Sliding mode-based adaptive learning in dynamical-filter-weights neurons

HEBERTT SIRA-RAMÍREZ†*, ELIEZER COLINA-MORLES
and FRANCKLIN RIVAS-ECHEVERRÍA

A sliding mode control strategy is proposed for the synthesis of an adaptive learning algorithm in a neuron whose weights are constituted by first-order dynamical filters with adjustable parameters, which in turn allows the representation of dynamical processes in terms of a set of such neurons. The approach is shown to exhibit robustness characteristics and fast convergence properties. A simulation example, dealing with an analog signal tracking task, is provided which illustrates the feasibility of the approach.

1. Introduction

The discrete-time context has dominated all proposed adaptive learning strategies in neuron-based feedforward neural networks of the multilayer perceptron type. The celebrated Widrow–Hoff delta rule (Widrow and Lehr 1990) constitutes a least-mean-square learning error minimization algorithm by which, under certain conditions, an asymptotically stable linear dynamics is imposed on the underlying discrete-time error dynamics. Using quasi-sliding mode control ideas (Sira-Ramírez 1991), a modification of the Delta Rule was proposed in Sira-Ramírez and Żak (1991), whereby a switching weight adaptation strategy is shown also to impose a discrete time asymptotically stable linear learning error dynamics. This algorithm is the basis of recently proposed dynamical systems identification and control schemes based on neural networks (Colina-Morles and Mort 1993, Kuschewsky *et al.* 1993). An entirely different viewpoint in neuron-based adaptive learning has been recently proposed by considering a class of problems defined on analogue (i.e., continuous time) adaptive neurons. In correspondence with such a setting, continuous time—rather than discrete time – adaptive weight adjustment needs to be tackled. From such a continuous time viewpoint, the design of learning strategies in adaptive analogue neurons, using the perspective of sliding mode control, has been addressed in the work by Sira-Ramírez and Colina-Morles (1995). The relevance of ordinary differential equations with discontinuous right-hand sides, or variable structure systems (Utkin 1978), was analysed in Li *et al.* (1989), also in the context of analogue neural networks of the Hopfield type with infinite gain non-linearities. In that work, it is established under what circumstances sliding mode trajectories do not appear in the behaviour of such a class of neurons.

In this article the continuous time sliding mode control approach for the adaptation of time-varying neuron weights is briefly revisited, closely following the exposition in Sira-Ramírez and Colina-Morles (1995). Motivated by the dynamical character of the resulting sliding mode control solution, we proceed to propose a new type of neuron, referred to as the 'dynamical filter-weights neuron', where all weights are substituted by first-order, linear, time-varying dynamical systems. The weight adjustment manoeuvres, from a sliding mode perspective, are now to be carried upon the time-varying 'gains' and the time-varying 'time constants' of the proposed 'dynamical filter weights', which in turn simplifies the representation of dynamical processes in terms of a set of dynamical-filter-weights neurons. On such a dynamical structure for the neuron weights, the sliding mode control strategy results in a versatile, simple, and easy to implement adaptation algorithm. The basic features of the proposed approach are not only fast convergence but also robustness with respect to unknown external perturbation inputs. Such advantageous features are, in general, characteristic of sliding mode control adaptive schemes.

Section 2 contains the fundamental definitions, assumptions and derivations of the main characteristics of a sliding mode control approach for adaptation in dynamical-weights neurons. Section 3 includes a formulation to show the robustness characteristics of the adaptation algorithm with respect to bounded external perturbation inputs. Section 4 contains an illustrative example exploring the behaviour of the proposed dynamical weight adjustment algorithm in an output signal tracking problem. Section 5 contains the conclusions and suggestions for further research.

Accepted in final form June 1999.

† Departamento Sistemas de Control, Escuela de Ingeniería de Sistemas, Universidad de Los Andes, Mérida, 5101, Venezuela.

* Author for correspondence. e-mail: isira@ing.ula.ve

International Journal of Control ISSN 0020-7179 print/ISSN 1366-5820 online © 2000 Taylor & Francis Ltd
http://www.tandf.co.uk/journals

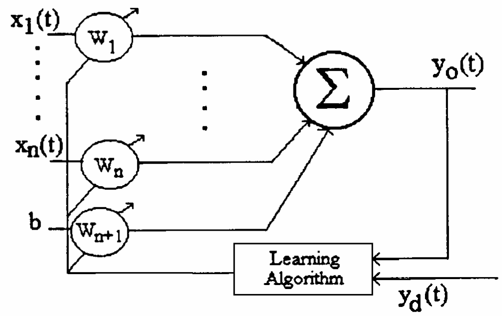

Figure 1. Neuron model.

