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Sliding Mode Rest-to-Rest Stabilization and Trajectory Tracking for a Discretized Flexible Joint Manipulator

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Abstract. In this article, a *difference flatness* approach is used for trajectory tracking tasks of an approximately (Euler) discretized model of a nonlinear, single link, flexible joint manipulator. The system's *flat output* is commanded to follow a prescribed trajectory achieving a desired angular position maneuver. A new robust discrete time feedback controller design technique, of the sliding mode type, is then proposed for the closed loop regulation of the link position around the prescribed trajectory. The effectiveness of the approach is illustrated by means of digital computer simulations in a rest-to-rest stabilization maneuver and in a sinusoidal reference trajectory tracking task.

Keywords: difference flatness, flexible joint robot, trajectory planning

1. Introduction

Flexible joint manipulators constitute a step closer to the modeling of realistic robotic systems than that achieved by the rigid joint model. Elastic, or flexible, joint manipulators have been extensively studied in the past by many researchers. We refer the reader to the works of Book [3], Spong [10] and the recent book chapter by De Luca [4], for historical and technical details of this area. Most of the contributions in theoretical studies, as well as in many experimental implementations, include continuous-time models, based on nonlinear ordinary differential equations (see, for example [9]). Controller design, in these instances, has been greatly facilitated by the fact that most of treated models are *exactly linearizable* and, hence, *differentially flat*. This fact holds, in an extended sense, even for the infinite dimensional models (see for instance, Fliess et al. [5]). In spite of being of crucial importance in the experimental implementation of designed nonlinear controllers, no articles, within our knowledge, deal with either exact or, approximately, discretized models of flexible joint manipulators. Exploitation of the associated *difference flatness* has remained open for contributions, so far.

Difference flatness for discrete time nonlinear systems is a concept that directly stems from the concept of *differential flatness*, introduced, within the context of continuous nonlinear controlled systems by Prof. Michel Fliess and his colleagues in a series of articles (Fliess et al. [6]–[8]). The fundamental property of flatness, in discrete time nonlinear systems, is analogous to its continuous time counterpart: It allows for a complete *difference parametrization* of all system variables, including the inputs, in terms of a special set of independent variables, called the *flat outputs*, exhibiting the same cardinality as the set of control inputs, which are *difference functions* of the state, i.e. they are functions of the state and of a finite number of advances of the state. The relations between difference flatness, forward accessibility and static and dynamic feedback linearization have been explored in the works of Aranda-Bricaire et al. [1]–[2] from an algebraic viewpoint.

Here, we are particularly interested in exploiting the difference flatness associated with the discretized single-link flexible joint manipulator. A reference trajectory tracking controller, based on flatness and sliding mode control, is proposed which equally allows for a rest-to-rest stabilization maneuver, or, if desired, a large amplitude sinusoidal reference tracking task for the manipulator's angular position.

Section 2 provides some basic definitions and fundamental generalities about the class of difference flat systems. The formulation, as presented, follows directly from the continuous time counterpart. Section 3 presents the discretized model of the single-link flexible joint manipulator and obtains the difference parametrization of all system variables in terms of the flat output, constituted by the link angular position. We proceed to specify two types of desired trajectories for the flat output. The first one results in a rest-to-rest maneuver covering a large angular displacement, devoid of oscillations. The second trajectory entitles the tracking of a periodic signal, for the link position, of monotonically increasing amplitude until it reaches a prescribed steady state sinusoidal oscillation of fixed amplitude. A new sliding mode based feedback control strategy is then proposed to have the system accurately track the proposed off-line planned trajectory and exhibit a certain degree of robustness with respect to initial and un-modeled perturbations causing temporary deviations from the nominal trajectory. Section 4 presents the simulation results evaluating the performance of the proposed sliding mode plus flatness control scheme. Section 5 is devoted to the conclusions and suggestions for further research in this field. An appendix, at the end of the article, collects the basic background results on a new sliding surface nonlinear dynamics paradigm which guarantees finite time reachability of the sliding surface and exhibits robustness with respect to bounded perturbations.

2. Difference Flat Systems

Let us consider a discrete-time single-input single-output nonlinear dynamic system described by

$$
\begin{aligned}
x(k+1) &= \phi(x(k), u(k)) \\
y(k) &= h(x(k))
\end{aligned}
\tag{1}
$$

where $x(\cdot) \in R^n$ is the state, $u(\cdot) \in R$ is the input and $y(\cdot) \in R$ is the output. The functions $\phi\colon R^n \times R \to R^n$ and $h\colon R^n \to R$, are assumed to be analytic functions on their domains.

In order to simplify the notation, we denote that value of the variable D at stage k by D_k, that is $D_k = D(k)$. We now introduce the following definition.

Definition 1. A scalar function ρ: $R^{L+1} \rightarrow R$ is a *difference function* of a state vector x_k if ρ can be expressed as $\rho_k = \rho(x_k, x_{k+1}, \ldots, x_{k+L})$, for some finite integer L, i.e., ρ is a function of x_k and a finite number of *advances* of x_k.

From the system equations (1) it follows that the value at time k of a scalar quantity, which is a difference function of the state, necessarily involves a finite number of advances of the control input u_k. Following the continuous time case, introduced by Prof. M. Fliess and his colleagues [6]–[8], we propose the following definition.

Definition 2. System (1) is *difference flat* if there exists, for some finite integer M, a scalar difference function of x_k denoted by $F_k = \mu(x_k, u_k, \ldots, u_{k+M})$, called the *flat output*, such that the following relations are valid for all k and for some finite integer J

$$\begin{aligned} x_k &= \psi(F_k, F_{k+1}, \ldots, F_{k+J}) \\ y_k &= (h \circ \psi)(F_k, F_{k+1}, \ldots, F_{k+J}) \\ u_k &= \vartheta(F_k, F_{k+1}, \ldots, F_{k+J+1}) \end{aligned} \tag{2}$$

We refer to relations of the form (2) as *difference parametrizations* of the state, output and control input variables, in terms of the flat output F. The difference parametrization (2), of the variables of the system (1) in terms of a flat output F, contains all the structural information about the system, such as the minimum or non-minimum phase character of the output and state variables, the detectability of the system as well as some other useful information including a convenient *static* parametrization of the systems equilibria in terms of the flat output equilibrium values.

