INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2000; **10**:301–320

A Liouvillian systems approach for the trajectory planning-based control of helicopter models

Hebertt Sira-Ramírez[1,*,†], Rafael Castro-Linares[1] and Eduardo Licéaga-Castro[2]

[1] *CINVESTAV-IPN, Departamento de Ingeniería Eléctrica, Apartado Postal 14-740, 07300 México, D.F., México*
[2] *Department of Mathematics, Glasgow Caledonian University, Glasgow G4 OBA, Scotland, U.K.*

SUMMARY

A feedback regulation scheme, based on off-line trajectory planning and an approximate state linearization, is proposed for the hover-to-hover stabilization of simplified, underactuated, models of a helicopter system. The approach, based on the 'Liouvillian' character of the helicopter kinematic equations, advantageously uses the total, or partial, differential flatness property of the system models. The controller performance is evaluated through digital computer simulations which include initial state setting errors of significant magnitudes. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: underactuated systems; helicopter regulation; Liouvillian systems; trajectory planning

1. INTRODUCTION

The regulation of helicopter systems has received sustained attention in the past. In Reference [1], a state-space feedback linearization approach was followed in order to calculate the inverse of a helicopter model. Nevertheless, the simplifying assumptions lead to maneuvers constrained to very low bandwidths. A sliding mode control approach, using dynamical feedback linearization techniques and generalized canonical forms, proposed by Fliess [2], was carried out in Reference [3] for an actual laboratory scaled helicopter. In that work, the zero dynamics of the system was

* Correspondence to: Hebertt Sira-Ramírez, CINVESTAV-IPN, Departamento de Ingeniería Eléctrica, Apartado Postal 14-740, 07300 México, D.F., México.
† E-mail: hsira@mail.cinvestav.mx

Contract/grant sponsor: Centro de Investigación y de Estudios Avanzados of the Instituto Politécnico Nacional (CINVESTAV-IPN) of México
Contract/grant sponsor: Consejo Nacional de Ciencia y Tecnologia (CONACYT) of México; contract/grant number: 32681-A
Contract/grant sponsor: Department of Mathematics of Glasgow Caledonian University, Glasgow, Scotland, U.K.

suitably 'absorved' by the sliding mode controller dynamics. More recently, the concept of outer differential flatness was introduced and used in Reference [4] for trajectory generation in an experimental model helicopter (see Fliess *et al.* [5–7] for basic definitions and results on differentially flat systems). While some traditional models of helicopters turn out to be differentially flat, some more recently developed models of either real (see the work of Thomson and Bradley [8, 9], or 'toy' helicopters (called toycopters), turn out to be non-differentially flat. For interesting studies dealing with a toycopter stabilization problem, which resorts to a physically motivated approximation to a flat system, the reader is referred to the work of Mullhaupt and his colleagues [10].

In this article we propose a trajectory planning approach. combined with approximate (Jacobian) linearization around the specified trajectory, to regulate hover-to-hover maneuvers in several simplified under-actuated models of a helicopter system. The Liouvillian character of the system. i.e. the presence of an integrable defect (see [11]), is exploited to carry out an off-line trajectory planning resulting in nominal state and nominal control input trajectories. Jacobian linearization around the nominal trajectories results in a controllable time-varying system for which a stabilizing feedback controller is specified using linear systems theory.

In any of the treated cases. the proposed controller synthesis method entitles an off-line trajectory planning on the basis of the the desired displacement trajectories. This forces us to off-line solve ordinary differential equations with appropriate initial (hovering) conditions with the desired displacement trajectories viewed as given data. It should be pointed out, however, that the longitudinal dynamics model of the helicopter adopted here is indeed ***differentially flat*** and it can be exactly linearized by means of dynamic state feedback. This may be inferred from the closely related work of Martin *et al.* (see [12]) where a PVTOL aircraft model is considered. The approach in Reference [12] requires a dynamic extension of the original model by the introduction of appropriately defined auxiliary state variables. The main difference between our Liouvillian system-based approach and one entirely based on the differential flatness of the suitably extended model, is that the computation needed for the ideal open-loop control involves the off-line solution of a second-order nonlinear differential equation. The flatness-based approach, on the other hand, would not require such an off-line calculation, but, instead, an equivalent associated burden is transferred to the on-line solution of a nonlinear second-order differential equation representing the dynamic feedback controller. Our controller, on the other hand, is linear and static, though time varying, and somehow 'simpler' in nature than the one we would have obtained based on the full-fledged differential flatness approach.

Section 2 revisits a ninth order model for the helicopter dynamics developed already in Reference [13]. The proposed simplifications leading to more tractable, but still underactuated, models are directly obtained from such a model under the assumptions of constant lateral velocity and constant lateral and normal velocities (see also the works of Liceaga *et al.* [1]). Section 3 contains a brief introduction to Liouvillian systems and discusses the unstable nature of the 'remaining or zero dynamics. The proposed feedback control scheme, based on approximate linearization and off-line trajectory planning, is also presented in this section and a linear time varying controller is derived which complements the off-line computed nominal control input. This guarantees some robustness to the proposed systematic control scheme. In Section 4, simulation tests are performed for the closed-loop system which include initial state perturbations. The conclusions and proposals for further research are presented in the last section.

2. SIMPLIFIED MATHEMATICAL MODELS OF A HELICOPTER

2.1. The full model

In this paper, we consider simplified models of a, so-called, *Lynx* helicopter which has been fully reported in various works (see [13, 14]). Such a helicopter is modelled by the following set of first-order ordinary differential equations:

$$
\begin{aligned}
\dot{u} &= -[\cos(\psi)\sin(\theta)\cos(\phi)+\sin(\psi)\sin(\phi)]\frac{1}{M}u_1-[\cos(\psi)\cos(\theta)]\frac{1}{M}u_2\\
&\quad -[\cos(\psi)\sin(\theta)\sin(\phi)-\sin(\psi)\cos(\phi)]\frac{1}{M}u_3\\
\dot{v} &= -[\sin(\psi)\sin(\theta)\cos(\phi)-\cos(\psi)\sin(\phi)]\frac{1}{M}u_1-[\sin(\psi)\cos(\theta)]\frac{1}{M}u_2\\
&\quad -[\sin(\psi)\sin(\theta)\sin(\phi)+\cos(\psi)\cos(\phi)]\frac{1}{M}u_3\\
\dot{\omega} &= g-[\cos(\theta)\cos(\phi)]\frac{1}{M}u_1+\sin(\theta)\frac{1}{M}u_2-[\cos(\theta)\sin(\phi)]\frac{1}{M}u_3\\
\dot{p} &= \frac{i_{yy}-i_{zz}}{i_{xx}}qr-\frac{l_h}{i_{xx}}u_3\\
\dot{q} &= \frac{i_{zz}-i_{xx}}{i_{yy}}rp+\frac{l_h}{i_{yy}}u_2\\
\dot{r} &= \frac{i_{xx}-i_{yy}}{i_{zz}}pq+\frac{l_t}{i_{zz}}u_4\\
\dot{\phi} &= p+[q\sin(\phi)+r\cos(\phi)]\tan(\theta)\\
\dot{\theta} &= q\cos(\phi)-r\sin(\phi)\\
\dot{\psi} &= [q\sin(\phi)+r\cos(\phi)]\sec(\theta)
\end{aligned}
\qquad (1)
$$

where u, v and ω are the forward, lateral and normal velocities (also called *translational velocities*) the angles ϕ, θ and ψ are the roll angle, the pitch angle, and the yaw angle, (referred to as the *attitude angles*) and the quantities, p, q and r are the roll rate, the pitch rate and the yaw rate. u_1, u_2, u_3 and u_4 are assumed to be the control variables; u_1, u_2 and u_3 are related to the components of the main 'rotor' thrust and may be associated with collective, longitudinal cycle and lateral cycle, respectively, while u_4 represents a torque produced by the tail 'rotor' thrust. The constants i_{xx}, i_{yy} and i_{zz} represent moments of inertia, l_h is the distance between the rotor hub and the fuselage centre of mass, l_t is the distance between the tail hub and the fuselage centre of mass, M is the helicopter mass and g is the gravitational force.

