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Sliding Mode Control of Nonlinear Mechanical Vibrations[1]

In this article we illustrate how the property of differential flatness can be advantageously joined to the sliding mode controller design methodology for the active stabilization of nonlinear mechanical vibration systems. The proposed scheme suitably combines off-line trajectory planning and an on-line "smoothed" sliding mode feedback trajectory tracking scheme for regulating the evolution of the flat output variables toward the desired equilibria. [S0022-0434(00)00404-4]

1 Introduction

Differential flatness is a useful structural property exhibited by many nonlinear systems of practical, or realistic, significance. The theoretical background of "flatness" has been established in several articles by Prof. M. Fliess and his colleagues (see Fliess et al. [1,2]) from the viewpoints of differential algebra and Lie-Bäcklund transformations. A system is *flat* if it can be completely differentially parameterized by a set of state functions (called the flat outputs) which are differentially independent and equal, in number, to the control inputs. This property not only greatly facilitates the feedback controller design task, from any particularly desired synthesis methodology (passivity, back-stepping, feedback linearization, etc), but it is also an interesting analysis tool (see Sira-Ramírez [3]). Sliding mode control, on the other hand, represents a quite robust and simple approach which has enjoyed well gained respect and popularity over the years (see the book by Utkin [4]). In this article, we use, in the context of a nontrivial nonlinear multivariable example, the sliding mode feedback controller design option in suitable combination with the differential flatness property.

We deal with a nonlinear vibration mechanical system which is *differential flat*. This system has been studied in an article by Astolfi and Meini [5] and its linear version represents a frequently chosen example for the classical, frequency domain, analysis of mechanical systems and, also, for the active vibration control design in mechanical systems (see for instance the books by Thomson [6] and by Inman [7]). Our motivation stems from an exposition by Lévine, [8] in which active vibration damping is proposed as a fruitful application area of control theory and flatness. We proceed to design a feedback controller based on the advantageous combination of the intrinsic robustness properties of sliding mode control and the several conceptual advantages of differential flatness. We specifically propose a trajectory planning approach, which is natural for flat systems, for the active stabilization of the highly oscillatory, underactuated, nonlinear mechanical vibration system variables. The system may also be subject to persistent and significant unmodeled external perturbations. The proposed scheme suitably combines: 1) the flatness property of the nonlinear multivariable system allowing a useful differential parameterization of all system variables and control inputs, directly leading to specifications of feasible equilibria and nominal control inputs; 2) an off-line trajectory planning for position stabilization, using an intermediate resting equilibrium which effectively introduces damping to all free, uncontrolled, system vibrations; 3) an on-line "smoothed-switch" sliding mode feedback trajectory tracking controller for regulating the evolution of the flat output variables from the achieved intermediate equilibria toward the final desired resting equilibria located at the origin of coordinates.

Section 2 presents the model of the nonlinear mechanical vibration system and demonstrates the flatness of the system. Section 3 develops the exact linearization based feedback controller and argues its robustness. A sliding mode feedback control scheme is then proposed which is based on the differential flatness of the system. We carry out an off-line trajectory planning which solves the stabilization problem in two stages: An initial, clutched, intermediate stabilization and a smooth transition toward the resting desired equilibrium. Thanks to the flatness property, we show that the proposed sliding mode controller is robust with respect to external additive perturbations even if they directly act on the non-actuated mass. Section 4 presents some simulation results testing the performance of the designed controller with respect to the control objective and proving the robustness of the feedback scheme to unmodeled sustained oscillatory perturbations. The last section is devoted to conclusions and suggestions for further research.

2 The Nonlinear Mechanical Vibration System

2.1 A Nonlinear Mass-Spring System. Consider the mechanical system, shown in Fig. 1, constituted by three identical blocks, of mass m, and three identical nonlinear springs. The system is acted upon by two independent forces, denoted by u_1 and u_2, directly pushing the first and the second blocks. We assume, following Astolfi and Menini [5] from where the example is taken, that the coupling springs are nonlinear springs, characterized by the following nonlinear "deformation to applied tension" static relationship:

$$T(x)=kx+k_p x^3$$

with k and k_p being known constants.