2. Adaptation in dynamical-weights neuron

2.1. *Background results*

Consider the neuron model depicted in figure 1, where $x(t) = (x_1(t), \ldots, x_n(t))$ represents a vector of bounded time-varying inputs, assumed also to exhibit bounded time derivatives, i.e.

$$\left.\begin{aligned} \|x(t)\| &= \sqrt{x_1^2(t) + \cdots + x_n^2(t)} \le V_x \ \forall t \\ \|\dot{x}(t)\| &= \sqrt{\dot{x}_1^2(t) + \cdots + \dot{x}_n^2(t)} \le V_{\dot{x}} \ \forall t \end{aligned}\right\} \quad (1)$$

where V_x and $V_{\dot{x}}$ are known positive constants.

We denote by $\tilde{x}(t)$ the vector of augmented inputs, which includes a constant input of value $b \neq 0$, affecting the 'bias' or 'threshold' weight w_{n+1} in the neuron model, i.e.

$$\tilde{x}(t) = \operatorname{col}(x_1(t), \ldots, x_n(t), b) = \operatorname{col}(x(t), b) \quad (2)$$

The vector $\omega(t) = \operatorname{col}(\omega_1(t), \ldots, \omega_n(t))$ represents a set of dynamical weights. It will be assumed that, due to physical constraints, the magnitude of the vector $\omega(t)$ is bounded, $\|\omega(t)\| \le W \ \forall t$, for some constant W. We also define the vector of augmented weights $\tilde{\omega}$ by including the bias weight component

$$\begin{aligned} \tilde{\omega}(t) &= \operatorname{col}(\omega_1(t), \ldots, \omega_n(t), \omega_{n+1}(t)) \\ &= \operatorname{col}(\omega(t), \omega_{n+1}(t)) \end{aligned} \quad (3)$$

Similarly, $\tilde{\omega}(t)$ is assumed to be bounded at each instant of time t by means of

$$\|\tilde{\omega}(t)\| = \sqrt{\omega_1^2(t) + \cdots + \omega_n^2(t) + \omega_{n+1}^2(t)} \le \tilde{W} \ \forall t \quad (4)$$

for some constant $\tilde{W}$.

The scalar signal $y_d(t)$ represents the time-varying desired output of the neuron. It will be assumed that $y_d(t)$ and $\dot{y}_d(t)$ are also bounded signals, i.e.

$$\left.\begin{aligned} |y_d(t)| &\le V_y \ \forall t \\ |\dot{y}_d(t)| &\le V_{\dot{y}} \ \forall t \end{aligned}\right\} \quad (5)$$

The neuron output signal $y_0(t)$ is a scalar quantity defined as:

$$\begin{aligned} y_0(t) &= \sum_{i=1}^{n} \omega_i(t) x_i(t) + \omega_{n+1}(t) b \\ &= \omega^{\mathrm{T}}(t) x(t) + \omega_{n+1}(t) b = \tilde{\omega}^{\mathrm{T}}(t) \tilde{x}(t) \end{aligned} \quad (6)$$

We define the 'learning error' $e(t)$ as the scalar quantity obtained from

$$e(t) = y_0(t) - y_d(t) \quad (7)$$

Using the theory of 'sliding mode control of variable structure systems' (Utkin 1992), we propose to consider the zero value of the learning error coordinate $e(t)$ as a time-varying sliding surface, i.e.

$$s(e(t)) = e(t) = 0 \quad (8)$$

Equation (8) is, therefore, deemed as a desired condition for the learning error signal $e(t)$ and one which guarantees that the neuron output $y_0(t)$ coincides with the desired output signal $y_d(t)$ for all time $t > t_h$ where t_h is called the 'hitting time'.