2.2. A simplified model with constant lateral velocity

If one considers flight with constant lateral velocity one sets $\psi(t)=\phi(t)=0$ and $\dot{r}(t)=\dot{p}(t)=0$ with $r(t)=0$, $p(t)=0$ for all $t\geqslant 0$. Under these assumptions, we may consider all motions to take place on the plane $z = 0$. These assumptions readily lead to $u_3 = u_4 = 0$ in Equations (1). The following set of simplified differential equations for the longitudinal dynamics of the helicopter are obtained:

$$\dot{u} = -\frac{1}{M}\sin(\theta)u_1 - \frac{1}{M}\cos(\theta)u_2$$

$$\dot{\omega} = g - \frac{1}{M}\cos(\theta)u_1 + \frac{1}{M}\sin(\theta)u_2$$

$$\dot{\theta} = q$$

$$q = \frac{l_h}{i_{yy}} u_2 \tag{2}$$

If we denote by x the forward position of the rotor-craft and by y its vertical height, then the first-order model (2) is rewritten as the following 'second-order' model:

$$\ddot{x} = -\frac{1}{M}\sin(\theta)u_1 - \frac{1}{M}\cos(\theta)u_2$$

$$\ddot{y} = g - \frac{1}{M}\cos(\theta)u_1 + \frac{1}{M}\sin(\theta)u_2$$

$$\ddot{\theta} = Lu_2 \tag{3}$$

where the constant parameter L is defined as $L = l_h/i_{yy}$. Note that under null horizontal displacement x = constant, $\dot{x} = u = 0$ and $\ddot{x} = \dot{u} = 0$. It thus follows that the vertical displacement dynamics is governed by

$$\ddot{y} = g - \frac{1}{M\cos(\theta)} u_1 \tag{4}$$

A free fall condition implies that $\ddot{y} = g$ and $u_1 = 0$. A hovering condition implies that $\ddot{y} = 0$ and $u_1 = Mg\cos(\theta)$ and, finally, an ascending condition implies that $\ddot{y} < 0$ and $u_1 = M(g - \ddot{y})\cos(\theta)$. A reasonable assumption is that descent maneuvers are never faster than a free fall condition, i.e. $g > \ddot{y}$. This condition is, in turn, guaranteed if we let u_1 be strictly positive and θ bounded, in absolute value, to angles strictly smaller than $\pi/2$. However, these assumptions entitle an implication on the unconstrained dynamics (3). Eliminating u_2 from the first two equations in (3) and solving for u_1 we obtain

$$u_1 = M[(g - \ddot{y})\cos(\theta) - \ddot{x}\sin(\theta)] \tag{5}$$

We, therefore, assume that the following conditions are valid for any given maneuver:

$$(g - \ddot{y})\cos(\theta) - \ddot{x}\sin(\theta) > \delta > 0, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \tag{6}$$

where δ is a strictly positive constant.

Solving for u_2, from the first two equations in (3), we obtain the following expression:

$$u_2 = -M[(g - \ddot{y})\sin(\theta) + \ddot{x}\cos(\theta)] \tag{7}$$

We easily see that no restrictions may apply to the sign of u_2 which are compatible with elementary maneuvers such as hovering, null vertical or horizontal displacement. We thus merely assume that u_2 is a bounded control input and, hence

$$|(g - \ddot{y})\sin(\theta) + \ddot{x}\cos(\theta)| < \kappa \tag{8}$$

for a strictly positive constant κ.

A final assumption is made, regarding the need for continuous controls u_1. We shall demand that the absolute value of the time derivative of u_1 be bounded for any maneuver, this is

$$|\dot{u}_1| < \varepsilon \tag{9}$$

2.3. *A simplified model with constant normal and lateral velocity*

When one considers flight with constant normal and lateral velocities, one further sets $\dot{\omega} = 0$, for all $t \geqslant 0$. The motions are now restricted to the line $z = 0$, $y =$ constant. In this case, the control input u_1 is automatically determined as $u_1 = Mg/\cos(\theta) + u_2\tan(\theta)$. These additional simplifications lead to the uncontrollable model found in Reference [4], which is at variance with real-life experience. Further, imposing the condition $\omega = 0$, one is lead to the following set of differential equations for the helicopter model:

$$\begin{aligned} \dot{\theta} &= q \\ \dot{q} &= Lu_2 \\ \dot{x} &= u \\ \dot{u} &= -g\tan(\theta) - \frac{1}{M\cos(\theta)}u_2 \end{aligned} \tag{10}$$

where x denotes, as before, the forward position of the rotor-craft and $L = l_h/i_{yy}$. It is worthwhile to notice that this model is different from the one analysed in Reference [4] where the forward velocity dynamics, in straight level flight, reduces to $\ddot{x} = -g\tan(\theta)$. We rewrite the model (10) as the 'second-order' model

$$\begin{aligned} \ddot{\theta} &= Lu_2 \\ \ddot{x} &= -g\tan(\theta) - \frac{1}{M\cos(\theta)}u_2 \end{aligned} \tag{11}$$

Straightforward manipulations of Equation (11) yield the relationship

$$g - \ddot{x}\tan(\theta) = \sec^2(\theta)\left[g + \frac{1}{M}u_2\sin(\theta)\right] \tag{12}$$

The right-hand side of Equation (12) must always be positive for reasonable maneuvers that do not exceed an attitude angular displacement restriction $-\theta_{max} \leqslant \theta \leqslant \theta_{max}$ with $\theta_{max} < \pi/2$. It then follows that the quantity $g - \ddot{x}\tan(\theta)$ can be assumed to be non-negative. In fact, for the flight conditions considered in this section it will be assumed that there exists a strictly positive scalar μ such that $g - \ddot{x}\tan(\theta) \geqslant \mu$.

3. REGULATION OF THE SIMPLIFIED HELICOPTER MODELS

3.1. Liouvillian systems

Differentially flat systems, or *flat* systems in short, were introduced by Professor M. Fliess and his coworkers in a series of articles [5–7]. Flat systems are characterized by the fact that all system variables, including the inputs, can be expressed in terms of *differential functions of the flat outputs*. The flat outputs constitute a set of independent differential functions of the state, possibly involving the inputs and a finite number of their time derivatives. The set of flat outputs has the same cardinality as the set of control inputs. Differentially flat systems constitute a subclass of the set of controllable nonlinear systems which are equivalent to a linear system in Brunovsky's form by means of *endogenous* feedback, i.e. one that does not require external variables to the system for its synthesis.

A non-flat system may still be controllable, but not all its variables can be expressed as differential functions of a particular set of independent outputs. The number of variables, not expressible in terms of the flat outputs, is known as the *defect* of the non-flat system. Liouvillian systems constitute a natural extension of *differentially flat systems* into the area of systems which are non-linearizable by means of endogenous feedback. The class of Liouvillian systems contains a subset of the class of non-flat systems with an identifiable flat subsystem of maximal dimension. A non-flat system is said to be Liouvillian, or *integrable by quadratures* if the variables not belonging to the flat subsystem are expressible as elementary integrations of the flat outputs and a finite number of their time derivatives. The introduction of this class of systems has been recently given by Chelouah in [11] from the perspective of Differential Galois theory in the context of Piccard–Vessiot extensions of differentially flat fields. The idea has also been shown to have interesting implications on finitely discretizable nonlinear systems, as inferred from the work of Chelouah and Petitot [15].

