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# On the Control of the Hovercraft System

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**Abstract.** A simplified model of the hovercraft system, used in the literature to illustrate nonlinear control options in underactuated systems, is shown to be *differentially flat*. The flat outputs are given by the position coordinates with respect to the fixed earth frame. This fact is here exploited for the design of a dynamic feedback controller for the global asymptotic stabilization of the system's trajectory tracking error with respect to off-line planned position trajectories.

**Keywords:** hovercraft, flat systems, trajectory planning

## 1. Introduction

The control of a ship having two independent thrusters, located at the aft, has received sustained attention in the last few years. The interest in devising feedback control strategies for the underactuated ship model stems from the fact that the system does not satisfy Brockett's necessary condition for stabilization to the origin by means of time-invariant state feedback (see Brockett, [1]). Reyhanoglu [13] proposes a discontinuous feedback control which locally achieves exponential decay towards a desired equilibrium. A feedback linearization approach was proposed by Godhavn [6] for the regulation of the position variables without orientation control. In an article by Pettersen and Egeland [8], a time-varying feedback control law is proposed which exponentially stabilizes the state towards a given equilibrium point. Time-varying quasi-periodic feedback control, as in Pettersen and Egeland [10], has been proposed exploiting the homogeneity properties of a suitably transformed model achieving simultaneous exponential stabilization of the position and orientation variables. A remarkable experimental set-up has been built which is described in the work of Pettersen and Fossen [11]. In that work, the time-varying feedback control, found in [8], is extended to include integral control actions, with excellent experimental results. High frequency feedback control signals, in combination with averaging theory and

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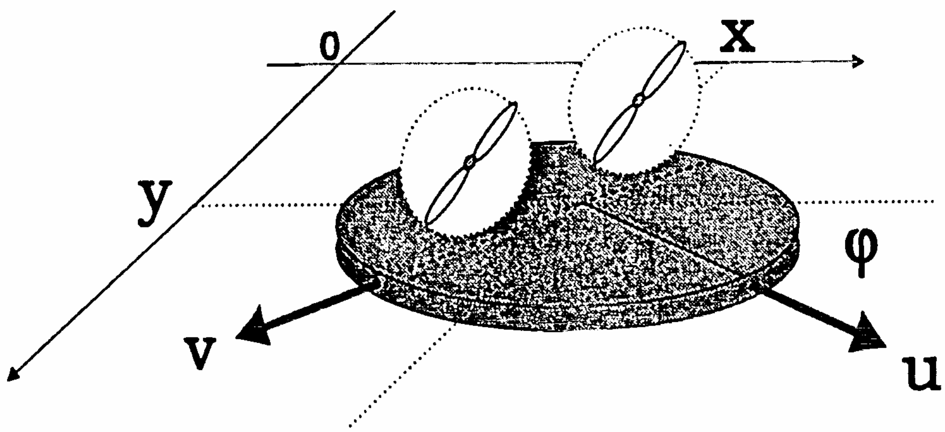


*Figure 1.* The simplified hovercraft system.

coordinates. The vector $\tau = [\tau_1, \tau_2, \tau_3]$ denotes the control forces in surge and sway and the control torque in yaw. The matrix $C(\nu)$ is the matrix of Coriolis and centripetal forces.

Consider the simplified version of the underactuated hovercraft, shown in Figure 1. A mathematical model for such a vessel can be directly derived, as already done in Fantoni *et al.* [9], from equations (2.1)–(2.3) by enforcing the following simplifying assumptions

$$m_{11} = m_{22}, \quad \tau_1 = m_{11}\tau_u, \quad \tau_2 = 0, \quad \tau_3 = m_{33}\tau_r, \quad d_{11} = d_{33} = 0, \quad \beta = \frac{d_{22}}{m_{22}} \tag{2.4}$$

The assumption, $m_{11} = m_{22}$, implies that the vessel is assumed to be *symmetric* with respect to the axes $u$ and $v$. We have also assumed that the hydrodynamic damping coefficients, $d_{11}$ and $d_{33}$, are both zero. This latter assumption does not affect the *flatness property* of the system, on which we base our feedback control design. Secondly, if these damping terms were actually present, they can be readily compensated by partial state feedback through the matched control input forces, $\tau_1$ and $\tau_3$.