Definition 1: A sliding motion is said to exist on a sliding surface $s(e(t)) = e(t) = 0$, after time t_h, if the condition $s(t)\dot{s}(t) = e(t)\dot{e}(t) < 0$ is satisfied for all t in some non-trivial semi-open subinterval of time of the form $[t, t_h) \subset (-\infty, t_h)$

It is desired to devise a dynamical feedback adaptation mechanism, or adaptation law, for the augmented vector of variable weights $\tilde{\omega}(t)$ such that the sliding mode condition of Definition 1 is enforced.

Let 'sign $e(t)$' stand for the signum function, defined as

$$\operatorname{sign} e = \begin{cases} +1 & \text{for } e(t) > 0 \\ 0 & \text{for } e(t) = 0 \\ -1 & \text{for } e(t) < 0 \end{cases} \quad (9)$$

We then have the following result:

Theorem 1: *If the adaptation law for the augmented weight vector $\tilde{\omega}(t)$ is chosen as*

$$\begin{aligned} \dot{\tilde{\omega}}(t) &= -\left(\frac{\tilde{x}(t)}{\tilde{x}^{\mathrm{T}}(t)\tilde{x}(t)}\right) k \operatorname{sign} e(t) \\ &= -\left(\frac{\begin{bmatrix} x(t) \\ b \end{bmatrix}}{b^2 + x^{\mathrm{T}}(t) x(t)}\right) k \operatorname{sign} e(t) \end{aligned} \quad (10)$$

with k being a sufficiently large positive design constant satisfying

$$k > \tilde{W} V_{\dot{x}} + V_{\dot{y}} \tag{11}$$

then, given an arbitrary initial condition $e(0)$, the learning error $e(t)$ converges to zero in finite time, t_h, estimated by

$$t_h \leq \frac{|e(0)|}{k - \tilde{W} V_{\dot{x}} - V_{\dot{y}}} \tag{12}$$

and a sliding motion is sustained on $e(t) = 0$ for all $t > t_h$.

Proof: See Sira-Ramírez and Colina-Morles (1995). □

Note that the proposed dynamical feedback adaptation law for the vector of weights in equation (10) results in a continuous regulated evolution of the vector of dynamical weights $\tilde{\omega}(t)$.

Note also that if the quantity $\dot{\tilde{x}}(t)$ is measurable, one can obtain a more relaxed variable structure feedback control strategy than the one obtained in (10). Generally speaking, such an adaptive feedback strategy for the dynamical weights requires smaller design gains k to obtain a corresponding sliding motion (see Sira-Ramírez and Colina-Morles 1995).

The proposed solution for $\dot{\tilde{\omega}}(t)$ in (10) is, necessarily, aligned with the augmented vector of inputs $\tilde{x}(t)$. The total disregard for the effect of the scalar signal $\dot{y}_d(t)$ in the above adaptation scheme arises from the implicit assumption that such a signal is not, generally speaking, measurable in practice, nor can it be estimated with sufficient precision. The previous theorem indicates that as long as $\dot{y}_d(t)$ is bounded, the adaptation policy always manages to bring the learning error to zero in finite time.

2.2. *Dynamical-filter-weights neuron*

Consider a neuron in which the traditional adjustable weights have been substituted by first-order, linear, time-varying, dynamical filters described by

$$\dot{y}_i = a_i(t) y_i + K_i(t) x_i(t); \quad i = 1, \ldots, n \tag{13}$$

where the time-varying scalar functions $a_i(t)$; $i = 1, \ldots, n$ and $K_i(t); i = 1, \ldots, n$ play the role of adjustable parameters. For lack of better names, we will improperly refer to such parameters as 'time constants' and 'gains', respectively, in parallel with the traditional terms associated with time-invariant counterparts (see figure 2).

As in traditional neurons, $x(t) = (x_1(t), \ldots, x_n(t))$ represents a vector of bounded time-varying inputs, also assumed to possess bounded time derivatives. We define the vectors $a(t)$ and $K(t)$ as n-dimensional vectors constituted by the 'time constants' and 'gains', i.e.