Some models of helicopter systems are differentially flat and they are linearizable by means of dynamic state feedback. However, they may also be regarded as Liouvillian systems. This is explained by the fact that, in such cases, 'flat' outputs of a particular subsystem may be proposed in terms of the original variables of the system. These variables have the additional property that the 'remaining variables' may still be expressible in terms of elementary quadratures of such 'flat' outputs. Hence, strictly speaking, in such cases, a *differential parameterization* of all the system variables is no longer possible, but, instead, an *integral–differential parameterization* could be established. This last property, characterizing Liouvillian systems, may be suitably exploited at the off-line stage of a control scheme based on trajectory planning. The advantage with respect to existing flatness-based schemes is confined to the fact that a static, linear, time varying controller can be synthesized while solutions of nonlinear differential equations are conveniently relegated to the off-line stage of the controller design, rather than to the on-line stage where a set of similarly complex dynamic controller equations must be solved.

3.2. The helicopter model as a Liouvillian system: simplified model with constant lateral velocity

Let us now consider the model (3) of the helicopter. This system is differentially flat, and, hence, linearizable by means of dynamic state feedback. The flat outputs are given by

$$\mathscr{P} = x + \frac{1}{LM}\sin(\theta), \quad \mathscr{Q} = y + \frac{1}{LM}\cos(\theta) \tag{13}$$

which, as pointed out in the closely related work of Martin *et al.* (see [12]), define the equivalent to the *Huygens centre of oscillation* in a set of similar equations describing a pendulum.

By defining the input-dependent auxiliary state variable

$$\vartheta = u_1 + \frac{1}{L}(\dot{\theta})^2 \tag{14}$$

system (3) may be shown, after some lengthy algebraic manipulations, to be equivalent to the following set of elementary linear *pure integration* systems:

$$\mathscr{P}^{(4)} = v_1, \quad \mathscr{Q}^{(4)} = v_2 \tag{15}$$

with v_1 and v_2 representing new external control inputs (see also [16]).

The helicopter model (3), however, can also be regarded as a Liouvillian system, with the flat subsystem being represented by the state variables $(\theta, \dot{\theta}, y, \dot{y})$. The 'flat' outputs for this subsystem are given by the attitude angle θ and by the vertical displacement y, which we denote by F and R, respectively. The following partial *differential parameterization* of the system variables allows for some elementary equilibrium analysis and also establishes the main features of the system to be controlled:

$$\theta = F, \quad \dot{\theta} = \dot{F}, \quad u_2 = \frac{\ddot{F}}{L}, \quad y = R, \quad \dot{y} = \dot{R}, \quad u_1 = \frac{M}{\cos(F)}\left(g + \frac{\ddot{F}}{LM}\sin(F) - \ddot{R}\right) \tag{16}$$

The 'remaining' system variables, represented by the horizontal displacement variables $(x, \dot{x})$, are expressible in terms of quadratures of the proposed flat outputs F and R and its second-order time derivatives $\ddot{F}$, $\ddot{R}$. Indeed, from (3) and the previous considerations, we obtain, modulo initial conditions and specific integration limits,

$$\dot{x} = -\int \tan(F)\left(g + \frac{\ddot{F}}{LM}\sin(F) - \ddot{R}\right)\mathrm{d}t - \frac{1}{LM}\int \ddot{F}\cos(F)\mathrm{d}t$$

$$x = -\int\int \tan(F)\left(g + \frac{\ddot{F}}{LM}\sin(F) - \ddot{R}\right)\mathrm{d}\sigma\,\mathrm{d}t - \frac{1}{LM}\int\int \ddot{F}\cos(F)\,\mathrm{d}\sigma\,\mathrm{d}t \tag{17}$$

This is, system (3) qualifies as a Liouvillian system with flat subsystem outputs given by F and R.

The *zero dynamics* of the system, corresponding to a resting hovering position, characterized by x = constant, y = constant, is given, according to (3), by the zero dynamics

$$\ddot{F} = -LgM\sin(F) \tag{18}$$

System (18) is a locally stable oscillatory system with equilibria located at the origin and, also, at the attitude angles of the form $F = \pm k\pi$, $k = 1, 2, \ldots$. Thus, as expected, the system is ***weakly minimum phase*** with respect to the horizontal and vertical co-ordinates x, y, taken as the system outputs.

3.2.1. Off-line trajectory planning. Suppose a rest-to-rest maneuver is demanded which transfers the helicopter system from a given equilibrium (or hovering) position towards a second desired equilibrium position. It should be clear that it is far simpler (while perhaps, physically speaking, more appealing) to specify the desired rest-to-rest maneuver in terms of the vertical and horizontal displacements than it is to specify it in terms of one of the displacements and the suitable attitude angular trajectory. For this reason, suppose we select suitable trajetories $x^*(t)$ and $y^*(t)$ for the horizontal and vertical position variables in order to specify a certain displacement maneuver. Such a selection allows, for example, to take the helicopter from an initial equilibrium hovering position, given by $(x(t_{hi}), y(t_{vi}))$, towards a final hovering position specified as $(x(T_{hf}), y(T_{vf}))$ where the quantities $t_{hi}, t_{vi}, T_{hf}, T_{vf}$ are, in principle, independent (finite) instants of time. One may also include intermediate hovering positions, 'backing-ups', advancements, and ascents and descents of arbitrary lengths and durations. Also, the time evolution of the 'off- line' planned trajectories $x^*(t)$, $y^*(t)$ are assumed to start with a sufficient number of zero initial and final time derivatives, extending these properties to intermediate resting positions.

The partial differential flatness properties of the variables $\theta = F$ and $y = R$, allows one to express the control inputs u_1 and u_2 as the open-loop control laws

$$u_1 = \frac{M}{\cos(F)}\left(g - \ddot{R} + \frac{\ddot{F}}{LM}\sin(F)\right), \quad u_2 = \frac{\ddot{F}}{L} \tag{19}$$

Thus, for a desired displacement maneuver given by $x^*(t)$, $R(t) = y^*(t)$, the corresponding attitude angular trajectory may be computed by finding the solution $F^*(t)$ of the following nonlinear second-order differential equation:

$$\ddot{F}^* = -LM[\ddot{x}^*(t)\cos(F^*) + \ddot{y}^*(t)\sin(F^*)] \tag{20}$$

with initial conditions given in complete accordance with the desired maneuver. Notice that, for an equilibrium-to-equilibrium maneuver, the angular position $F^*(t)$ and the accelerations, $\ddot{x}^*(t)$ and $\ddot{y}^*(t)$ must be zero at the initial and at the final times. Under such conditions $\ddot{F}^*(t)$ would also be identically zero for all times *before* the transfer motions are started and also *after* the transfer has been accomplished. From (19), the corresponding control input u_2 would also be zero during these time intervals while u_1 should remain constant. It thus remains to be proven that the solutions of (20) from zero initial conditions, with $x^*(t)$ and $y^*(t)$ representing rest-to-rest maneuvers in a certain open time interval, $[\min\{t_{hi}, t_{vi}\} \max\{t_{hf}, t_{vf}\}]$, actually yield $F^*(t) = 0$ for all $t > \max\{t_{hf}, t_{vf}\}$.