We thus obtain the following model of the underactuated hovercraft vessel system,

$$\begin{aligned}
\dot{x} &= u\cos\varphi - v\sin\varphi \\
\dot{y} &= u\sin\varphi + v\cos\varphi \\
\dot{\varphi} &= r \\
\dot{u} &= vr + \tau_u \\
\dot{v} &= -ur - \beta v \\
\dot{r} &= \tau_r
\end{aligned} \tag{2.5}$$

We have the following proposition

PROPOSITION 2.1 *The model (2.5) is differentially flat, with flat outputs given by x and y i.e.,* all *system variables in (2.5) can be* differentially parametrized *solely in terms of x and y, as*

$$
\begin{aligned}
\varphi &= \arctan\left(\frac{\ddot{y}+\beta\dot{y}}{\ddot{x}+\beta\dot{x}}\right)\\
u &= \frac{\dot{x}(\ddot{x}+\beta\dot{x})+\dot{y}(\ddot{y}+\beta\dot{y})}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}}\\
v &= \frac{\dot{y}\ddot{x}-\dot{x}\ddot{y}}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}}\\
r &= \frac{y^{(3)}(\ddot{x}+\beta\dot{x})-x^{(3)}(\ddot{y}+\beta\dot{y})+\beta^2(\ddot{x}\dot{y}-\ddot{y}\dot{x})}{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}\\
\tau_u &= \frac{\ddot{x}(\ddot{x}+\beta\dot{x})+\ddot{y}(\ddot{y}+\beta\dot{y})}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}}\\
\tau_r &= \frac{y^{(4)}(\ddot{x}+\beta\dot{x})-x^{(4)}(\ddot{y}+\beta\dot{y})+\beta\left(y^{(3)}\ddot{x}-x^{(3)}\ddot{y}\right)-\beta^2\left(x^{(3)}\dot{y}-y^{(3)}\dot{x}\right)}{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}\\
&\quad -2\frac{\left[y^{(3)}(\ddot{x}+\beta\dot{x})-x^{(3)}(\ddot{y}+\beta\dot{y})-\beta^2(\ddot{x}\dot{y}-\ddot{y}\dot{x})\right]\times\left[(\ddot{x}+\beta\dot{x})(x^{(3)}+\beta\ddot{x})+(\ddot{y}+\beta\dot{y})(y^{(3)}+\beta\ddot{y})\right]}{\left[(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2\right]^2}
\end{aligned}
\tag{2.6}
$$

**Proof:** From the first two equations in (2.5) we readily obtain

$$
\begin{aligned}
v &= \dot{y}\cos\varphi-\dot{x}\sin\varphi\\
u &= \dot{x}\cos\varphi+\dot{y}\sin\varphi
\end{aligned}
\tag{2.7}
$$

Differentiating now the first two equations in (2.5) with respect to time. This yields, after use of (2.5) and (2.7)

$$
\begin{aligned}
\ddot{x} &= \dot{u}\cos\varphi-u\dot{\varphi}\sin\varphi-\dot{v}\sin\varphi-v\dot{\varphi}\cos\varphi\\
&= \tau_u\cos\varphi+\beta v\sin\varphi\\
\ddot{y} &= \dot{u}\sin\varphi+u\dot{\varphi}\cos\varphi+\dot{v}\cos\varphi-v\dot{\varphi}\sin\varphi\\
&= \tau_u\sin\varphi-\beta v\cos\varphi
\end{aligned}
\tag{2.8}
$$

Multiplying the first equation in (2.8) by $\sin\varphi$ and the second equation by $\cos\varphi$ and then subtracting the obtained expressions we obtain, after use of (2.5),

$$\ddot{x}\sin\varphi-\ddot{y}\cos\varphi=\beta v \tag{2.9}$$

Similarly, multiplying the first equation in (2.8) by $\cos\varphi$ and the second by $\sin\varphi$ and adding, we obtain

$$\tau_u = \ddot{x}\cos\varphi + \ddot{y}\sin\varphi \tag{2.10}$$

Substituting now the first of (2.7) into (2.9) one obtains, after some algebraic manipulations

$$\tan\varphi = \frac{\ddot{y} + \beta\dot{y}}{\ddot{x} + \beta\dot{x}} \longrightarrow \varphi = \arctan\left(\frac{\ddot{y} + \beta\dot{y}}{\ddot{x} + \beta\dot{x}}\right) \tag{2.11}$$