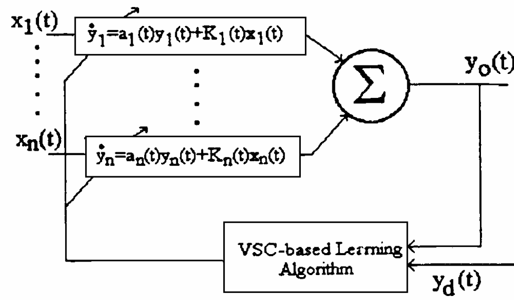

Figure 2. Dynamical-filter-weights neuron.

$$\left.\begin{aligned} a(t) &= \mathrm{col}\,(a_1(t), \ldots, a_n(t)) \\ K(t) &= \mathrm{col}\,(K_1(t), \ldots, K_n(t)) \end{aligned}\right\} \tag{14}$$

The vector $y(t)$ is constituted by the outputs of the dynamical filters acting as weights, $y(t) = (y_1(t), \ldots, y_n(t))$. The scalar signal $y_d(t)$ represents the desired output of the neuron and constitutes the signal to be tracked by the neuron scalar output $y_0(t)$. The output of the neuron, $y_0(t)$, is given by

$$y_0(t) = \sum_{i=1}^{n} y_i(t) \tag{15}$$

It is assumed that the set of dynamical filter weights, characterized by the vector $y(t)$, has an initial condition vector given by $y(t_0)$. The learning error, denoted by $e(t)$, is the scalar quantity defined by

$$e(t) = y_0(t) - y_d(t) \tag{16}$$

As in the traditional case, it is desired to derive a feedback adaptation law for the adjustable parameter vectors $a(t)$ and $K(t)$, such that the learning error $e(t)$ reaches the value zero for any initial condition – represented by the vector $y(t_0)$ – of the dynamical filter weights. Moreover, it is desired that once the learning error reaches the value zero, such a value is sustained for the remaining time horizon.

In the following theorem we assume that the external signals $y_d(t)$ and $\dot{y}_d(t)$ are bounded as in equation (5).

Theorem 2: *If the adaptation laws for the adjustable parameters of the dynamical filter weights are chosen as*

$$\begin{bmatrix} a(t) \\ K(t) \end{bmatrix} = -\left(\frac{W \operatorname{sign} e(t)}{\|y(t)\|^2 + \|x(t)\|^2} \right) \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} \tag{17}$$

with W being a sufficiently large positive design constant satisfying $W > V_{\dot{y}}$ then, given any arbitrary initial condition $y_0(t_0)$, the learning error $e(t)$ converges to zero in finite time t_h, estimated as

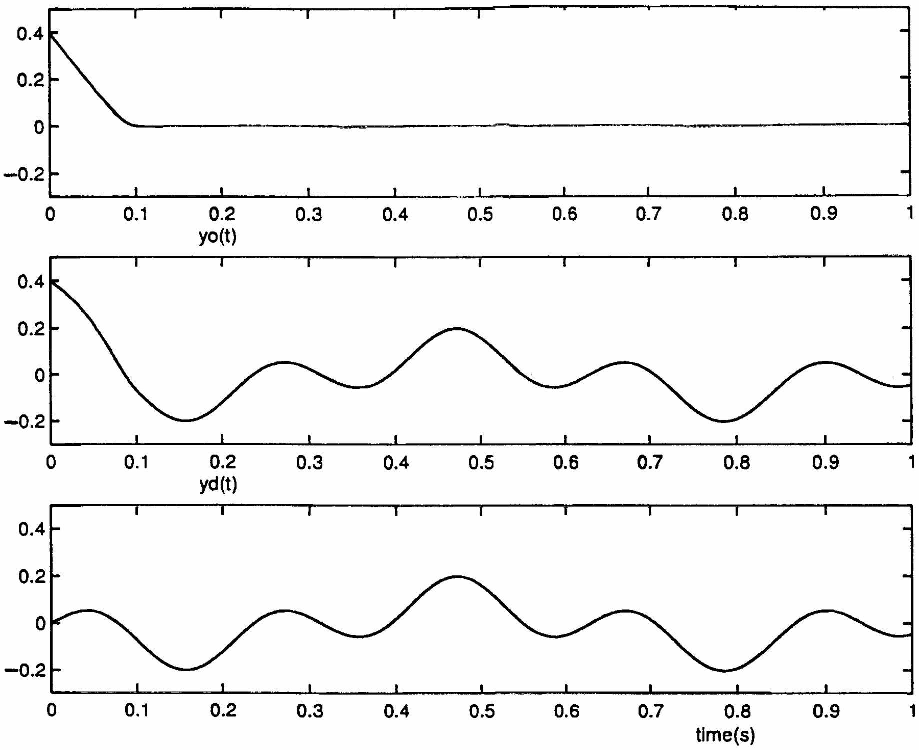

Figure 3. Computer simulation results ($W = 5$): learning error ($e(t)$), neuron output ($y_o(t)$), desired output ($y_d(t)$).

$$t_h \leq \frac{|y_0(t_0) \quad y_d(t_0)|}{W - V_{\dot{y}}}$$

and a sliding motion is sustained on $e(t) = 0$ *for all* $t > t_h$.