The preceding statement is proved by considering a suitable *energy function* for the system (20). Let $t_0 < \min\{t_{hi}, t_{vi}\}$ and $T > \max\{T_{hf}, T_{vf}\}$. Define an energy-like function of the form

$$V(F^*(t), \dot{F}^*(t)) = \tfrac{1}{2}(\dot{F}^*(t))^2 + \left\{1 + LM\int_{t_0}^{t}[\ddot{x}(\sigma)\cos F^*(\sigma) + (g - \ddot{y}(\sigma))\sin F^*(\sigma)]\dot{F}^*(\sigma)\mathrm{d}\sigma\right\} \tag{21}$$

The time derivative of V, along the trajectories of system (20), is identically zero for all t. For $t = t_0$ the system is assumed to be in equilibrium, with $F^*(t_0) = 0$ and $\dot{F}^*(t_0) = 0$, and since $V(F^*(t_0), \dot{F}^*(t_0)) = 1$, then, the expression $V(F^*(t), \dot{F}^*(t)) = 1$ represents a first integral, or an integral manifold of the nonlinear system (20). This manifold is given by the graph, in the co-ordinate plane $(\dot{F}, F)$, of the implicit function

$$\tfrac{1}{2}(\dot{F}^*(t))^2 + LM\int_{t_0}^{t}[\ddot{x}(\sigma)\cos F^*(\sigma) + (g - \ddot{y}(\sigma))\sin F^*(\sigma)]\dot{F}^*(\sigma)\mathrm{d}\sigma = 0 \tag{22}$$

For any open interval of time $(t_1, t) \subset (T, t)$ in which $\ddot{x} = \ddot{y} = 0$ with $F(t_1) = 0$, one has that $(\frac{1}{2})(\dot{F}^*(t))^2 - LMg\cos F^*(t) = -LMg\cos F^*(t_1)$, i.e. $(\frac{1}{2})(\dot{F}^*(t))^2 = LMg(\cos F^*(t) - \cos F^*(t_1))$. This implies that for any $t > t_1 > T$, $\cos F^*(t) > \cos F^*(t_1)$ as long as there is angular displacement. The argument, being valid for any t_1 implies that the evolution of $F^*(t)$ is such that

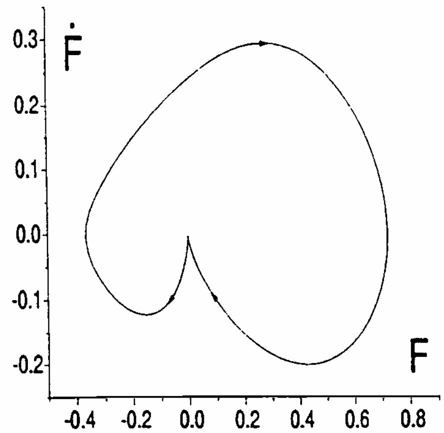

Figure 1. Integral manifold for rest to rest maneuver in the phase plane $(F, \dot{F})$.

it monotonically approaches the value zero. Once $F^*(t) = 0$, for some $t > t_1$, then the angular velocity also becomes zero. We conclude that the time-parameterized integral manifold (22) necessarily departs from the origin and arrives back at the origin of coordinates of the plane $(F, \dot{F})$ for any rest-to-rest maneuver (see Figure 1).

In this work we emphasize the use of *polynomial splines*, also known as *Bezier* polynomials, for the specification of the desired trajectories, $x^*(t)$ and $y^*(t)$, although some other options for trajectory planning are equally possible.

Let $\xi(t, t_{hi}, T_{hf})$ and $\eta(t, t_{vi}, T_{vf})$ be polynomials in t, satisfying the following conditions:

$$\begin{array}{ll} \xi(t_{hi}, t_{hi}, T_{hf}) = 0, & \left\{\begin{array}{l}\eta(t_{vi}, t_{vi}, T_{vf}) = 0 \\ \eta(T_{vf}, t_{vi}, T_{vf}) = 1\end{array}\right. \\ \xi(T_{hf}, t_{hi}, T_{hf}) = 1, & \end{array} \tag{23}$$

where we also demand that a finite number of time derivatives of the polynomials, $\xi(t, t_{hi}, T_{hf})$ and $\eta(t, t_{vi}, T_{vf})$, be equal to zero at the initial and final maneuver times, t_{hi}, t_{vi} and T_{hf}, T_{vf}, respectively. Then, the functions given by $x(t_{hi}) + \xi(t, t_{hi}, T_{hf})(x(t_{hf}) - x(t_{hi}))$ and $y(t_{vi}) + \eta(t, t_{vi}, T_{vf})(y(t_{vf}) - y(t_{vi}))$, suitably interpolate between the initial and final values of the horizontal and vertical displacements. The horizontal and vertical displacement maneuvers may be assumed to occur independently during the finite intervals of time given by $t_{hi} \leqslant t \leqslant T_{hf}$, and $t_{vi} \leqslant t \leqslant T_{vf}$, i.e.

$$x^*(t) = \begin{cases} x(t_{hi}) & \text{for } t < t_{hi} \\ x(t_{hi}) + \xi(t, t_{hi}, T_{hf})(x_{hf} - x_{hi}) & \text{for } t_{hi} \leqslant t \leqslant t_{hf} \\ x(t_{hf}) & \text{for } t > t_{hf} \end{cases} \tag{24}$$

$$y^*(t) = \begin{cases} y(t_{vi}) & \text{for } t < t_{vi} \\ y(t_{vi}) + \eta(t, t_{vi}, T_{vf})(y_{vf} - y_{vi}) & \text{for } t_{vi} \leqslant t \leqslant t_{vf}. \\ y(t_{vf}) & \text{for } t > t_{vf} \end{cases} \tag{25}$$

3.2.2. A trajectory tracking feedback controller. Define the state variable and control input tracking errors

$$\begin{aligned}
x_{1\delta} &= x - x^*(t),\ x_{2\delta} = \dot{x} - \dot{x}^*(t) \\
x_{3\delta} &= y - R^*(t),\ x_{4\delta} = \dot{y} - \dot{R}^*(t) \\
x_{5\delta} &= \theta - F^*(t),\ x_{6\delta} = \dot{\theta} - \dot{F}^*(t) \\
u_{1\delta} &= u_1 - \frac{M}{\cos(F^*(t))}\left(g - \ddot{R}^*(t) + \frac{\ddot{F}^*(t)}{LM}\sin(F^*(t))\right) \\
u_{2\delta} &= u_2 - \frac{\ddot{F}^*(t)}{L}
\end{aligned} \tag{26}$$

The linearized dynamics, around the ideally regulated open-loop trajectories, is given by

$$\begin{aligned}
\dot{x}_{1\delta} &= x_{2\delta} \\
\dot{x}_{2\delta} &= (\ddot{R}^* - g)x_{5\delta} - \frac{1}{M}\sin(F^*)u_{1\delta} - \frac{1}{M}\cos(F^*)u_{2\delta} \\
\dot{x}_{3\delta} &= x_{4\delta} \\
\dot{x}_{4\delta} &= -\ddot{x}^* x_{5\delta} - \frac{1}{M}\cos(F^*)u_{1\delta} + \frac{1}{M}\sin(F^*)u_{2\delta} \\
\dot{x}_{5\delta} &= x_{6\delta} \\
\dot{x}_{6\delta} &= Lu_{2\delta}
\end{aligned} \tag{27}$$

The linearized system is of the form $\dot{x}_\delta = A(t)x_\delta + B(t)u_\delta$. The system is found to be controllable, as the following well-known *Silverman's controllability rank condition*

$$\text{rank}\left[B(t), \left(A(t) - \frac{d}{dt}\right)B(t), \ldots, \left(A(t) - \frac{d}{dt}\right)^5 B(t)\right] = 6 \tag{28}$$

is satisfied.