Using (2.11) in (2.7) we obtain,

$$v = \frac{\dot{y}\,(\ddot{x} + \beta\dot{x}) - \dot{x}\,(\ddot{y} + \beta\dot{y})}{\sqrt{(\ddot{x} + \beta\dot{x})^2 + (\ddot{y} + \beta\dot{y})^2}} = \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\sqrt{(\ddot{x} + \beta\dot{x})^2 + (\ddot{y} + \beta\dot{y})^2}} \tag{2.12}$$

and

$$u = \frac{\dot{x}(\ddot{x} + \beta\dot{x}) + \dot{y}(\ddot{y} + \beta\dot{y})}{\sqrt{(\ddot{x} + \beta\dot{x})^2 + (\ddot{y} + \beta\dot{y})^2}} \tag{2.13}$$

Substituting in (2.10) the value of $\varphi$, obtained in (2.11), leads to the expression for the force input, $\tau_u$, given in the proposition. Finally, we make use of the fact that $r = \dot{\varphi}$ and $\tau_r = \ddot{\varphi}$. ■

*Remark 2.2.* Notice that once $\varphi$ and $v$ are obtained as differential functions of $x$ and $y$, the rest of the hovercraft system variables can also be expressed as differential functions of $\varphi$ and $v$. Indeed, from (2.5) we obtain,

$$\begin{aligned} r &= \dot{\varphi} \\ u &= -\frac{\dot{v}}{\dot{\varphi}} \\ \tau_u &= -\left(\frac{\ddot{v}\dot{\varphi} - \dot{v}\ddot{\varphi}}{\dot{\varphi}^2}\right) + v\dot{\varphi} \\ \tau_r &= \ddot{\varphi} \end{aligned} \tag{2.14}$$

It is clear that *all* system variables are expressible as differential functions of the flat outputs.

The differential parametrization of the input torque $\tau_r$ depends up to the fourth order time derivatives of, both, the flat outputs, $x$ and $y$. Notice, however, that the corresponding parametrization of the control input $\tau_u$ only depends up to the second order time derivatives of $x$ and $y$. This simple fact clearly reveals an "obstacle" to achieve *static* feedback linearization and points to the need for a second order *dynamic* extension of the control input $\tau_u$ in order to exactly linearize the system.

*Remark 2.3.* Use of (2.5) allows the following (simpler) expressions for the control inputs $\tau_r$ and $\ddot{\tau}_u$, in terms of the system's state variables, the highest order derivatives of the flat outputs $x$ and $y$, and first order extensions of the control input $\tau_u$.

$$\tau_r = \frac{y^{(4)}\cos\varphi - x^{(4)}\sin\varphi - \beta r\tau_u - 2r\dot{\tau}_u - 2\beta r^2 v - \beta^2 ur + \beta^3 v}{\beta u + \tau_u} \tag{2.15}$$

$$\ddot{\tau}_u = x^{(4)}\cos\varphi + y^{(4)}\sin\varphi + 2\beta ur^2 + 2\beta^2 rv - \beta v\tau_r + r^2\tau_u \tag{2.16}$$

## 3. Trajectory Tracking for the Hovercraft System

Suppose a desired trajectory is given for the position coordinates $x$ and $y$ in the form $x^*(t)$ and $y^*(t)$, respectively. The following proposition gives a dynamic feedback solution to the trajectory tracking problem based on flatness and exact tracking error linearization.

PROPOSITION 3.1 *Let the set of constant real coefficients* $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ *and* $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ *represent independent sets of* Hurwitz coefficients. *Then, given a set of desired trajectories* $x^*(t)$ *and* $y^*(t)$, *for the position coordinates, the following dynamic feedback controller*

$$\tau_r = \frac{\phi\cos\varphi - \xi\sin\varphi - \beta r\tau_u - 2r\dot{\tau}_u - 2\beta r^2 v - \beta^2 ur + \beta^3 v}{\beta u + \tau_u} \tag{3.1}$$

$$\ddot{\tau}_u - r^2\tau_u = \xi\cos\varphi + \phi\sin\varphi + 2\beta ur^2 + 2\beta^2 rv - \beta v\tau_r \tag{3.2}$$