Example: The nonlinear system

$$x_{1,k+1} = x_{1,k} + x_{2,k} + \theta x_{1,k}^2, \quad x_{2,k+1} = x_{2,k} + u_k$$

where θ is a constant parameter, is difference flat, with flat output $F_k = x_{1,k}$. The difference parametrization (2) is readily obtained as

$$x_{2,k} = F_{k+1} - F_k - \theta F_k^2, \quad u_k = F_{k+2} - 2F_{k+1} + F_k - \theta \left(F_{k+1}^2 - F_k^2\right)$$

On the other hand, notice that the system

$$x_{1,k+1} = x_{1,k} + \theta x_{1,k}^2, \quad x_{2,k+1} = x_{2,k} + x_{3,k}, \quad x_{3,k+1} = x_{3,k} + u_k$$

is *not* difference flat. To demonstrate this assertion, notice that the trajectories of the state variable, x_1, solely depend upon its initial conditions. Suppose, contrary to what we want to establish, that a difference parametrization, of the form (2) *does* exists for the system state and the input variables. Chose an arbitrary input trajectory, say u_k^* $k = 0, 1, \ldots$ and compute, on the basis of the difference parametrization of the control input. the solution F_k^*, for all k's, of the implicit difference equation $u_k^* = \vartheta(F_k, F_{k+1}, \ldots F_{k+J+1})$. This can be done for any arbitrary set of initial conditions for the flat output F and a number of its advances. i.e., for $F_0^*, F_1^*, \ldots F_J^*$. Use the generated flat output trajectory F_k^*, $k = 0, 1, 2, \ldots$, in the difference parameterization for x_1, given by, say, $\psi_1(F_k, F_{k+1}, \ldots, F_{k+J})$, and evaluate the corresponding $x_{1,k}^*$, $k = 0, 1, \ldots$. It is clear that for each possible specified trajectory u_k^*, $k = 0, 1, 2, \ldots$, and each set of possible initial conditions for F^* and its advances, a new corresponding trajectory, $x_{1,k}^*$, is generated for x_1. This means that somehow, the control input variable u_k has a certain influence on the behavior of $x_{1,k}$. This is a contradiction to the fact that. clearly, $x_{1,k}$ only depends upon its initial conditions. Such an hypothesized difference parametrization cannot exist and the system is not difference flat.

Nevertheless, the examined system contains a largest flat subsystem represented by the variables x_2 and x_3, with flat output. $F_k = x_{2,k}$ and difference parametrization given by

$$x_{2,k} = F_k, \quad x_{3,k} = F_{k+1} - F_k, \quad u_k = F_{k+2} - 2F_{k+1} + F_k$$

The non-flat variable x_1 is usually called the *defect* (see [7]).

It follows from (2), that complete knowledge of a flat output trajectory $\{F_k^*\}$, $k = 0, 1, \ldots$ immediately determines the corresponding state, output and control input trajectories $\{x_k^*\}$. $\{y_k^*\}$, and $\{u_k^*\}$, for all times k.

Remark. Notice that, in the difference parametrization (2), the integer J, specifying the number of advances of F_k which are needed to determine the state vector x_k, cannot be strictly smaller than $n - 1$. For, suppose that only F_k and, say, the first $p - 1$ advances of F_k suffice for the difference parametrization (2) to hold valid, with $p < n$. Then, letting $z_k = (z_{1,k}, \ldots, z_{p,k}) = (F_k, F_{k+1}, \ldots, F_{k+p-1})$, it follows that, upon elimination of the p variables z in the equation $x_k = \psi(z_k)$, there are $n - p$ state variables which are locally uniquely determined by the rest of the state variables. This contradicts the fundamental algebraic independece among the state variables.

Let $z_k = (z_{1,k}, \ldots, z_{n,k})$ be defined as an n-dimensional vector $z_k = (F_k, \ldots, F_{k+n-1})$. Then, if $\psi(z)$ is a locally invertible map, it can be regarded as a locally invertible state coordinate transformation linking x_k and z_k. Thus, given x_k, the flat output and its advances, comprised in z_k, would be completely determined. The relation $u_k = \vartheta(F_k, F_{k+1}, \ldots, F_{k+n-1}, F_{k+n}) = \vartheta(z_k, F_{k+n})$ allows. in principle, to determine F_{k+n} if and only if u_k is specified. Thus, $F_{k+n} = z_{n,k+1}$ plays the role of an independent control input of a similar nature than u_k. Let $v_k = F_{k+n}$ denote the local solution of $u_k = \vartheta(z_k, v_k)$ for a given u_k. i.e., let $v_k = z_{n,k+1} = \theta(z_k, u_k)$. The local state representation of z_k in terms of the new input v_k

is thus given by

$$\begin{aligned} z_{1,k+1} &= z_{2,k} \\ z_{2,k+1} &= z_{3,k} \\ &\vdots \\ z_{n-1,k+1} &= z_{n,k} \\ z_{n,k+1} &= v_k \\ F_k &= z_{1,k} \end{aligned} \tag{3}$$

Under all the previous assumptions, the original system (1) is therefore locally equivalent, under state coordinate transformation $x = \psi(z)$ and static state feedback $u_k = \vartheta(z_k, v_k)$, to a linear controllable system of the form $z_{1,k+n} = F_{k+n} = v_k$.

3. The Flexible Joint Robot

Consider the following Euler-discretization model of a flexible joint robot, shown in Figure 1. [10]

$$\begin{aligned} x_{1,k+1} &= x_{1,k} + Tx_{2,k} \\ x_{2,k+1} &= x_{2,k} + \frac{mgLT}{I}\sin(x_{1,k}) - \frac{K_aT}{I}(x_{1,k} - x_{3,k}) \\ x_{3,k+1} &= x_{3,k} + Tx_{4,k} \\ x_{4,k+1} &= x_{4,k} + \frac{K_aT}{J}(x_{1,k} - x_{3,k}) + \frac{T}{J}u_k \end{aligned} \tag{4}$$

where x_1 is the link angular position, x_2 is the link angular velocity, x_3 is the motor axis angular position and x_4 is the motor axis angular velocity. The control input u represents the motor applied torque while the fixed parameter T is the duration of the sampling interval. I is the inertia of the link, J denotes the motor inertia, mgL is the nominal lad in the link and K_a is the flexible joint stiffness coefficient. The model (4) does not take into account the motor viscous friction nor the inertia of the actuator about the three independent axes. However, it has been shown that it represents the manipulator dynamics and also it is suitable for control design [10].