A linear time-varying state feedback controller for this system, of the 'proportional plus derivative' (PD) type, including time-varying compensation terms, results in the incremental correction inputs

$$\begin{bmatrix} u_{1\delta} \\ u_{2\delta} \end{bmatrix} = \begin{bmatrix} M\sin(F^*) & M\cos(F^*) \\ M\cos(F^*) & -M\sin(F^*) \end{bmatrix} \begin{bmatrix} (\ddot{R}^* - g)x_{5\delta} + k_{xp}x_{1\delta} + k_{xd}x_{2\delta} \\ -\ddot{x}^* x_{5\delta} + k_{yp}x_{3\delta} + k_{yd}x_{4\delta} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{L}(k_{\theta p}x_{5\delta} + k_{\theta d}x_{6\delta}) \end{bmatrix} \tag{29}$$

where k_{xp}, k_{xd}, k_{yp}, k_{yd}, $k_{\theta p}$, $k_{\theta d}$, are stricly positive design constants.

The closed-loop linearized system is given by

$$\begin{aligned}
\dot{x}_{1\delta} &= x_{2\delta} \\
\dot{x}_{2\delta} &= -k_{xp}x_{1\delta} - k_{xd}x_{2\delta} + \frac{1}{LM}\cos(F^*)\,[k_{\theta p}x_{5\delta} + k_{\theta d}x_{6\delta}]
\end{aligned}$$

$$\dot{x}_{3\delta} = x_{4\delta}$$

$$\dot{x}_{4\delta} = -k_{yp}x_{3\delta} - k_{yd}x_{4\delta} - \frac{1}{LM}\sin(F^*)[k_{\theta p}x_{5\delta} + k_{\theta d}x_{6\delta}]$$

$$\dot{x}_{5\delta} = x_{6\delta}$$

$$\dot{x}_{6\delta} = -\left[k_{\theta p} + LM\left((g - \ddot{R}^*)\cos(F^*) - \ddot{x}^*\sin(F^*)\right)\right]x_{5\delta} - k_{\theta d}x_{6\delta}$$

$$+ LM\cos(F^*)\left[k_{xp}x_{1\delta} + k_{xd}x_{2\delta}\right] - LM\sin(F^*)\left[k_{yp}x_{3\delta} + k_{yd}x_{4\delta}\right] \tag{30}$$

It is not difficult to show, in the light of the result given in the appendix, that the closed-loop system (30) may be rendered exponentially stable to zero under the physically meaningful assumptions stated in Section 2.2, and for a set of suitably chosen controller design constants and displacement reference trajectories. Notice, first of all, that the closed-loop time varying linear system (30) is of the 'interconnected' form

$$\frac{d}{dt}\begin{bmatrix} \xi \\ \zeta \end{bmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}\begin{bmatrix} \xi \\ \zeta \end{bmatrix} \tag{31}$$

with $\xi = [x_{1\delta}\ x_{2\delta}\ x_{3\delta}\ x_{4\delta}]^T \in R^4$ and $\zeta = [x_{5\delta}\ x_{6\delta}]^T \in R^2$ and

$$A_{11}(t) = A_{11} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_{xp} & -k_{xd} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -k_{yp} & -k_{yd} \end{bmatrix}$$

$$A_{22}(t) = \begin{bmatrix} 0 & 1 \\ -[k_{\theta p} + LM((g - \ddot{R}^*)\cos(F^*) - \ddot{x}^*\sin(F^*))] & -k_{\theta d} \end{bmatrix} \tag{32}$$

while the 'interconnection' matrices are given by

$$A_{12}(t) = \frac{1}{LM}\begin{bmatrix} 0 & 0 \\ \cos(F^*)k_{\theta p} & \cos(F^*)Sk_{\theta d} \\ 0 & 0 \\ -\sin(F^*)k_{\theta p} & -\sin(F^*)k_{\theta d} \end{bmatrix}$$

$$A_{21}(t) = LM\begin{bmatrix} 0 & 0 & 0 & 0 \\ \cos(F^*)k_{xp} & \cos(F^*)k_{xd} & -\sin(F^*)k_{yp} & -\sin(F^*)k_{yd} \end{bmatrix} \tag{33}$$

It is, first of all, evident that the decoupled matrix A_{11} is (uniformly) exponentially stable for any set of positive design constants k_{xp}, k_{xd}, k_{yp} and k_{yd}. To see that there exists controller design constants and attitude and horizontal displacement trajectories which render the matrix $A_{22}(t)$ (uniformly) exponentially stable, notice the following facts:

(1) Assumption (6) particularised for $\theta = F^*$, $y = R$ and $x = x^*$, and the positivity of the design constants $k_{\theta d}$ and $k_{\theta p}$, imply that the real parts of the pointwise-in-time eigenvalues

of $A_{22}(t)$ are bounded above by a strictly negative constant given by the real part of the expression

$$\frac{-k_{\theta d} + \sqrt{k_{\theta d}^2 - 4(k_{\theta p} + \mu LM)}}{2}$$

(2) The norm of the matrix $A_{22}(t)$ is clearly bounded by a positive constant. This is trivially true for any maneuver entitling *bounded* horizontal and *bounded* vertical accelerations.

(3) According to Equation (5), the quantity $M((g - \ddot{R}^*)\cos(F^*) - \ddot{x}^*\sin(F^*))$ coincides with u_1^* and therefore $A_{22}(t)$ may be rewritten as

$$A_{22}(t) = \begin{bmatrix} 0 & 1 \\ -(k_{\theta p} + Lu_1^*) & -k_{\theta d} \end{bmatrix}$$

Theorem A.1 in the appendix states that if the norm of the time derivative of $A_{22}(t)$ is sufficiently small, then the matrix $A_{22}(t)$ is uniformly exponentially stable. This implies, in compliance with the last fact and Equation (9), that $|L\dot{u}_1^*| < L\varepsilon$ should be a sufficiently small quantity. Generally speaking, this condition is fulfilled for reasonably 'slow' maneuvers, i.e. those which produce suitably bounded magnitude values for the time derivatives of R^* and x^* up to a third order, and corresponding small attitude angular magnitude values for F^*. Our design approach, based on off-line trajectory planning, is quite convenient for complying with this type of requirement.

The *linear* interconnections are indeed time-varying maps which remain *bounded* in the region of operation of the system. The exponential asymptotic stability of A_{11} and $A_{22}(t)$, and the fact that the norms of these matrices are bounded by positive constants imply, in accordance with the Corollary A.3 in the appendix, that there exists a positive constant bounding the norm of the interconnection matrix for which the trajectories of the linear system (30) are exponentially stable from arbitrary initial conditions. It is easy to see that the squared norm of the interconnection matrices is bounded by the sum of the squared norms of the following vectors of design parameters: $LM[k_{xp}\ k_{xd}\ k_{yp}\ k_{yd}]^{\mathrm{T}}$ and $(1/LM)\,[k_{\theta p}\ k_{\theta d}]^{\mathrm{T}}$. Positive design constants, k_{xp}, k_{xd}, k_{yp}, k_{yd}, $k_{\theta p}$, $k_{\theta d}$, can always be found, such that the sum of the squared norms of these vectors is as small as required, without destroying the exponential asymptotic stability of A_{11} and $A_{22}(t)$, which, as it was demonstrated, only demanded strict positivity of the controller design parameters and the physically plausible assumptions of Section 2.2.