$$\begin{aligned} \xi &= x^{*(4)}(t) - \alpha_4(x^{(3)} - x^{*(3)}(t)) - \alpha_3(\ddot{x} - \ddot{x}^*(t)) - \alpha_2(\dot{x} - \dot{x}^*(t)) \\ &\quad - \alpha_1(x - x^*(t)) \end{aligned} \tag{3.3}$$

$$\begin{aligned} \phi &= y^{*(4)}(t) - \gamma_4(y^{(3)} - y^{*(3)}(t)) - \gamma_3(\ddot{y} - \ddot{y}^*(t)) - \gamma_2(\dot{y} - \dot{y}^*(t)) \\ &\quad - \gamma_1(y - y^*(t)) \end{aligned} \tag{3.4}$$

*with*

$$\begin{aligned} \dot{x} &= u\cos\varphi - v\sin\varphi \\ \dot{y} &= u\sin\varphi + v\cos\varphi \\ \ddot{x} &= \beta v\sin\varphi + \tau_u\cos\varphi \\ \ddot{y} &= \tau_u\sin\varphi - \beta v\cos\varphi \\ x^{(3)} &= -\left[r\,(\beta u + \tau_u) + \beta^2 v\right]\sin\varphi + (\beta rv + \dot{\tau}_u)\cos\varphi \\ y^{(3)} &= \left[r\,(\beta u + \tau_u) + \beta^2 v\right]\cos\varphi + (\beta rv + \dot{\tau}_u)\sin\varphi \end{aligned} \tag{3.5}$$

*globally exponentially asymptotically stabilizes the tracking errors* $e_x = x - x^*(t)$ *and* $e_y = y - y^*(t)$ *to zero.*

**Proof:** Subtracting the controller expression, for $\ddot{\tau}_u$ in (3.1), from the open loop expression in the Remark 2.3 we obtain, after some simple algebra,

$$\left[e_x^{(4)} + \alpha_4 e_x^{(3)} + \alpha_3 \ddot{e}_x + \alpha_2 \dot{e}_x + \alpha_1 e_x\right]\cos\varphi + \left[e_y^{(4)} + \gamma_4 e_y^{(3)} + \gamma_3 \ddot{e}_y + \gamma_2 \dot{e}_y + \gamma_1 e_y\right]\sin\varphi = 0 \tag{3.6}$$

Proceeding in a similar fashion with respect to the corresponding closed and open loop expressions for $\tau_r$, one finds:

$$-\left[e_x^{(4)} + \alpha_4 e_x^{(3)} + \alpha_3 \ddot{e}_x + \alpha_2 \dot{e}_x + \alpha_1 e_x\right]\sin\varphi + \left[e_y^{(4)} + \gamma_4 e_y^{(3)} + \gamma_3 \ddot{e}_y + \gamma_2 \dot{e}_y + \gamma_1 e_y\right]\cos\varphi = 0 \tag{3.7}$$

Then, clearly, the tracking errors satisfy the exponentially asymptotically stable fourth order dynamics

$$\begin{aligned} &e_x^{(4)} + \alpha_4 e_x^{(3)} + \alpha_3 \ddot{e}_x + \alpha_2 \dot{e}_x + \alpha_1 e_x = 0 \\ &e_y^{(4)} + \gamma_4 e_y^{(3)} + \gamma_3 \ddot{e}_y + \gamma_2 \dot{e}_y + \gamma_1 e_y = 0 \end{aligned} \tag{3.8}$$

## 4. Simulation Results

Simulations were carried out to evaluate the performance of the proposed feedback controller for two common trajectory tracking tasks: The first trajectory consisted of a straight line passing through the origin of earth fixed coordinates. A second trajectory was proposed as a circular trajectory, defined in the earth fixed coordinate frame, of radius $\rho$, centered around the origin.