Proof: Compute the time derivative of the learning error as

$$\begin{aligned} \dot{e}(t) &= \sum_{i=1}^{n} \dot{y}_i(t) - \dot{y}_d(t) \\ &= \sum_{i=1}^{n} (a_i(t)y_i(t) + K_i(t)x_i(t)) - \dot{y}_d(t) \\ &= a^T(t)y(t) + K^T(t)x(t) - \dot{y}_d(t) \\ &= [a(t) \quad K(t)] \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} - \dot{y}_d(t) \\ &= -\dot{y}_d - W \operatorname{sign} e(t) \end{aligned}$$

where the last equality is obtained after substitution of the proposed parameter adaptation laws (17). Evidently, for all $e(t) \neq 0$,

$$\begin{aligned} e(t)\dot{e}(t) &= -e(t)\dot{y}_d(t) - W|e(t)| \\ &\leq |e(t)|V_{\dot{y}} - W|e(t)| \\ &= -|e(t)|(W - V_{\dot{y}}) < 0 \end{aligned}$$

The learning error $e(t)$, thus satisfies a differential equation with discontinuous right-hand side whose solution exhibits a sliding regime in finite time t_h (Utkin 1992). □

A relaxed version of Theorem 2 is obtained if one assumes that the signal $\dot{y}_d(t)$ is measurable.

Theorem 3: *If the adaptation laws for the adjustable parameters of the dynamical filter weights are chosen as*

$$\begin{bmatrix} a(t) \\ K(t) \end{bmatrix} = \left(\frac{\dot{y}_d(t) - W \operatorname{sign} e(t)}{\|y(t)\|^2 + \|x(t)\|^2} \right) \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} \tag{18}$$

with W *being a sufficiently large positive design constant, then, given an arbitrary initial condition* $y_0(t_0)$*, the learning error* $e(t)$ *converges to zero in finite time* t_h*, given by*

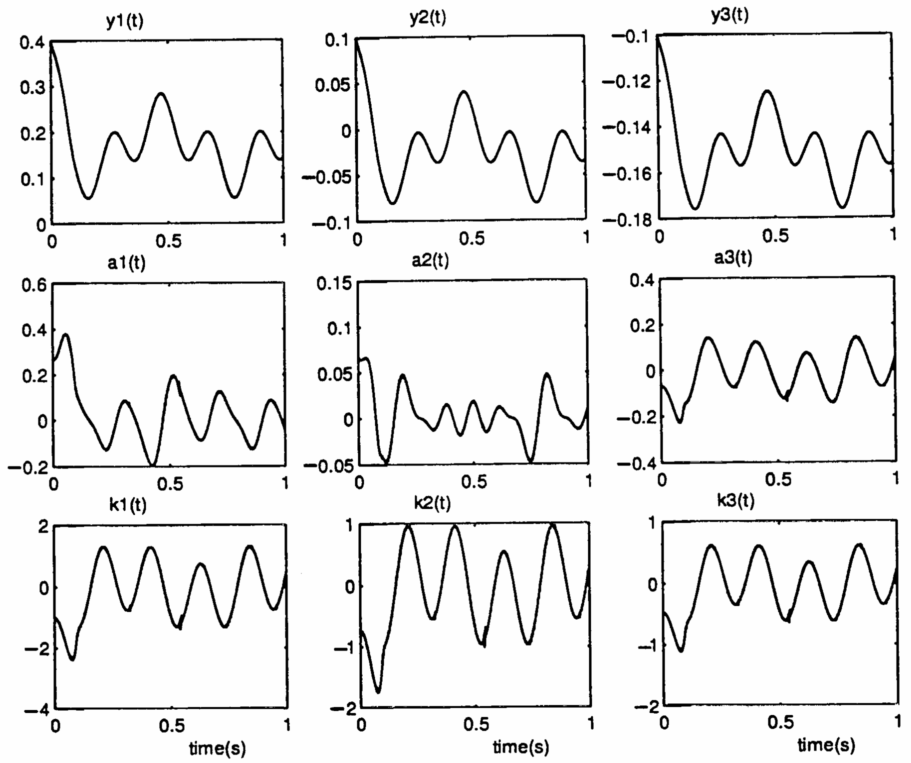

Figure 4. Dynamic neuron parameters and filter output $(W = 5)$: filter outputs $(y_i(t))$, filter time constants $(a_i(t))$, filter gains $(k_i(t))$.