Based on the above arguments, one finally has the following full feedback controller:

$$u_1 = \frac{M}{\cos(F^*(t))}\left(g - \ddot{R}^*(t) + \frac{\ddot{F}^*(t)}{LM}\sin(F^*(t))\right) + u_{1\delta}$$

$$u_2 = \frac{\ddot{F}^*(t)}{L} + u_{2\delta} \tag{34}$$

3.3. *The helicopter model as a Liouvillian system: simplified model with constant normal and lateral velocity*

Let us consider the simplified model of the helicopter longitudinal dynamics given by Equations (10). The system is *not* linearizable by means of static state feedback and, hence, according to the results of Charlet *et al.* [2], it is also non-linearizable by dynamic state feedback either. The system is clearly Liouvillian, with the flat subsystem being represented by the kinematic state pair

(θ, q). The following partial *differential parameterization* of the system variables allows for some elementary equilibrium analysis and also establishes the main features of the system:

$$\theta = F, \quad q = \dot{F}, \quad u_2 = \frac{\ddot{F}}{L}$$
$$\ddot{F} = -LM\ddot{x}\cos(F) - LgM\sin(F) \tag{35}$$

Thus, the *zero dynamics*, or remaining dynamics, corresponding to a resting hovering position, characterized by $x =$ constant, $\dot{x} = u_2 = 0$ and $\ddot{x} = \dot{u}_2 = 0$, is given, according to (35), by the dynamics (18). Thus, as expected, the system is *weakly minimum phase* around the origin, with respect to the horizontal position co-ordinate x, taken as a system output.

3.3.1. Off-line trajectory planning. As in the previous case, a desired displacement maneuver is specified as $x^*(t)$ for the horizontal position variable x. The maneuver takes the initial equilibrium hovering position, located at $x(t_{\rm hi})$, towards a final hovering horizontal co-ordinate value, specified as $x(T_{\rm hf})$.

The partial differential flatness property of the variable θ allows one to express the control input u_2 as the quantity $\ddot{F}\ L$. The following nonlinear second-order differential equation is readily obtained for a given displacement trajectory.

$$\ddot{F}^* = -LM[\ddot{x}^*(t)\cos(F^*) + g\sin(F^*)] \tag{36}$$

with initial conditions given by the ideal hovering condition $F^*(t_{\rm hi}) = 0$, $\dot{F}^*(t_{\rm hi}) = 0$. Notice that (36) can be obtained from (20) by letting $\ddot{y}^*(t)$ be identically zero. Hence, the same arguments related to the integral manifold of (20) can be now repeated for the corresponding first integral of (36). The solutions of (36) departing from zero initial conditions and performing a rest-to-rest maneuver for the horizontal displacement, $x^*(t)$, will return the origin of co-ordinates in the plane $(F, \dot{F})$ after the maneuver has been accomplished.

3.3.2. A trajectory tracking feedback controller. Define the state tracking errors as

$$x_{1\delta} = \theta - F^*(t) \quad x_{2\delta} = \dot{\theta} - \dot{F}^*(t), \quad x_{3\delta} = x - x^*(t), \quad x_{4\delta} = \dot{x} - \dot{x}^*(t)$$
$$u_2 = u_{2\delta} + \frac{\ddot{F}^*(t)}{L} \tag{37}$$

The linearized dynamics, under the assumption of small deviations from the planned trajectories, is given by

$$\dot{x}_{1\delta} = x_{2\delta}$$
$$\dot{x}_{2\delta} = Lu_{2\delta}$$
$$\dot{x}_{3\delta} = x_{4\delta}$$
$$\dot{x}_{4\delta} = -[g - \ddot{x}^*\tan(F^*)]x_{1\delta} - \frac{1}{M\cos(F^*)}u_{2\delta} \tag{38}$$

The time-varying linear system (38) is found to be controllable using Silverman's criterion (38). A linear state feedback controller results in the following incremental correction input, which is a sort of 'proportional-derivative' type of feedback controller, with appropriate time varying compensation terms:

$$u_{2\delta} = -M\cos(F^*)[[g - \ddot{x}^*\tan(F^*)]x_{1\delta} - [k_{xp}x_{3\delta} + k_{xd}x_{4\delta}]] - \frac{1}{L}[k_{\theta p}x_{1\delta} + k_{\theta d}x_{2\delta}] \quad (39)$$

where $k_{\theta p}$, $k_{\theta d}$, k_{xp} and k_{xd} are strictly positive design constants. The closed-loop linearized system is given by

$$\dot{x}_{1\delta} = x_{2\delta}$$

$$\dot{x}_{2\delta} = -[k_{\theta p} + LM\cos(F^*(t))(g - \ddot{x}^*(t)\tan(F^*))]x_{1\delta} - k_{\theta d}x_{2\delta} + LM\cos(F^*(t))[k_{xp}x_{3\delta} + k_{xd}x_{4\delta}]$$

$$\dot{x}_{3\delta} = x_{4\delta}$$

$$\dot{x}_{4\delta} = -[k_{xp}x_{3\delta} + k_{xd}x_{4\delta}] + \frac{1}{LM\cos(F^*(t))}[k_{\theta p}x_{1\delta} + k_{\theta d}x_{2\delta}] \quad (40)$$

It is not difficult to show, in the light of the results given in the appendix, that the closed-loop system (40) is exponentially stable under the physically meaningful assumptions given in Section 2.3 which determine the operating region.

A full feedback controller for the helicopter model. based on the above considerations, is thus given by

$$u_2 = \frac{1}{L}[\ddot{F}^*(t)] + u_{2\delta} \quad (41)$$

with $u_{2\delta}$ given by (39).

4. SIMULATION RESULTS

Numerical simulations were carried out in order to evaluate the performance of the controllers designed for each simplified model. These simulations are presented in this section with the following values of the helicopter system parameters:

$$M = 4313\,\text{kg}, \quad g = 9.8\,\text{m/s}^2, \quad L = 1.0456 \times 10^{-4}\,\text{rad/N s}^2$$

4.1. Simplified model with constant lateral velocity

4.1.1 Off-line trajectory planning. For the simplified model with constant lateral velocity (3) we first consider a trajectory planning example. corresponding to the off-line computations represented by Equation (20) for given desired displacements $x^*(t)$ and $y^*(t)$ starting and ending with ideal hovering conditions while requiring a position transfer between two known equilibrium values in the x–y plane. The desired horizontal and vertical displacement maneuvers were also specified as polynomial splines in the manner given in (20) and (21) with

$$x(t_{\text{hi}}) = 100\,\text{m}. \quad x(T_{\text{hi}}) = 300\,\text{m}. \quad t_{\text{hi}} = 20\,\text{s} \quad T_{\text{hf}} = 40\,\text{s}$$

$$y(t_{\text{vi}}) = 30\,\text{m}. \quad y(T_{\text{vi}}) = 200\,\text{m}. \quad t_{\text{vi}} = 20\,\text{s} \quad T_{\text{vf}} = 40\,\text{s}$$

For simplicity, we used the same polynomial function for both the horizontal and vertical displacement trajectories and assumed they occurred during the same transfer intervals. For a transfer in the interval. $[t_i, T_f]$, we considered a polynomial of the form

$$\eta(t, t_i, T_f) = \left[\frac{t-t_i}{T_f-t_i}\right]^5 \left\{ r_1 - r_2 \frac{t-t_i}{T_f-t_i} + r_3\left(\frac{t-t_i}{T_f-t_i}\right)^2 - r_4\left(\frac{t-t_i}{T_f-t_i}\right)^3 + r_5\left(\frac{t-t_i}{T_f-t_i}\right)^4 - r_6\left(\frac{t-t_i}{T_f-t_i}\right)^5 \right\} \quad (42)$$

with

$$r_1 = 252, \quad r_2 = 1050, \quad r_3 = 1800, \quad r_4 = 1575, \quad r_5 = 700, \quad r_6 = 126$$

Figure 2 shows the nominal horizontal and vertical displacements $x^*(t)$, $y^*(t)$ and the corresponding off-line computed attitude angle trajectory $\theta^*(t) = F^*(t)$. The figure also shows the nominal control inputs $u_1^*(t)$ and $u_2^*(t)$.