3.1. *Difference Flatness of the Flexible Joint Manipulator*

The system is difference flat, with flat output given by the link angular position x_1. This means, in particular, that all system variables, including the input u, are expressible as difference functions of x_1. The system equations (4) lead to the following *difference*

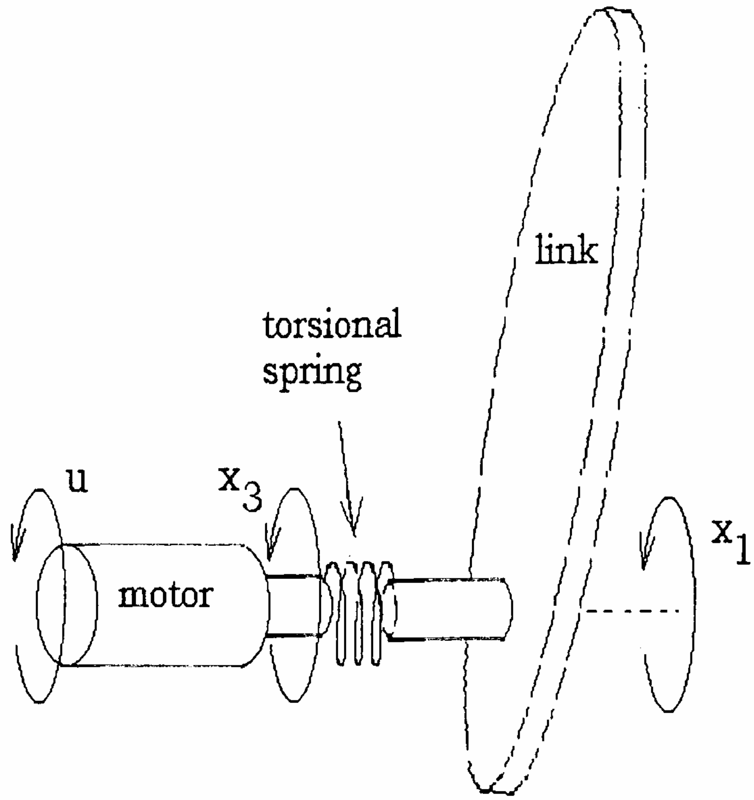

Figure 1. Single link flexible joint manipulator.

parametrization

$$
\begin{aligned}
x_{1,k} &= x_{1,k} \\
x_{2,k} &= \frac{x_{1,k+1} - x_{1,k}}{T} \\
x_{3,k} &= x_{1,k} + \frac{I}{K_a}\left(\frac{x_{1,k+2} - 2x_{1,k+1} + x_{1,k}}{T^2}\right) - \frac{mgL}{K_a}\sin(x_{1,k}) \\
x_{4,k} &= \frac{\left(x_{1,k+1} - x_{1,k}\right)}{T} + \frac{I}{K_a}\left(\frac{x_{1,k+3} - 3x_{1,k+2} + 3x_{1,k+1} - x_{1,k}}{T^3}\right) \\
&\quad - \frac{mgL}{K_a T}\left(\sin(x_{1,k+1}) - \sin(x_{1,k})\right) \\
u_k &= (J + I)\left(\frac{x_{1,k+2} - 2x_{1,k+1} + x_{1,k}}{T^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{JI}{K_a}\left(\frac{x_{1,k-4} - 4x_{1,k+3} + 6x_{1,k+2} - 4x_{1,k-1} + x_{1,k}}{T^4}\right) \\
&- \frac{J}{K_a}\frac{mgL}{T^2}\left(\sin(x_{1,k+2}) - 2\sin(x_{1,k+1}) + \sin(x_{1,k})\right) \\
&- mgL\sin(x_{1,k}) \qquad (5)
\end{aligned}
$$

The last equation immediately suggests the following state-dependent input coordinate transformation

$$
\begin{aligned}
u_k \;=\; &(J+I)\left(\frac{x_{1,k-2} - 2x_{1,k+1} + x_{1,k}}{T^2}\right) \\
&+ \frac{JI}{K_a}\left(\frac{v_k - 4x_{1,k+3} + 6x_{1,k+2} - 4x_{1,k+1} + x_{1,k}}{T^4}\right) \\
&- \frac{J}{K_a}\frac{mgL}{T^2}\left(\sin(x_{1,k+2}) - 2\sin(x_{1,k+1}) + \sin(x_{1,k})\right) - mgL\sin x_{1,k} \qquad (6)
\end{aligned}
$$

where v_k represents the new, or transformed, input coordinate. Thus, the system is seen to be equivalent, after state feedback and an input coordinate transformation, to the *linear* system

$$x_{1,k+4} = v_k \qquad (7)$$

3.2. *Off-line Trajectory Planning*

3.2.1. *A Rest-to-Rest Stabilization via Trajectory Tracking*

Suppose it is desired to bring the link angular position variable x_1 from an initial equilibrium value, $\overline{x}_1^{initial}$, at time $k = K_1$, towards a final equilibrium position, $\overline{x}_1^{final}$, at time $k = K_2$, along a prescribed path $x^*_{1,k}$ satisfying the initial and final conditions. We prescribe such a desired trajectory as

$$x^*_{1,k} = \overline{x}_1^{initial} + \left(\overline{x}_1^{final} - \overline{x}_1^{initial}\right)\varphi(k, K_1, K_2) \qquad (8)$$

with $\varphi(K_1, K_1, K_2) = 0$ and $\varphi(K_2, K_1, K_2) = 1$. Specifically, we choose, as an interpolating polynomial, a *Bezier polynomial* in discrete time. The expression

$$
\begin{aligned}
\varphi(k, K_1, K_2) \;=\; &\left(\frac{k-K_1}{K_2-K_1}\right)^5 \Bigg[r_1 - r_2\left(\frac{k-K_1}{K_2-K_1}\right) \\
&+ r_3\left(\frac{k-K_1}{K_2-K_1}\right)^2 - r_4\left(\frac{k-K_1}{K_2-K_1}\right)^3 \\
&+ r_5\left(\frac{k-K_1}{K_2-K_1}\right)^4 - r_6\left(\frac{k-K_1}{K_2-K_1}\right)^5\Bigg] \qquad (9)
\end{aligned}
$$