### 4.1. *Tracking of a Straight Line*

We particularized the developed dynamic feedback controller for the case of tracking a straight line passing through the origin of the fixed earth frame. The hovercraft must follow this line at constant surge speed while moving away from the origin of coordinates. The path is given by the following parametric equations,

$$x^*(t) = at, \quad y^*(t) = bt \tag{4.1}$$

For this particular choice of $x$ and $y$, the nominal hovercraft orientation angle $\varphi^*(t)$ is given by the constant value,

$$\varphi^*(t) = \varphi^* = \arctan\left(\frac{b}{a}\right) \tag{4.2}$$

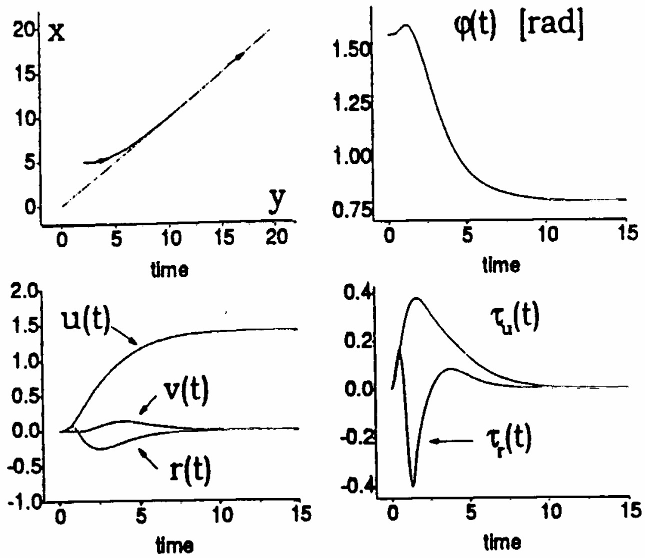


*Figure 2.* Feedback tracking of a straight line trajectory.

The nominal surge, sway velocities and the nominal yaw angular velocity are given, by

$$u^* = \sqrt{a^2+b^2}\cos(\varphi^* - \theta) = \sqrt{a^2+b^2},$$
$$v^*(t) = \sqrt{a^2+b^2}\sin(\varphi^* - \theta) = 0 \quad r^*(t) = 0 \tag{4.3}$$

with $\theta = \varphi^* = \arctan(b/a)$.

The nominal applied inputs are found to be given by the following constant values

$$\tau_u^*(t) = 0, \quad \tau_r^*(t) = 0 \tag{4.4}$$

We have chosen the following parameters for the reference trajectory, the system, and the feedback controller

$$a = 1, \ b = 1, \ \beta = 1.2, \ a_4 = b_4 = 4, \ a_3 = b_3 = 6, \ a_2 = b_2 = 4, \ a_1 = b_1 = 1$$

which result in $\theta = \pi/4$ rad. Figure 2 shows the closed loop trajectories for the state and control input variables of the hovercraft system. We set the initial conditions for the system significantly off the initial point, (0, 0), of the desired trajectory. We used, $x(0) = 5$, $y(0) = 2$. The initial hovercraft orientation angle was taken to be $\varphi(0) = \pi/2$ Notice that along the prescribed trajectory, $\beta u^*(t) + \tau_u^*(t) = \beta\sqrt{a^2+b^2} \neq 0$.

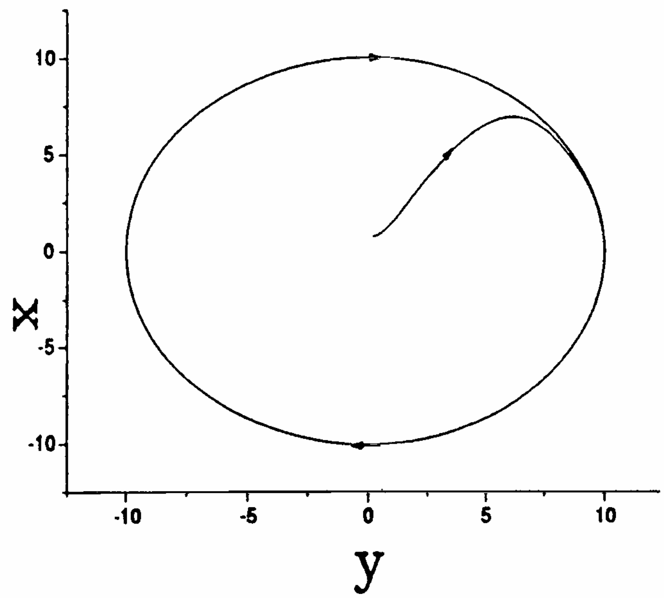


*Figure 3.* Feedback controlled position coordinates for circular path tracking.