The kinetic and potential energies of the system are given by,

$$\mathcal{T}(\dot q)=\frac{1}{2}m(\dot q_1^2+\dot q_2^2+\dot q_3^2)$$

$$\mathcal{V}(q)=\frac{1}{2}k[q_1^2+(q_2-q_1)^2+(q_3-q_2)^2]+\frac{1}{4}k_p[q_1^4+(q_2-q_1)^4+(q_3-q_2)^4]$$

Application of the Euler-Lagrange formalism, leads to the following multivariable nonlinear controlled system,

$$m\ddot q_1=k(-2q_1+q_2)+k_p[-q_1^3+(q_2-q_1)^3]+u_1$$

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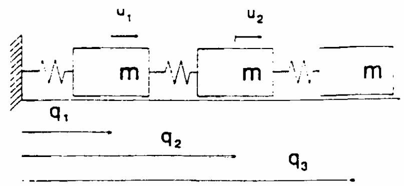

Fig. 1 Nonlinear mechanical vibration system

$$m\ddot{q}_2=k(q_1-2q_2+q_3)+k_p[(q_3-q_2)^3-(q_2-q_1)^3]+u_2 \tag{2.1}$$

$$m\ddot{q}_3=k(q_2-q_3)+k_p(q_2-q_3)^3$$

2.2 Flatness of the Underactuated Mechanical System. The system (2.1) is differentially flat with flat outputs given by the displacement coordinates of the first and the third mass: (q_1, q_3), which we express, respectively, as (F, L). In general, flatness means that all system variables (i.e., states and control inputs) can be differentially parameterized in terms of the independent set of flat outputs and a finite number of their time derivatives. The number of flat outputs is equal to the number of inputs.

Indeed, under the assumption of perfect knowledge of, $q_3=L$, the last equation in (2.1) actually represents a *reduced* cubic algebraic equation from where the second mass position coordinate q_2 can be obtained. The only real root of such a cubic equation is readily obtained as

$$q_2=L+\frac{1}{6k_p}\left[k_p^2\left(108\,m\ddot{L}+12\sqrt{3\left(\frac{4k^3+27m^2k_p(\ddot{L})^2}{k_p}\right)}\right)\right]^{1/3}-2k\left[k_p^2\left(108\,m\ddot{L}+12\sqrt{3\left(\frac{4k^3+27m^2k_p(\ddot{L})^2}{k_p}\right)}\right)\right]^{-1/3} \tag{2.2}$$

Notice that the differentially parameterized expression for q_2, in (2.2), implies that its second time derivative, $\ddot{q}_2$, can be expressed as a function denoted by: $\phi(\ddot{L}, L^{(3)}, L^{(4)})$. The control inputs, u_1 and u_2, can thus be also parameterized in terms of differential functions of F and L as

$$u_1=m\ddot{F}-k(-2F+q_2(L,\ddot{L}))+k_p[F^3-(q_2(L,\ddot{L})-F)^3] \tag{2.3}$$

$$u_2=m\ddot{L}+m\phi(\ddot{L},L^{(3)},L^{(4)})-k(F-q_2(L,\ddot{L}))-k_p(F-q_2(L,\ddot{L}))^3$$

Therefore, *all* system variables are expressible as *differential functions* of the flat outputs. Some system properties may be obtained from the analysis of such a differential parameterization.

2.2.1 Parameterization of Equilibria. Let $F=\bar{F}$ and $L=\bar{L}$ be constant equilibrium values for the flat outputs F and L, respectively. The expressions for the corresponding equilibrium values for the second mass displacement, $q_2=\bar{q}_2$, and the control input forces, $u_1=\bar{u}_1$, $u_2=\bar{u}_2$, can be directly found from (2.2) and (2.3). One obtains

$$\bar{q}_1=\bar{F},\quad \bar{q}_2=\bar{L},\quad \bar{q}_3=\bar{L}$$

$$\bar{u}_1=-\bar{u}_2+k\bar{F}+k_p\bar{F}^3 \tag{2.4}$$

$$\bar{u}_2=-k(\bar{F}-\bar{L})-k_p(\bar{F}-\bar{L})^3$$

Notice that for the particular equilibria: $\bar{F}=0$, $\bar{L}=0$, we have $\bar{q}_1=\bar{q}_2=\bar{q}_3=0$, and also $\bar{u}_1=\bar{u}_2=0$.

3 Sliding Mode Feedback Controller Based on Flatness

3.1 Flatness and Exact Feedback Linearization. Differential flatness is naturally related to exact feedback linearization and it also allows for a rather direct means of trajectory planning in nonlinear control problems. We illustrate this point by first deriving the exact linearization controller.