$$t_{\mathrm{h}} = \frac{|y_0(t_0) - y_{\mathrm{d}}(t_0)|}{W} \tag{19}$$

and a sliding motion is sustained on $e(t) = 0$ *for all* $t > t_{\mathrm{h}}$.

Proof: The proof is immediate upon realizing that the controlled learning error satisfies the following differential equation with discontinuous right-hand side

$$\dot{e}(t) = -W \operatorname{sign} e(t) \tag{20}$$

and hence a sliding regime exists on $e(t) = 0$, since $e(t)\dot{e}(t) = -W|e(t)| < 0$ for all non-zero $e(t)$. The sliding regime is reachable in finite time t_{h} given by $|e(0)|/W$, as can be inferred from the time integration of (20). □

3. Robustness features

Consider an unmeasurable norm-bounded perturbation vector $\eta(t) = (\eta_1(t), \ldots, \eta_n(t))^{\mathrm{T}}$ which additively affects the neuron input vector $x(t)$ (i.e., the neuron input vector $x(t)$ corresponds to the state vector of a dynamical system). The magnitude of $\eta(t)$, is not larger than the magnitude of $x(t)$, i.e.

$$\|\eta(t)\| = \sqrt{\eta_1^2(t) + \cdots + \eta_n^2(t)} \le V_\eta < V_x \, \forall t \tag{21}$$

Note that the ith component of the measurable input vector may be represented as

$$\xi_i(t) = x_i(t) + \eta_i(t) \tag{22}$$

which is also a bounded signal. This is

$$\|\xi(t)\| = \sqrt{\xi_1^2(t) + \cdots + \xi_n^2(t)} \le V_\xi \tag{23}$$

Under these circumstances, if equation (17) is rewritten as

$$\begin{bmatrix} a(t) \\ K(t) \end{bmatrix} = -\left(\frac{W \operatorname{sign} e(t)}{\|y(t)\|^2 + \|\xi(t)\|^2} \right) \begin{bmatrix} y(t) \\ \xi(t) \end{bmatrix} \tag{24}$$

then Theorem 2 holds true.

On the other hand, note that equation (17) suggests that both $a(t)$ and $K(t)$ are bounded vectors (i.e.