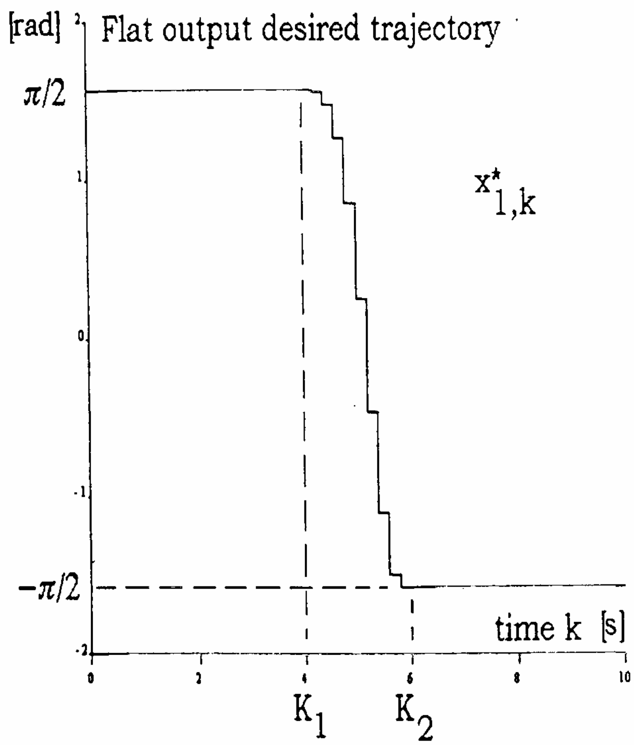

Figure 2. Rest-to-rest planned maneuver for link angular position (flat output).

with

$$r_1 = 252, \quad r_2 = 1050, \quad r_3 = 1800, \quad r_4 = 1575, \quad r_5 = 700, \quad r_6 = 126$$

defines a rather "smooth" interpolation between the inital value of zero, at time $k = K_1$, and the final value of 1, at time $k = K_2$.

Figure 2 shows the shape of the prescribed angular trajectory, $x^*_{1,k}$, for the flat output x_1. In this particular instance, the initial value of the angular position was taken to be $x_{1,K_1} = \pi/2$ and the final value of the angular position was taken to be $x_{1,K_2} = -\pi/2$, with $K_1 = 4$ s and $K_2 = 6$ s.

The difference parametrization (5) allows to off-line obtain the behavior of all the state variables and of the control input once the trajectory for the flat output x_1 has been prescribed as $x^*_{1,k}$. This aspect represents an important design asset of flatness, specially when hard limitations, of the saturation type, are imposed on the control input amplitudes. The off-line prescribed trajectory can be suitably modified in order to satisfy the control input restrictions.

Simulations were performed to obtain the open loop behavior of the system state variables and control inputs in accordance with the off-line planned trajectory $x^*_{1,k}$, as given by (8), (9). A flexible joint manipulator model with the following parameter values was used for the simulations:

$$m = 0.4 \text{ Kg}, \quad g = 9.81 \text{ m/s}^2, \quad L = 0.185 \text{ m}, \quad J = 0.002 \text{ N} - \text{ms}^2/\text{rad},$$

$$I = 0.0059 \text{N} - \text{ms}^2/\text{rad}, \quad K_a = 1.61 \text{ N} - \text{m} - \text{s/rad}$$

The proposed maneuver entitled starting the motions at time $K_1 = 4$, from an equilibrium position located at $x_{1,K_1} = \pi/2$, with no initial velocity, and to perform a rotation of the link, during a time interval of only 2 s, towards a final position given by $x_{1,K_2} = -\pi/2$, arriving at the new position also with zero angular velocity. In order to properly initialize the states of the manipulator, the initial angular position for the motor axis, corresponding to the link equilibrium, was computed from the equilibrium condition

$$x^*_{3,K_1} = x^*_{1,K_1} - \frac{mgL}{K_a} \sin(x^*_{1,K_1}) \tag{10}$$

This value turned out to be $x_{3,K_1} = 1.12$ rad. The final resting equilibrium position for the motor axis, at time $k = K_2$, can be similarly computed. This yields, $x_{3,K_2} = -1.12$ rad. Figure 3 depicts the nominal (open loop) trajectories of the state variables and the control input variables behavior for the given planned angular position maneuver on the described flexible joint manipulator.

A Reference Trajectory Tracking Task

Suppose now, it is desired to make the link angular position variable x_1 to undergo a sinusoidal oscillatory maneuver which departs, with zero amplitude, from an initial equilibrium value for the link position, $\overline{x}_1^{initial}$, at time $k = K_1$. The amplitude of the sinusoidal reference signal will steadily increase until time $k = K_2$. From time $k = K_2$ onwards, the desired trajectory is a sinusoidal signal of constant amplitude M. We prescribe such a desired trajectory as

$$x^*_{1,k} = \begin{cases} \overline{x}_1^{initial} & \text{for} \quad k \leq K_1 \\ \overline{x}_1^{initial} + M\varphi(k, K_1, K_2)\sin(\omega(k - K_1)) & \text{for} \quad K_1 < k \leq K_2 \\ \overline{x}_1^{initial} + M\sin(\omega(k - K_1)) & \text{for} \quad k \geq K_2 \end{cases} \tag{11}$$

with $\varphi(k, K_1, K_2)$ given by (9).