### *4.2. Tracking a Circular Trajectory*

A circular trajectory, or radius $\rho$, is to be followed in a clockwise sense in the plane $(x, y)$, with a given constant angular velocity of value $\omega$. In other words, the flat outputs are nominally specified as,

$$x^*(t) = \rho \cos\omega t, \quad y^*(t) = \rho \sin\omega t \tag{4.5}$$

For this particular choice of $x$ and $y$, the nominal orientation angle $\varphi^*(t)$ is given by

$$\varphi^*(t) = \arctan\left(\frac{\omega\sin\omega t - \beta\cos\omega t}{\omega\cos\omega t + \beta\sin\omega t}\right) = \arctan(\tan(\omega t - \theta)) = \omega t - \theta \tag{4.6}$$

with $\theta = \arctan(\beta/\omega)$.

The nominal surge and sway velocities and the nominal yaw angular velocity are given, according to (2.7) and the fact that $r = \dot{\varphi}$, by the following constant values

$$u^*(t) = -\rho\omega\sin\theta, \quad v^*(t) = \rho\omega\cos\theta, \quad r^*(t) = \omega \tag{4.7}$$

Similarly, using (2.10) and the fact that $\tau_r = \ddot{\varphi}$ we obtain that the nominal applied inputs are given by the following constant values

$$\tau_u^*(t) = -\rho\omega^2\cos\theta, \quad \tau_r^*(t) = 0 \tag{4.8}$$

Notice that for the chosen trajectory, the nominal value of the quantity $\beta u + \tau_u$, appearing

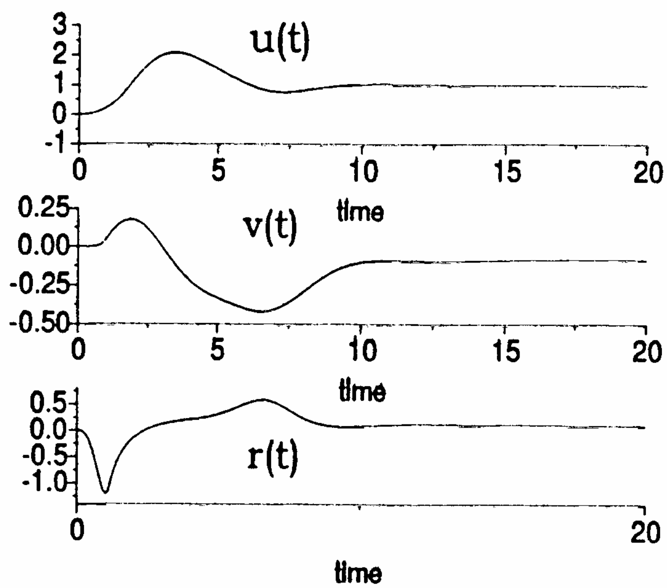


*Figure 4.* Feedback controlled velocity variables for circular path tracking.

in the denominator of the controller expression for $\tau_r$, is given by

$$\beta u + \tau_u = -\rho\omega(\omega\cos\theta + \beta\sin\theta) = -\rho\omega\sqrt{\rho^2 + \omega^2} \neq 0$$

We have chosen the following parameters for the reference trajectory, the system (with the same parameters previously used for the tracking error feedback controller)

$$\rho = 5, \quad \omega = 0.1, \quad \beta = 1.2$$

which result in $\theta = 1.487$ rad, $\tau_u = -4.18 \times 10^{-3}$

Figure 3 depicts the controlled evolution of the hovercraft position coordinates when the vessel motions are started significantly far away from the desired trajectory. In this case we set; $x(0) = 0.7$, $y(0) = 0$, $\varphi(0) = 1.3$ [rad]. Figure 4 shows the corresponding surge, and sway velocities as well as the yaw angular velocity. Figure 5 contains the angular position evolution and the applied external inputs.

### *4.2.1. Robustness with Respect to Unmodeled Perturbations*

In order to test the robustness of the proposed controller, used for the circular path maneuver, we introduced in the non-actuated dynamics (i.e., in the sway acceleration equation) an