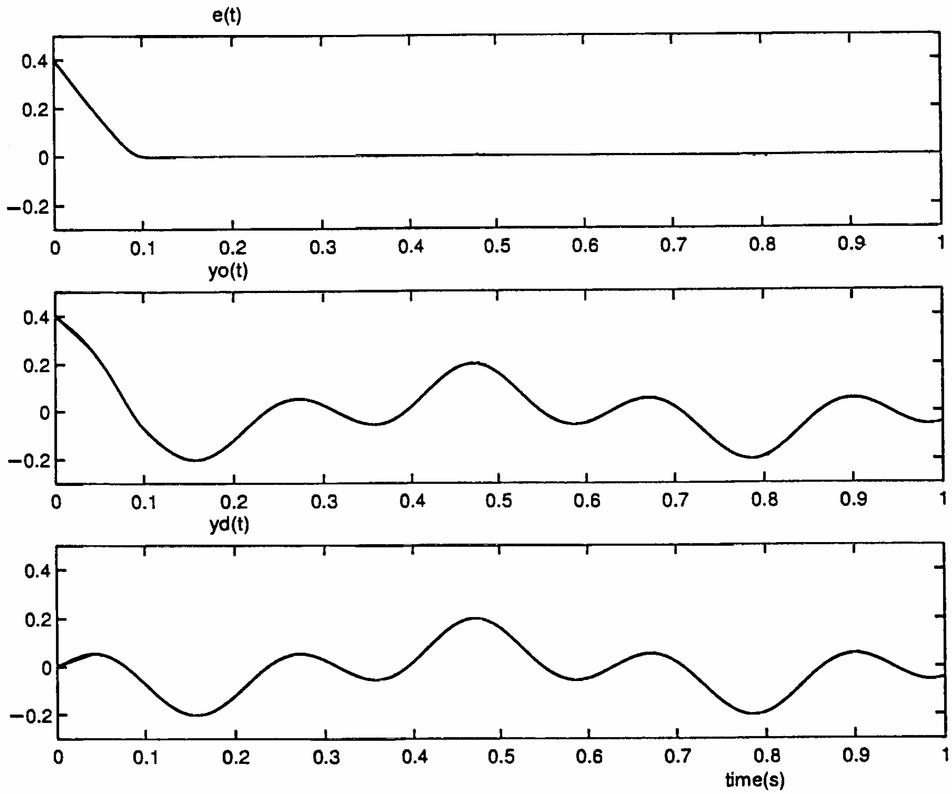

Figure 5. Computer simulation results ($W = 5$) in the presence of additive white noise.

$\|K(t)\| \leq V_K \, \forall t$ and $\|a(t)\| \leq V_a \, \forall t$) and therefore, if only the undisturbed input vector $x(t)$ is available for measurement, and the unmeasurable norm-bounded perturbation vector $\eta(t)$ still affects the neuron input in an additive manner, then Theorem 2 holds valid if the design gain W satisfies

$$W > V_{\dot{y}} + V_K V_\eta \tag{25}$$

In this case, the learning error $e(t)$ converges to zero in finite time t_h, estimated as

$$t_h \leq \frac{|y_o(t) - y_d(t)|}{W + V_{\dot{y}} - V_K V_\eta} \tag{26}$$

4. An illustrative simulation example

Consider a neuron consisting of three first-order, linear, time-varying filters acting as weights through their adjustable time-varying parameters. Suppose, furthermore, that the input signals x_1, x_2 and x_3 to the neuron are known constants. It is desired to track the scalar signal

$$y_d(t) = A \sin \omega t \cos 2\omega t$$

with $A = 0.4$, $\omega = 10$ [rad/s], by means of the output, $y_0(t) = y_1(t) + y_2(t) + y_3(t)$, of the neuron, specified by the sum of the filter outputs y_1, y_2, y_3 constituting the dynamical-weights neuron. The adaptive algorithm used to adjust the 'gains' and the 'time constants' was not fed with any information regarding the time derivative of the signal $y_d(t)$. Figures 3 and 4 depict the computer simulation results for this example when the design gain W was selected as $W = 5$. The components y_1, y_2 and y_3 of the neuron are seen to be bounded signals as well as the adaptation parameters constituting the three-dimensional vectors $a(t)$ and $K(t)$. The learning error $e(t)$ is seen to converge rapidly to zero in spite of lack of knowledge of $\dot{y}_d(t)$.

Finally, figures 5 and 6 show the performance of the dynamical-filter-weights neuron when its inputs are corrupted by additive white noise sequences of different amplitudes. In order to smooth out the natural chattering generated by the discontinuity present