4.1.2. Closed-loop feedback controller performance. A typical combination of desired horizontal and vertical displacement maneuvers, including initial states perturbation errors was used to verify the performance of controller (24). Figure (3) depicts the performance of the closed-loop system for the attitude angular displacement as well as the horizontal and vertical displacements corresponding to prescribed trajectories of the form (20), (21) with $t_{hi} = 20$ s, $T_{hf} = 40$ s and $t_{vi} = 20$ s and $T_{vf} = 40$ s. The simulation included initial discrepancies from the ideal hovering conditions. The feedback controller (24) is shown to effectively correct all initial discrepancies and

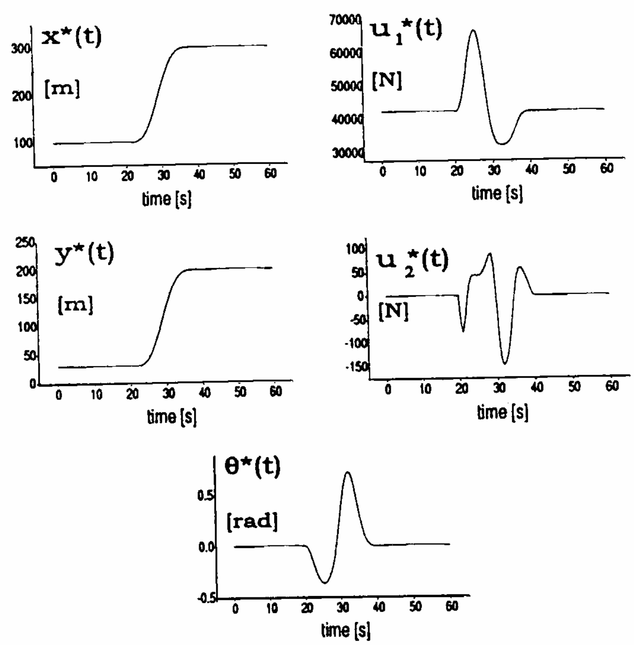

Figure 2. Off-line computed nominal state trajectories and nominal control inputs; simplified model with constant lateral velocity.

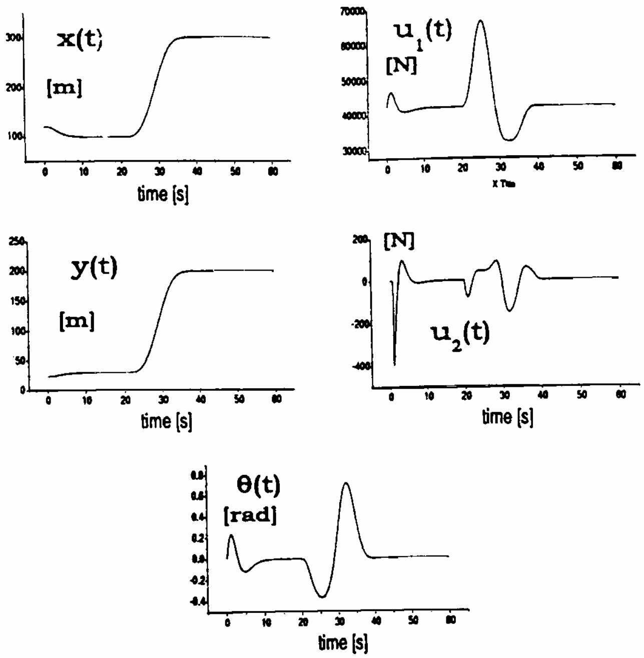

Figure 3. Closed-loop response in a hovering point-to-hovering point maneuver with initial state setting errors; simplified model with constant lateral velocity.

manages to achieve satisfactory tracking of, both, the computed attitude reference trajectory $F^*(t)$ and the originally given horizontal and vertical positions trajectories $x^*(t)$, $y^*(t)$.

4.2. *Simplified model with constant normal and lateral velocity*

4.2.1. Off-line trajectory planning behaviour. For the simplified model with constant normal and lateral velocity, a typical horizontal displacement maneuver was specified by a trajectory of the form (20) with an interpolating polynomial also given by (42). The initial and terminal times, as well as the initial and terminal points on the horizontal line co-ordinate were taken to be

$$x(t_{\mathrm{hi}}) = 100\,\mathrm{m}, \quad x(T_{\mathrm{hf}}) = 300\,\mathrm{m}, \quad t_{\mathrm{hi}} = 20\,\mathrm{s}, \quad T_{\mathrm{hf}} = 40\,\mathrm{s}$$

The off-line computed reference attitude angle trajectory $F^*(t)$ is obtained from the solution of the differential equation (36) with the given $x^*(t)$ and initial conditions chosen to exactly coincide with the ideal hovering conditions. The corresponding open-loop control input u_2^*, is given, according to the partial flatness property of the kinematic subsystem, by

$$u_2^* = \ddot{F}^*/L$$

Figure 4 shows the off-line computed nominal state and input trajectories.

4.2.2. Closed-loop feedback controller performance. The performance of the proposed controller (36), (39) and (41) was tested in a typical desired horizontal displacement maneuver, including

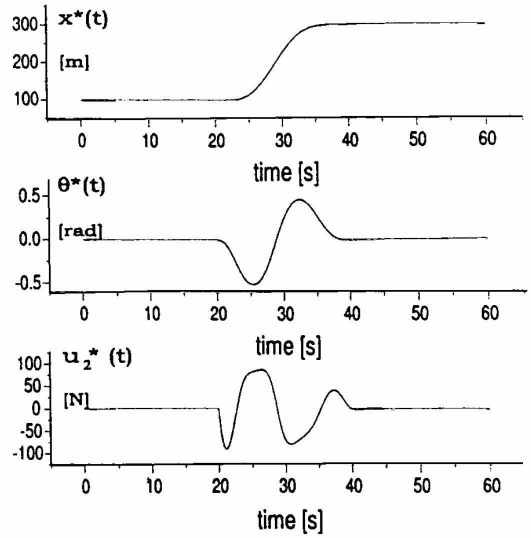

Figure 4. Off-line computed nominal state trajectories and nominal control inputs: simplified model with constant normal and lateral velocity.

initial states perturbation errors. Figure 5 depicts the performance of the closed-loop system for the attitude angular displacement and the horizontal displacement variables corresponding to a prescribed trajectory of the same form as in the previous section. The feedback controller (41) manages to effectively correct all initial discrepancies and achieve satisfactory tracking of, both, the computed attitude reference trajectory $F^*(t)$ and the originally given horizontal position maneuver $x^*(t)$. Tracking is achieved with insignificant discrepancies.

5. CONCLUSIONS

In this article we have proposed a linear time-varying state feedback controller complementing a nonlinear off-line (i.e. open-loop) nominal control input ideally solving a trajectory tracking task for simplified, underactuated, models of a helicopter system. The approach is based on exploiting the fact that the simplified models belongs to the class of 'Liouvillian' systems, which generalizes the class of differentially flat systems. This last property allows for an off-line trajectory planning of a chosen subsystem 'flat' output in terms of the required position displacement trajectories. Given such desired displacement trajectories, the corresponding attitude angle trajectory and the required control inputs are off-line computed using the partial differential flatness of the model. The ideal open-loop control is then completed with a linearization-based static, though time-varying, state feedback controller providing the required robustness to the open-loop control scheme with respect to initial setting errors. The proposed static feedback controller has been tested through computer simulations, with encouraging results.