Figure 4 shows the shape of the prescribed reference angular trajectory, $x^*_{1,k}$, for the flat output x_1. The initial value of the angular position was taken to be $x_{1,K_1} = \pi/2$, as in the rest-to-rest maneuver, while the sinusoidal motion was allowed to reach a maximum amplitude value of $M = 2$ rad. The transient phase of the tracking maneuver was set to take place during the interval given by $K_1 = 2.5$ s and $K_2 = 8.5$ s, with a total number l of complete oscillations. During this time interval we have chosen $l = 4$. Thus, $\omega = 2\pi l/(K_2 - K_1) = 4.188$ rad/s.

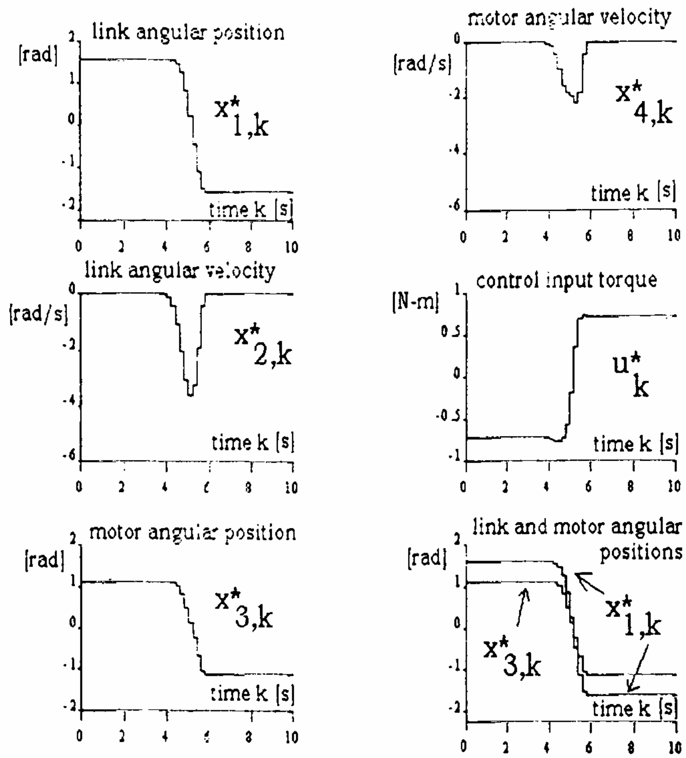

Figure 3. Nominal state and input trajectories for rest-to-rest angular maneuver.

As in the previous case, the difference parametrization (5) allows to off-line obtain the nominal behavior of all the state variables, and of the control input, for the proposed reference trajectory for the flat output x_1. Figure 5 depicts the nominal (open loop) trajectories of the flexible link state and control input variables, corresponding to the given sinusoidal reference angular motion.

3.3. *A Sliding Mode Based Feedback Controller Design*

In this section we propose a feedback, sliding mode-based, control scheme for the on-line regulation of the flexible joint manipulator model. The proposed controller processes the off-line prescribed information about the planned trajectory, $x^*_{1,k}$, in order to conform a sliding surface coordinate function, denoted by σ_k, and to directly compute the required feedback on the basis of the actual deviations of the closed loop flat output trajectory with respect to the off-line planned evolution. The reader is referred to the Appendix for the basic background related to a nonlinear discrete-time sliding surface dynamics paradigm, used at leisure within this section.

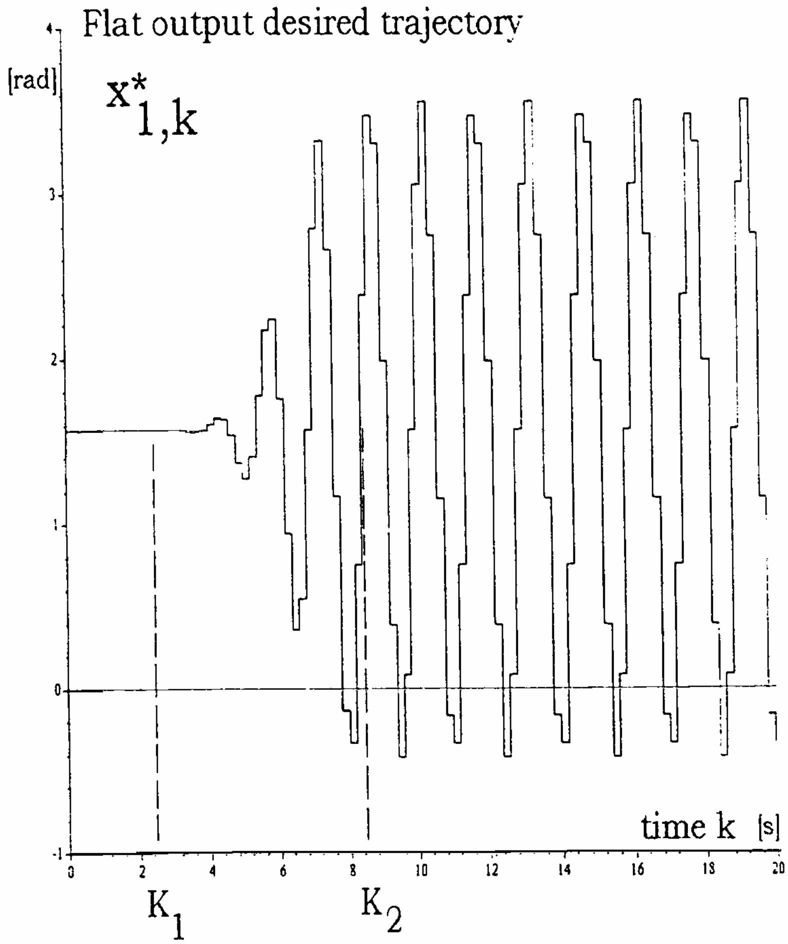

Figure 4. Reference signal for link angular position in trajectory tracking maneuver.