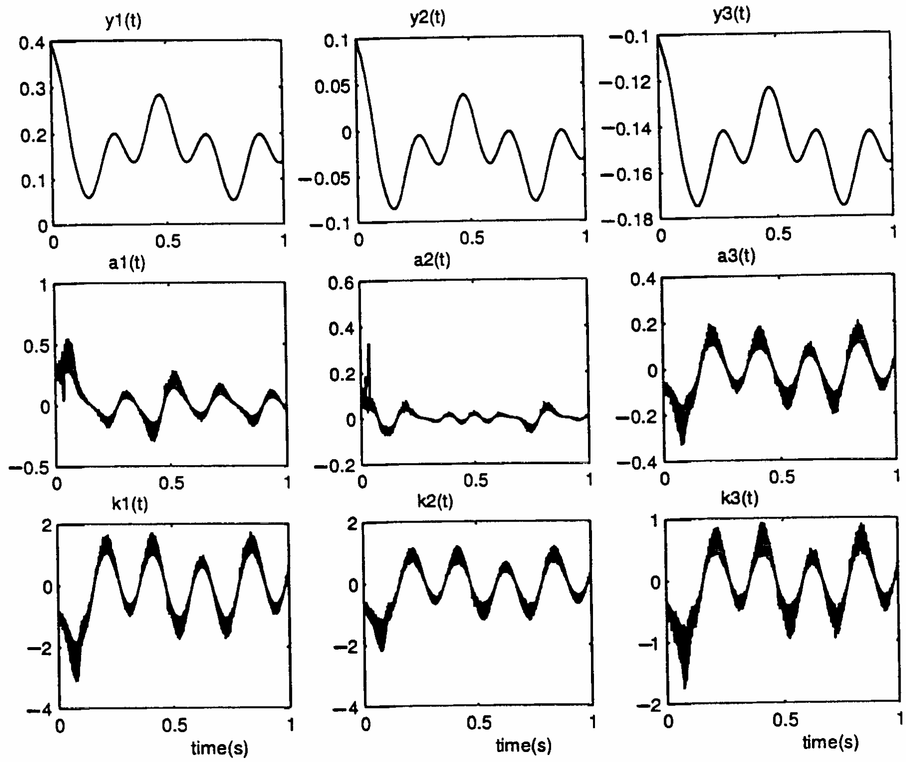

Figure 6. Dynamic neuron parameters and filter output $(W = 5)$ in the presence of additive white noise.

in the 'sign' function, we substituted such a function, as is customarily done, in sliding mode control practice by the approximating function

$$\frac{e(t)}{|e(t)| + \epsilon}$$

with ϵ taken to be a small constant of value $\epsilon = 0.01$

5. Conclusions

In this article, a sliding mode feedback adaptive learning algorithm has been proposed for a special class of neurons with first-order, linear, time-varying, dynamical filters acting as 'weights'. The sliding mode strategy was used here in the context of an output signal tracking problem. Research is under way to test the feasibility of implementing more complex tasks, such as direct and inverse dynamics identification, using a smoothed version of the parameters' behaviour, when the unknown process is subjected to external perturbations which introduce chattering into the estimated time-varying parameters of the filters. As in the traditional analogue neurons case, the sliding mode learning algorithm robustly drives the learning error to zero in finite time. The approach is also highly insensitive to bounded external perturbation inputs. The assumptions made about the bounded nature of external input signals and desired outputs, as well as of their time derivatives, are quite standard in relation to adaptive neuron elements.

Extensions of the results to more general classes of dynamical filters multilayer network arrangements is being pursued at the present time, with highly encouraging results.

Acknowledgments

This research was supported by the Consejo de Desarrollo Científico, Humanístico y Tecnológico of the Universidad de Los Andes under Research Grant I-611-98-02-A.

References

COLINA-MORLES, E., and MORT, N., 1993, Neural network-based adaptive control design. *Journal of Systems Engineering*, **1**, 9–14.

KUSCHEWSKI, J. G., HUI, S., and ŻAK, S. H., 1993, Application of feedforward networks to dynamical system identification and control. *IEEE Transactions on Control Systems Technology*, **1**, 37–49.

LI, J. H., MICHEL, A. N., and POROD, W., 1989, Analysis and synthesis of a class of neural networks: variable structure systems with infinite gain. *IEEE Transactions on Circuits and Systems*, **36**, 713–731.

SIRA-RAMÍREZ, H., 1991, Nonlinear discrete variable structure systems in quasi-sliding mode. *International Journal of Control*, **54**, 1171–1187.

SIRA-RAMÍREZ H., and COLINA-MORLES E., 1995, A sliding mode strategy for adaptive learning in adalines. *IEEE Transactions on Circuits and Systems—I: Fundamental Theory and Applications*, **42**, 1001–1012.

SIRA-RAMÍREZ, H., and ŻAK, S. H., 1991, The adaptation of perceptrons with applications to inverse dynamics identification of unknown dynamic systems. *IEEE Transactions on Systems, Man and Cybernetics*, **21**, 634–643.

UTKIN, V. I., 1978, *Sliding Regimes and their Applications in Variable Structure Systems* (Moscow: MIR).

UTKIN, V. I., 1992, *Sliding Modes in Control Optimization* (New York: Springer-Verlag).

WIDROW, B., and LEHR, M. A., 1990, 30 years of adaptive neural networks: perceptron, madaline, and back-propagation. *Proceedings IEEE*, **78**, 1415–1442.