zero. The problem is circumvented by suitably lowering the maximum value of the desired reference signal $z_d(t)$ as well as its slope on the 'descending' portion of the trapezoid.

3.4. *The boost-derived converter*

Generalized state representation of average input current dynamics

$$z(t_{k+1}) = \Psi_1^{1-\mu(t_k)} z(t_k) - \Psi_1^{1-\mu(t_k)} \Psi_2 + \Psi_2 + \tfrac{1}{2}\Psi_3(\Psi_1^{1-\mu(t_k)} \mu(t_k) + \mu(t_{k+1}) \tag{3.9}$$

Steady-state value of average input current

$$z_\infty = \frac{[\Psi_3 \mu_\infty(1 + \Psi_1^{(1-\mu_\infty)}) + 2\Psi_2(1 - \Psi_1^{(1-\mu_\infty)})]}{2(1 - \Psi_1^{(1-\mu_\infty)})} \tag{3.10}$$

which coincides with (2.29).

Desired closed loop linear dynamics

$$z(t_{k+1}) = \alpha(z(t_k) - z_d(t_k)) + z_d(t_{k+1}) \tag{3.11}$$

Dynamical duty ratio synthesizer

$$\mu(t_{k+1}) = \frac{2}{\Psi_3} \cdot \left[(\alpha - \Psi_1^{1-\mu(t_k)}) z(t_k) - \Psi_1^{1-\mu(t_k)} \left(\frac{1}{2} \Psi_3 \mu(t_k) - \Psi_2 \right) - \Psi_2 - \alpha z_d(t_k) + z_d(t_{k+1}) \right] \tag{3.12}$$

Remark: The initialization of the above controller is carried out in a manner similar to that corresponding to the buck-derived converter (see the Remark of §3.2).

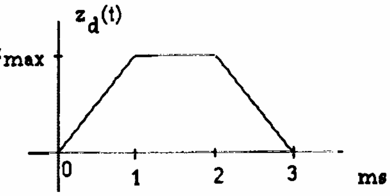

Figure 11. Desired trapezoidal input current signal.

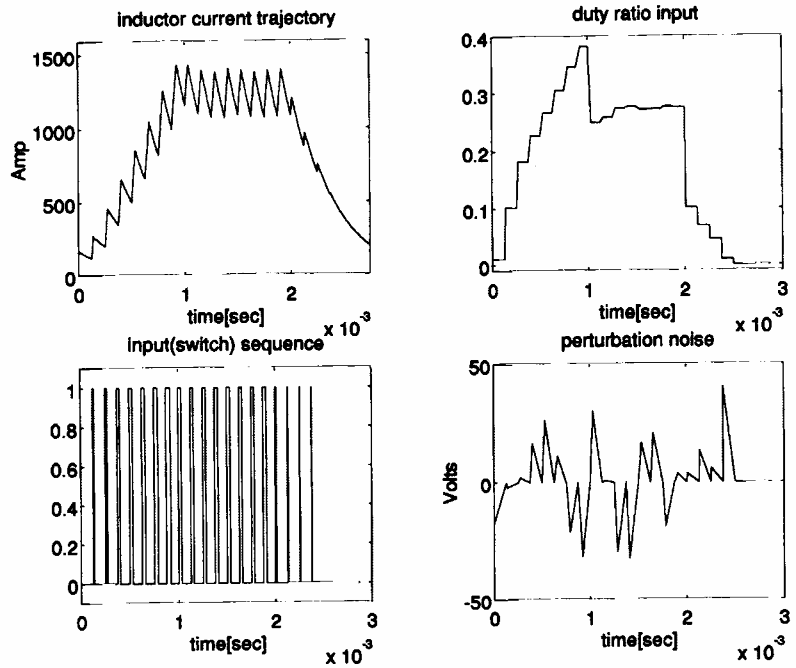

Figure 12. Simulation results for trapezoidal signal tracking problem for the perturbed buck-derived converter.

3.5. *Simulation results*

The previously derived PWM feedback tracking policy, based on exact discretization and the introduced average model, was used on the noise perturbed model (2.34) of the boost-derived converter presented in §2.8.