Using the results of the Appendix, a sliding mode controller can be proposed which asymptotically forces the system to track the given desired trajectory, $x^*_{1,k}$, for the flat output. We consider the sliding surface coordinate function

$$\sigma_k = \left(x_{1,k+3} - x^*_{1,k+3}\right) + a_3\left(x_{1,k+2} - x^*_{1,k+2}\right) + a_2\left(x_{1,k+1} - x^*_{1,k+1}\right) + a_1\left(x_{1,k} - x^*_{1,k}\right) \tag{12}$$

Let e_k denote the flat output reference tracking error $e_k = x_k - x^*_k$. Then, if the evolution of σ_k is indefinitely constrained to zero, the corresponding zero dynamics is characterized by the asymptotically stable linear dynamics

$$e_{k+3} + a_3 e_{k+2} + a_2 e_{k+1} + a_1 e_k = 0 \tag{13}$$

Imposing on the evolution of σ_k the nonlinear paradigm dynamics $\sigma_{k+1} = \Gamma(\sigma_k)$, described in the Appendix, one obtains from (7) the prescription of the transformed control

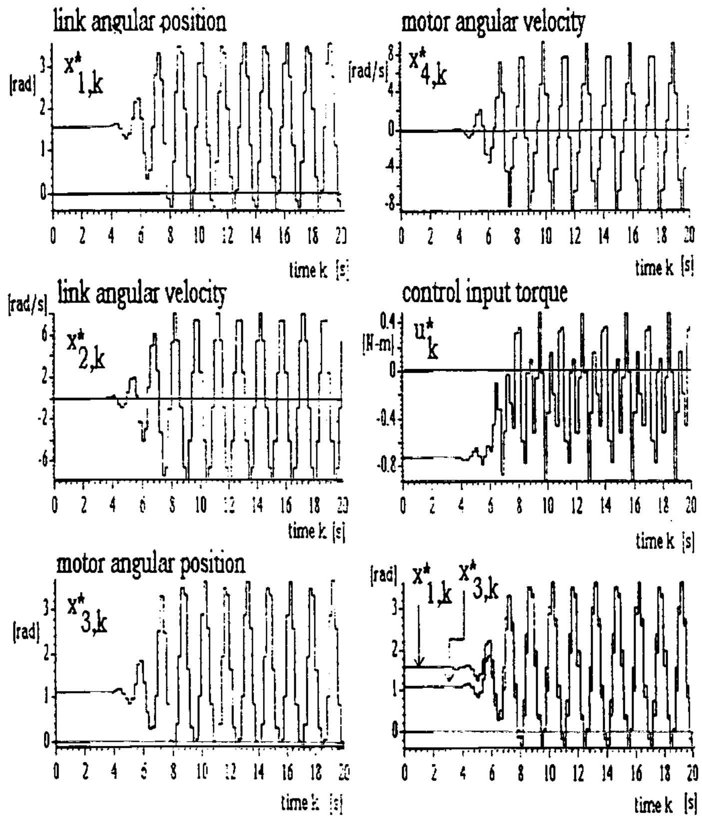

Figure 5. Nominal state and input trajectories for sinusoidal angular reference signal tracking.

input, v_k, as

$$\begin{aligned} v_k &= x^*_{1,k+4} - a_3\left(x_{1,k+3} - x^*_{1,k+3}\right) - a_2\left(x_{1,k+2} - x^*_{1,k+2}\right) \\ &\quad - a_1\left(x_{1,k+1} - x^*_{1,k-1}\right) - \Gamma(\sigma_k) \end{aligned} \tag{14}$$

where,

$$\begin{aligned} x_{1,k+1} &= x_{1,k} + T x_{2,k} \\ x_{1,k+2} &= \left(1 - \frac{K_a T^2}{I}\right) x_{1,k} + 2T x_{2,k} + \left(\frac{K_a T^2}{I}\right) x_{3,k} \\ &\quad + \left(\frac{mgLT^2}{I}\right) \sin(x_{1,k}) \end{aligned}$$

$$x_{1,k+3} = \left(1 - 3\frac{K_a T^2}{I}\right) x_{1,k} + \left(3T - \frac{K_a T^3}{I}\right) x_{2,k} + 3\left(\frac{K_a T^2}{I}\right) x_{3,k} + \left(\frac{K_a T^3}{I}\right) x_{4,k} + \frac{mgLT^2}{I}\left(2\sin(x_{1,k}) + \sin\left(x_{1,k} + Tx_{2,k}\right)\right) \quad (15)$$

The complete sliding mode feedback controller is constituted by the expressions in equations (6), (8), (9), (12)–(15) and (A.1).

4. Simulation Results

Numerical simulations were carried out for assessing the closed loop responses of the sliding mode controlled flexible joint manipulator represented by the previously given parameter values.

4.1. Rest-to-Rest Maneuver

First, a rest-to-rest trajectory traking task was considered which takes the link from the initial angular position $x_{1,K_1} = \pi/2$ at time $k = K_1$ towards the final desired position $x_{1,K_2} = -\pi/2$ at time $k = K_2$. The initial conditions for the numerical simulation were taken to be

$$x_{1,K_1} = 1 \text{ rad}, \quad x_{2,K_1} = 0 \text{ rad/s}, \quad x_3 = 0.62059 \text{ rad}, \quad x_4 = 0 \text{ rad/s}$$

wich represent a significant initial deviation from the prescribed trajectory.

The sliding mode controller parameters, as defined in the appendix, were set to be

$$A = 0.1, \quad B = 0.06, \quad K = 0.05$$

The auxiliary function σ was chosen in accordance with the stable characteristic polynomial coefficients given by

$$a_3 = 0.6, \quad a_2 = 0.12, \quad a_1 = 0.008$$

This is, after the sliding surface coordinate reaches zero, the tracking error signal, $e_k = x^*_{1,k} - x_{1,k}$, evolves according to the asymptotically stable linear dynamics

$$e_{k+3} + 0.6e_{k+2} + 0.12e_{k+1} + 0.008e_k = 0$$

whose characteristic polynomial has all its roots located at the point, 0.2, located inside the unit circle centered at the origin of the complex plane. The discretization interval was set to be $T = 0.2$ s.