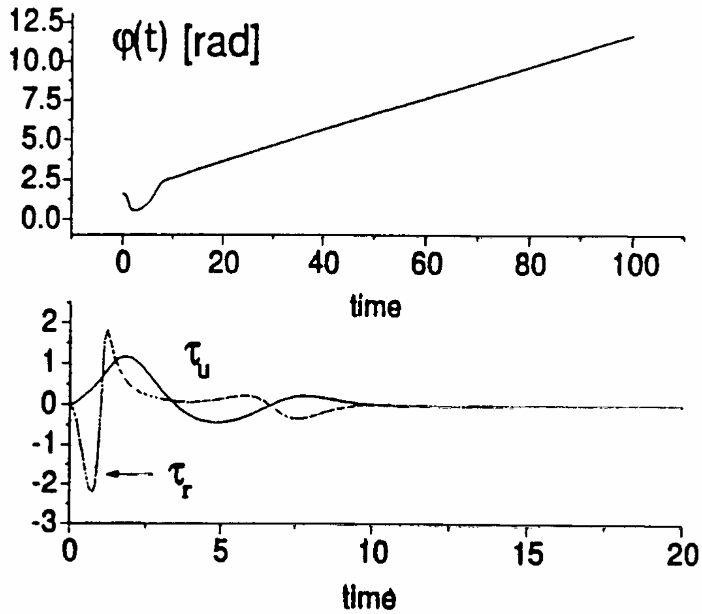


*Figure 5.* Feedback controlled angular orientation and applied inputs for circular path tracking.

unmodeled external perturbation force, simulating a "wave field" effect, of the form

$$\lambda(x) = A\left[\sin(fx) + \frac{1}{5}\cos(\pi fx)\right], \quad (\, \dot{v} = -ur - \beta v + \lambda(x) \,)$$

with $A = 0.6$ and $f = 10$. The results of the simulation are shown in Figure 6.

## 5. Conclusions

In this article, we have shown that the underactuated hovercraft system model, derived through some simplifying assumptions from the general surface vessel model, is differentially flat. This property immediately allows to establish the equivalence of the model, by means of dynamic state feedback, to a set of two decoupled controllable linear systems. A trajectory planning, combined with trajectory tracking error dynamic feedback linearization, allows to obtain a direct feedback controller synthesis for arbitrary position trajectory following. The design was shown to be robust with respect to significant perturbation forces affecting the non actuated dynamics.

The hovercraft system model is specially suitable for passivity based feedback control, as already remarked by Fossen [5] and, indirectly, carried out in [9], from a Lyapunov stability theory based control strategy. A fact that can be suitably exploited is that the hovercraft model can be placed in Generalized Hamiltonian form. The combination of differential flatness and total energy managing strategies may conveniently result in a simple and efficient feedback control option.

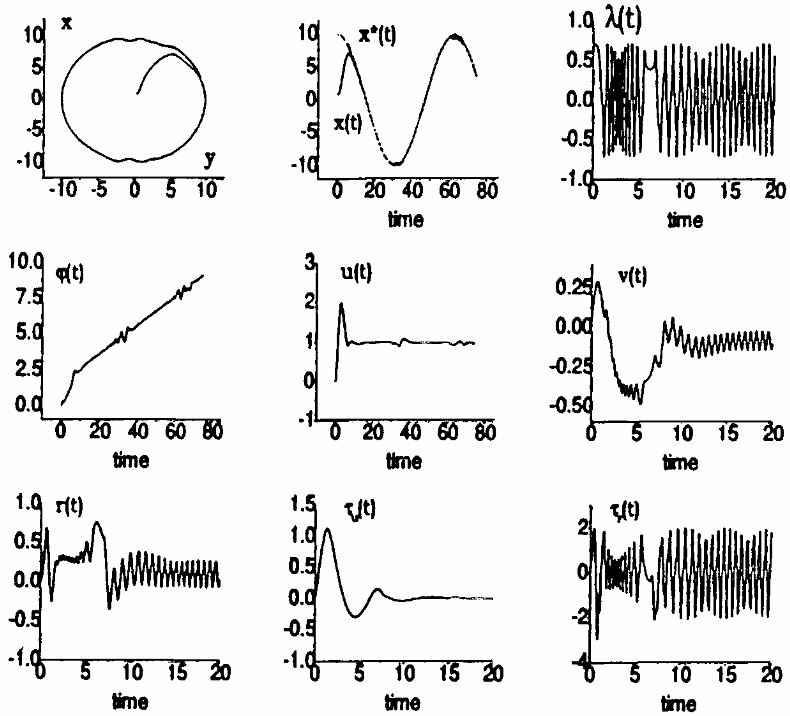


*Figure 6.* Circular path tracking performance under unmodeled sustained perturbations.

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