It was required to track a 'trapezoidal' reference input signal, similar to that shown in Fig. 11, with $z_{max} = 6000$ A. In accordance with the 'step-up' character of the derived-boost converter, the reference signal $z_d(t)$ to be tracked was specified by the following expression

$$z_d(t) = \begin{cases} 4500 + 1500t & \text{for } 0 < t \leqslant 1 \text{ ms} \\ 6000 & \text{for } 1 < t \leqslant 2 \text{ ms} \\ 6000 - 1500(t-2) & \text{for } 2 < t \leqslant 3 \text{ ms} \end{cases}$$

Figure 13 depicts the simulated PWM feedback controlled trajectory for the input current arising from the controlled perturbed model (2.34) of the boost-derived converter. This figure also shows the actual duty ratio function $\mu(t)$ and the corresponding switch position function $u(t)$. In this figure we also show the

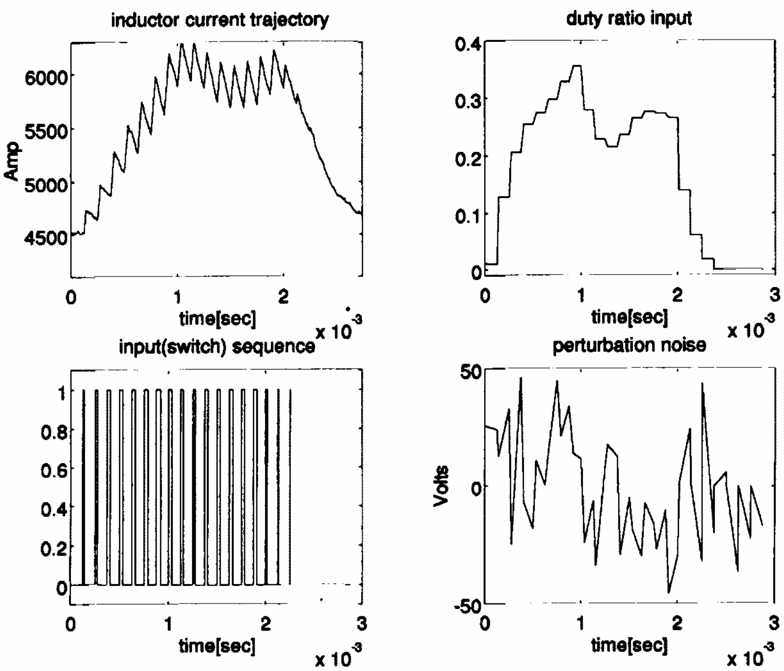

Figure 13. Simulation results for trapezoidal signal tracking problem for the perturbed boost-derived converter.

perturbation signal $v(t)$. As shown, in spite of the unmodelled perturbation signal, the derived nonlinear discrete-time duty ratio controller tracks remarkably well the desired reference signal, as long as the computed duty ratio function does not take values outside the interval [0, 1]. Note that close to the end of the tracking horizon the actual duty ratio function μ is seen to 'saturate' to the value of zero. The problem may be circumvented by suitably lowering the maximum value of the desired reference signal $z_d(t)$ as well as the slope on the 'descending' portion of the trapezoid.

4. Conclusions

In this article an exact discretization scheme has been proposed for the input current stabilization and tracking tasks in perfectly known derived DC-to-DC power supplies of the buck and boost types. The complexities arising in the stabilization problem associated with such devices are related, fundamentally, to the highly nonlinear form of the derived duty ratio compensators. For the boost converter, the controller cannot be found explicitly and a transcendental equation must be solved online, at each sampling instant, on the basis of the (sampled) state of the converter circuit. The signal tracking problem is solved by means of a non-kalmanian state representation of the average input current. Explicit dynamical controllers could then be found from exact discrete-time linearization schemes imposed on the average controlled models. The results are appropriate in high power systems where the sampling frequency may be limited.

Some of the difficulties encountered in the one-dimensional converter cases treated here become, not surprisingly, much harder when dealing with the two-dimensional models of traditional DC-to-DC power converters. In such cases, the symbolic manipulation tasks associated with the solution of the stabilization problems become particularly intricate, even with the help of very efficient computer packages such as Maple, or Mathematica.

Simulation studies have revealed a certain degree of sensitivity of the proposed exact control schemes to sudden, unmodelled, load parameter variations. As a topic for further research, the case of stabilization and tracking problems for derived converters with uncertain parameters is, therefore, of particular practical interest and one for which efficient nonlinear discrete-time adaptive and robust control techniques must be developed.

Acknowledgments

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