The closed loop responses of the flexible joint manipulator to the planned rest-to-rest maneuver are shown in Figure 6. In spite of the significant initial deviations, the sliding

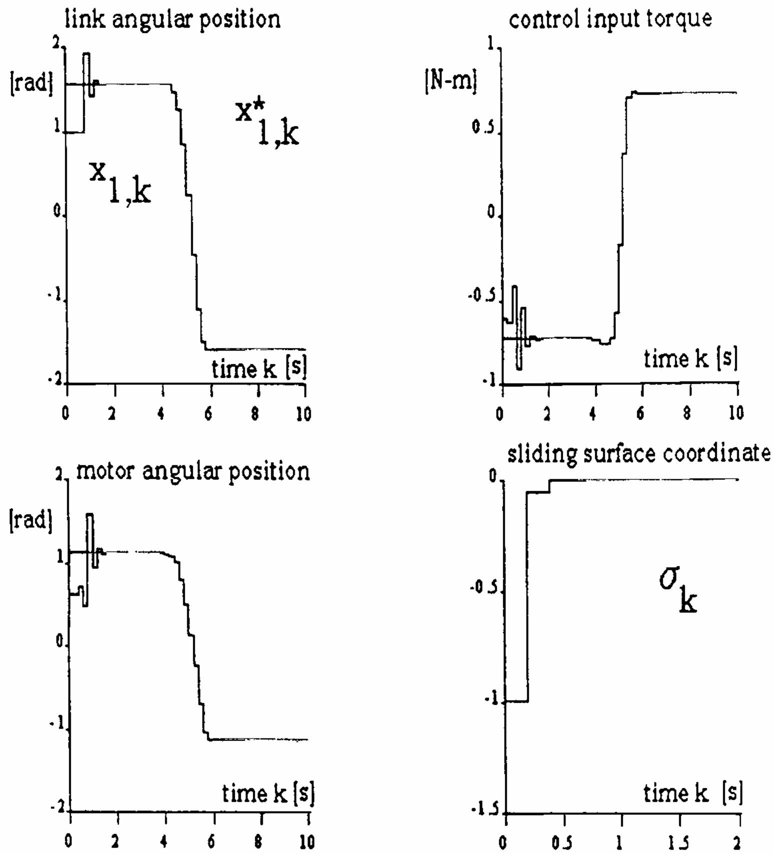

Figure 6. Closed loop sliding mode controlled responses for rest-to-rest angular maneuver, with initial deviations.

mode controller manages to stabilize the link to the correct initial settings and performs the subsequent desired trajectory tracking accomplishing the large angle equilibrium-to-equilibrium transfer in a total time span of 2 seconds. The closed loop equilibrium transfer is devoid of oscillations due to the joint flexibility.

4.2. Reference Trajectory Tracking

A reference trajectory traking task was next considered which takes the link from the initial angular position $x_{1,K_1} = \pi/2$, at time $k = K_1 = 2.5$ s, and induces a sinusoidal behavior of monotone increasing amplitude until time $k = K_2 = 8.5$ s, where the amplitude ceases to increase and becomes constant with a peak-to-peak amplitude of 4 rad. We set the initial conditions for the numerical simulations as

$$x_{1,K_1} = 1 \text{ rad}, \quad x_{2,K_1} = 0 \text{ rad/s}, \quad x_3 = 0.62059 \text{ rad}, \quad x_4 = 0 \text{ rad/s}$$

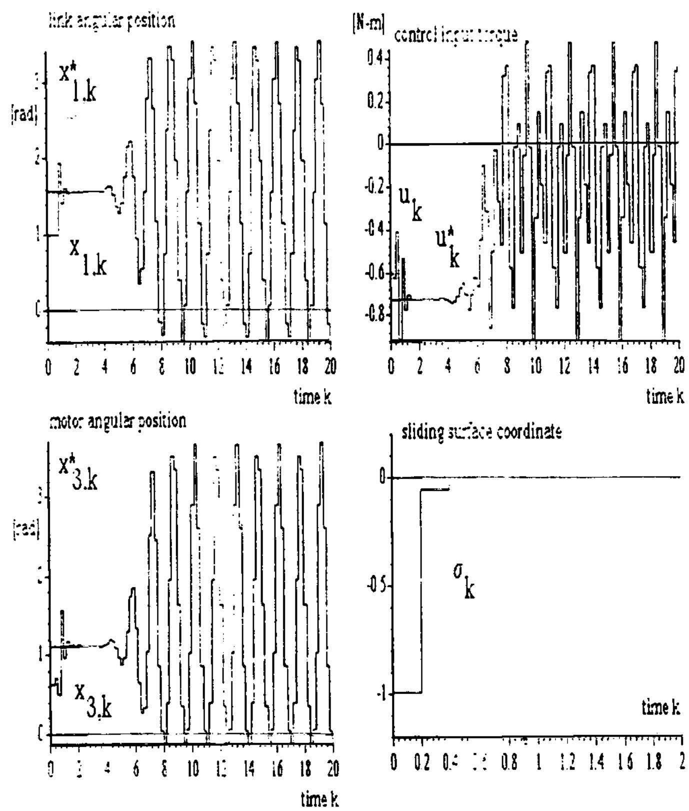

Figure Closed loop sliding mode controlled responses for the sinusoidal angular reference signal tracking task, with initial deviations.

which, again, represents a significant initial deviation from the prescribed trajectory.

The sliding mode controller parameters were set to be the same as in the rest-to-rest maneuver. The tracking features of the closed loop system towards the desired sinusoidal reference trajectory are shown in Figure 7. The sliding mode controller first stabilizes the link to the proper initial settings of the prescribed trajectory and subsequently drives the system to perfect tracking of the prescribed large amplitude sinusoidal swings.

5. Conclusions

In this article, we have examined the relevance of difference flatness for the regulation and tracking tasks, via trajectory planning and sliding mode control, of an approximately discretized flexible joint manipulator model. Difference flatness facilitates a systematic procedure for feedback controller synthesis directly from the associated difference parametrization provided by the flatness property. The difference parameterization represents an off-line computational asset for trajectory planning linked to the possibilities of complying with state variables and control input trajectory restrictions. Generally speaking, many other features of the given system are also determinable from such a difference parametrization. Issues such as minimum or non-minimum phase properties of state and output variables, detectability, passivity, etc. can be inferred from such a parametrization. These developments, however, will be reported elsewhere.