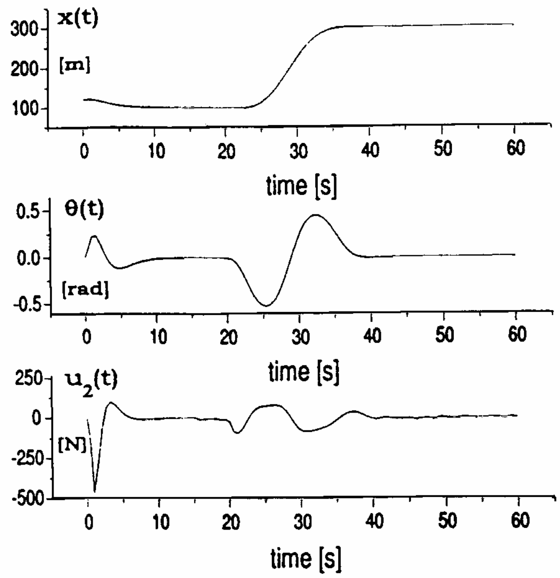

Figure 5. Closed-loop response in a hovering point-to-hovering point maneuver with initial state setting errors; simplified model with constant normal and lateral velocity.

A more complete study may be pursued using the full nonlinear model of the helicopter derived in Reference [14] and presented in Section 2. Also, the approach proposed in this article is useful for handling regulation tasks defined on several non-differentially flat systems examples, which are still 'Liouvillian'.

APPENDIX

Theorem A.1 (Rugh [18, pp. 135–138])

Suppose that for the linear time-varying system $\dot{x}(t) = A(t)\,x(t)$, the matrix $A(t)$ is continuously differentiable and there exist finite positive constants α and γ, such that for all t, $\|A(t)\| \leqslant \alpha$ and every pointwise eigenvalue of $A(t)$ satisfies $\mathrm{Re}[\lambda(t)] \leqslant -\gamma$. Then there exists a positive constant β such that if the time derivative of $A(t)$ satisfies $\|\dot{A}(t)\| \leqslant \beta$ for all t, the state equation is uniformly exponentially stable.

Theorem A.2 ([15, pp. 133–134]), and also Cellier and Desoer [19, pp. 190–192]).

Suppose the linear time-varying system $\dot{x}(t) = A(t)x(t)$ is uniformly exponentially stable and there exists a finite constant α such that $\|A(t)\| \leqslant \alpha$ for all t. Then, there exists a positive constant β such that the linear state equation

$$\dot{z}(t) = [A(t) + B(t)]\,z$$

is uniformly exponentially stable if $\|B(t)\| \leqslant \beta$ for all t.

The following is an immediate corollary of the above theorems.

Corollary A.3

Consider the unforced linear time-varying interconnected system defined in $R^{n_1+n_2}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \zeta\end{bmatrix}=\begin{bmatrix}A_{11}(t) & 0\\ 0 & A_{22}(t)\end{bmatrix}\begin{bmatrix}\xi\\ \zeta\end{bmatrix}+\begin{bmatrix}0 & A_{12}(t)\\ A_{21}(t) & 0\end{bmatrix}\begin{bmatrix}\xi\\ \zeta\end{bmatrix} \tag{A1}$$

with $\xi \in R^{n_1}$ and $\zeta \in R^{n_2}$. Let $A_{11}(t)$ and $A_{22}(t)$ be uniformly exponentially stable matrices whose norms are bounded by constant scalars. Then there exists a positive constant ρ such that the interconnected system (A1) is uniformly exponentially stable if $\|A_{12}(t)\| + \|A_{21}(t)\| \leqslant \rho$ for all t.

ACKNOWLEDGEMENTS

This research was supported by the Centro de Investigación y de Estudios Avanzados of the Instituto Politécnico Nacional (CINVESTAV-IPN) of México, by the Consejo Nacional de Ciencia y Technología (CONACYT) of México under Research Grant Number 32681-A, and by the Department of Mathematics of Glasgow Caledonian University, Glasgow, Scotland, U.K.

REFERENCES

1. Liceaga-Castro E, Bradley R, Castro-Linares R. Helicopter control design using feedback linearization techniques In *Proceedings of the 28th IEEE Conference on Decision and Control*, Tampa, FL, U.S.A., December 1989; 533–534.
2. Fliess M. Generalized controller canonical forms for linear and nonlinear dynamics. *IEEE Transactions on Automatic Control* 1990; AC-35:994–1001.
3. Sira-Ramirez H, Zribi M, Ahmad S. Dynamic sliding mode control approach fora vertical flight regulation in helicopters. *IEE Proceedings, Control Theory and Applications* 1994; **141**(1):19–24.
4. van Nieuwstadt M, Murray RM. Outer flatness: trajectory generation for a model helicopter. *Proceedings of the* 1997 *European Control Conference, ECC'97*, (CD-Rom version). Brussels. Belgium. July 1997.
5. Fliess M, Lévine J, Martin Ph, Rouchon P. Sur les systèmes nonlinéaires différentiellement plats 1992; I-315: 619–624.
6. Fliess M, Lévine J, Martin Ph, Rouchon P. Flatness and defect of nonlinear systems: introductory theory and examples. *International Journal of Control* 1995; **61**:1327–1361.
7. Fliess M, Lévine J, Martin Ph, Rouchon P. A Lie-Bäcklund approach to dynamic feedback equivalence and flatness. In *Lecture Notes in Control and Information Sciences*, Vol. 247, pp. 247–268, Garofalo F, Glielmo L. (eds.). Springer: London, 1996.
8. Thomson DG, Bradley R. Prediction of the dynamic characteristics of helicopters in Constrained flight. *Aeronautica Journal of The Aeronautical Royal Society*, December 1990.
9. Thomson DG, Bradley R. Recent developments in the calculation of inverse dynamic solutions of the helicopter equations of motion. *Proceedings of the U.K. Simulation Council Triennial Conference*, September 1987.
10. Mullhaupt P, Srinivasan B, Levine J, Bonvin D. A toy more difficult to control than the real thing. In *Proceedings* 1997 *European Control Conference, ECC'97*, (CD-Rom version). Brussels. Belgium, July 1997.
11. Chelouah A. Extensions of differential flat fields and Liouvillian systems. *Proceedings of the 36th Conference on Decision and Control*. San Diego, CA, U.S.A., December 1997; 4268–4273.
12. Martin Ph, Devasia S, Paden B. A different look at output tracking of a VTOL aircraft. *Automatica* [illegible]; **32**(1): 101–107.
13. Liceaga-Castro E. Dynamic analysis of an elementary rotor-craft: Part I. *Technical Report PR/MAT/*[illegible] *96-52*, Department of Mathematics, Glasgow Caledonian University, Scotland, 1996.
14. Bradley R. The flying brick exposed: nonlinear control of a basic helicopter model. *Technical Report TR/*[illegible] *RB/6*, Department of Mathematics, Glasgow Caledonian University, Scotland, 1996.

15. Chelouah A. Petitot M. Finitely discretizable nonlinear systems: concepts and definitions. *Proceedings of the 34th IEEE Conference on Decision and Control*, New Orleans. LA. U.S.A.. December 1995; 19–24.
16. Fliess M, Sira-Ramirez H, and Márquez-Contreras R. On the regulation of non-minimum phase outputs: a flatness based approach. In *Perspectives in Control* (Normand-Cyrot D. ed.). *Communications and Control* Springer: London, 1998.
17. Charlet B. Levine J, Marino R. Sufficient conditions for dynamic state feedback linearization *SIAM Journal of Control and Optimization* 1991; **29**:38–57.
18. Rugh W. *Linear System Theory* (2nd Edn.), Prentice-Hall: Upper Saddle River. NJ. 1996.
19. Cellier FM. Desoer ChA. *Linear System Theory*, Spinger Texts in Electrical Engineering, Springer: Berlin, 1991.