Appendix

In this appendix, we present some generalities about sliding mode control of nonlinear systems. Our devolopments are based on establishing a nonlinear autonomous dynamic system "paradigm" which exemplifies the sliding surface coordinate behavior. The idea is then to force a particular system output, like the flat output, to mimic the proposed prefered dynamics, with the aid of a suitable feedback control action.

A.1 A Robust Paradigm for Discrete Time Sliding Surface Dynamics

Let K, A, B be three strictly positive numbers with $A > B$. Consider a scalar nonlinear discrete-time dynamic system given by the following set of relations:

$$\sigma_{k+1} = \Gamma(\sigma_k) = \begin{cases} K \operatorname{sign} \sigma_k & \text{for } |\sigma_k| > A \\ \frac{K}{A-B}(|\sigma_k| - B) \operatorname{sign} \sigma_k & \text{for } B < |\sigma_k| < A \\ 0 & \text{for } |\sigma_k| < B \end{cases} \tag{A.1}$$

where "sign" stands for the *signum* function. We then have the following result.

THEOREM A.1 *The trajectories of system (A.1) are globally asymptotically stable to zero in finite time if and only if*

$$K < A$$

Moreover, σ_k *globally converges to zero in just one step (i.e., after* $k = 1$*), if and only if* $K < B$.

Proof: Consider a Lyapunov function candidate given by

$$V(\sigma) = \sigma^2, \quad \text{with} \quad V_k = V(\sigma_k) \tag{A.2}$$

Notice that V_k is strictly positive and it is bounded below by zero. Then, according to the scalar system dynamics (A.1), we have

$$V_{k+1} - V_k = \begin{cases} K^2 - \sigma_k^2 & \text{for } |\sigma_k| > A \\ \frac{K^2}{(A-B)^2}(|\sigma_k| - B)^2 - \sigma_k^2 & \text{for } B < |\sigma_k| < A \\ 0 - \sigma_k^2 & \text{for } |\sigma_k| < B \end{cases} \tag{A.3}$$

Suppose that, at some instant k, $|\sigma_k| > A$, then for the Lyapunov function candidate V_k to be strictly decreasing while this condition is valid, it is sufficient that $|K| < |A|$, since, then, $|K| < |A| < |\sigma_k|$ and therefore $V_{k+1} - V_k = K^2 - \sigma_k^2 < K^2 - A^2 < 0$. Under the above conditions, the evolution of σ_k reaches the region $B < |\sigma_k| < A$ in a single step. In this region, the condition $|K| < |A|$ implies that

$$\begin{aligned} V_{k+1} - V_k &= \frac{K^2}{(A-B)^2}(|\sigma_k| - B)^2 - \sigma_k^2 \\ &= \frac{K^2}{(A-B)^2}\left[(|\sigma_k| - B)^2 - \frac{(A-B)^2}{K^2}\sigma_k^2\right] \\ &< \frac{K^2}{(A-B)^2}\left[(|\sigma_k| - B)^2 - \frac{(A-B)^2}{A^2}\sigma_k^2\right] < 0 \end{aligned} \tag{A.4}$$

Therefore, in the region $B < |\sigma_k| < A$, the magnitude of $|\sigma|$ monotonically decreases with finite negative steps given by

$$\Delta_k = \sigma_{k+1} - \sigma_k = \left(\frac{K}{A-B} - 1\right)\sigma_k - \frac{K}{A-B}B \tag{A.5}$$

As it can be seen from (A.5), in the region $B < |\sigma_k| < A$, each element of the sequence of negative steps $\{\Delta_k\}$ is found within the interval

$$\min\{K - A \; -B\} < \Delta_k < \max\{K - A \; -B\}$$

The magnitude of $|\sigma|$ thus decreases until it eventually satisfies the condition $|\sigma(\mathcal{K})| < B$ at some finite instant $\mathcal{K}$. From the definition of the dynamics it follows that $\sigma_k = 0$ for $k = \mathcal{K} + 1, \mathcal{K} + 2, \ldots$ and the system is globally asymptotically stable in finite time.

To prove necessity, suppose the system is globally asymptotically stable to zero in finite time. It follows that, for each k, there exists a subsequence of integers $j_k \geq 1$, such that $V(k + j_k) - V_k < 0$. Then there exists a finite $\mathcal{K}$ such that for all $k > \mathcal{K}$ the sliding surface coordinate σ_k becomes zero after reaching the region $|\sigma| < B$. Suppose, contrary to what we want to establish that $|K| > |A|$, then motions starting on the region $|\sigma| > (A^2 + B(K - A))/K$ will never leave this region since $V_{k+1} - V_k \geq 0$ and the magnitude of $|\sigma|$ becomes constant (and equal to K) after the first step. We have a contradiction since the system is not globally asymptotically stable to zero.

It is clear that for any given initial condition, $\sigma(0)$, at $k = 0$, the next value of the surface satisfies $\sigma(1) \leq K$. Therefore, if $K < B$ then $\sigma_k = 0$ for $k \geq 2$.

A.2 Robustness of the Sliding Surface Evolution towards Zero

Let $\{\eta_k\}$ be a sequence of bounded perturbations taking values on the closed interval $[-N, N]$ of the real line, i.e., $\eta_k \in [-N, N]$ for all k. Consider then, the additively perturbed version of the nonlinear scalar sliding surface coordinate dynamics (A.1) given by

$$\sigma_{k+1} = \Gamma(u_k)$$
$$u_k = \sigma_k + \eta_k \tag{A.6}$$

We then have the following robustness result.

PROPOSITION A.2 *The sliding surface coordinate evolution (A.6) is asymptotically stable to zero in finite time if*

$$K + N < A \quad \text{and} \quad N < B \tag{A.7}$$

Proof: The proof is immediate after realizing that if the condition (A.7) is satisfied then, under the worst possible circumstances, the sliding surface coordinate σ_k reaches the region $B < |\sigma_k| < A$ after just one step. In this region, the value of $|\sigma|$ invariably decreases, in spite of the values of the perturbation η. The trajectory eventually reaches the region $|\sigma| < B$ at certain time $\mathcal{K}$. It is verified, from (A.1) that for $k > \mathcal{K}$, the perturbed value of the surface coordinate, σ_k, becomes zero and never leaves the region $[-B, B]$.

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