Let $s^4+\gamma_4s^3+\gamma_3s^2+\gamma_2s+\gamma_1$ and $s^2+\alpha_2s+\alpha_1$ be two *Hurwitz* polynomials i.e., they have all its roots in the open left half of the complex plane. Given a set of desired trajectories $F^*(t)$ and $L^*(t)$ for the flat outputs, a multivariable nonlinear feedback controller, based on exact tracking error linearization, may be synthesized as

$$u_1=m[\ddot{F}^*-\alpha_2(\dot{F}-\dot{F}^*)-\alpha_1(F-F^*)]-k(-2F+q_2(L,\ddot{L}))+k_p[F^3-(q_2(L,\ddot{L})-F)^3]$$

$$u_2=m\ddot{L}+m\phi(\ddot{L},L^{(3)},\Theta)-k(F-q_2(L,\ddot{L}))-k_p(F-q_2(L,\ddot{L}))^3$$

with

$$\Theta=(L^*)^{(4)}-\gamma_4(L^{(3)}-(L^*)^{(3)})-\gamma_3(\ddot{L}-\ddot{L}^*)-\gamma_2(\dot{L}-\dot{L}^*)-\gamma_1(L-L^*)$$

The use of this controller yields the following set of globally asymptotically stable closed-loop dynamics for the trajectory tracking errors, $e_L=L-L^*$ and $e_F=F-F^*$.

$$\ddot{e}_F+\alpha_2\dot{e}_F+\alpha_1e_F=0$$

$$e_L^{(4)}+\gamma_4e_L^{(3)}+\gamma_3\ddot{e}_L+\gamma_2\dot{e}_L+\gamma_1e_L=0$$

The above controller, although quite natural and of simple conception, is, nevertheless, quite sensitive to plant parameter variations and unmodelled external perturbation inputs. For these reasons, we resort to a sliding mode controller which is known to be quite robust with respect to these two classes of perturbations.

3.2 Sliding Mode Controller Design. Define two sliding surfaces as follows

$$s_F=\dot{F}-\dot{F}^*(t)+\lambda(F-F^*(t))=\dot{e}_F+\lambda e_F \tag{3.1}$$

$$s_L=L^{(3)}-(L^*(t))^{(3)}+\beta_3(\ddot{L}-\ddot{L}^*(t))+\beta_2(\dot{L}-\dot{L}^*(t))+\beta_1(L-L^*(t))=e_L^{(3)}+\beta_3\ddot{e}_L+\beta_2\dot{e}_L+\beta_1e_L$$

where λ and the set of real coefficients, $\{\beta_3,\beta_2,\beta_1\}$, constitute a set of design parameters to be suitably chosen.

By forcing the two sliding surface coordinates functions, s_F and s_L, to satisfy the discontinuous closed loop dynamics:

$$\dot{s}_F=-W_F\,\mathrm{sign}(s_F),\quad W_F>0 \tag{3.2}$$

$$\dot{s}_L=-W_L\,\mathrm{sign}(s_L),\quad W_L>0$$

where "sign" stands for the *signum* function, one then obtains the following sliding mode multivariable feedback controller:

$$u_1=m\Theta_F-k(-2F+q_2(L,\ddot{L}))+k_p[F^3-(q_2(L,\ddot{L})-F)^3] \tag{3}$$

$$u_2=m\ddot{L}+m\phi(\ddot{L},L^{(3)},\Theta_L)-k(F-q_2(L,\ddot{L}))-k_p(F-q_2(L,\ddot{L}))^3$$

with,

$$\Theta_F=\ddot{F}^*(t)-\lambda(\dot{F}-\dot{F}^*(t))-W_F\ \mathrm{sign}(s_F) \tag{3.4}$$

$$\Theta_L=(L^*)^{(4)}-\beta_3(L^{(3)}-(L^*)^{(3)})-\beta_2(\ddot{L}-\ddot{L}^*)-\beta_1(\dot{L}-\dot{L}^*)-W_L\ \mathrm{sign}(s_L)$$

... is a strictly positive constant, and let the *set of real constant coefficients*: $\{\beta_3, \beta_2, \beta_1\}$, *constitute a set of* Hurwitz *coefficients. Then, the nonlinear system* (2.1) *regulated by the feedback controller*: (3.3), (3.4), *with* s_F *and*, s_L, *given by* (3.1), *yields a globally asymptotically stable closed loop tracking error dynamics*,

$$\lim_{t\to\infty} e_F(t)=0,\quad \lim_{t\to\infty} e_L(t)=0$$

Proof.

The proof of this proposition is immediate upon realizing that the trajectories of the imposed closed loop sliding surface dynamics (3.2), reach the origin of the sliding surface space: $s_F=0$ and $s_F=0$, in a *finite amount of time*, which, incidentally, only depends upon the design parameters, W_F, W_L and the initial conditions for s_F and s_L. Moreover, the closed-loop motions of s_F and s_L stay, indefinitely, at the value zero. This implies that the ideal sliding motions of the tracking errors, $e_F=F-F^*$ and $e_L=L-L^*$, are governed by the globally asymptotically stable linear, time-invariant, set of decoupled dynamics:

$$\dot{e}_F+\lambda e_F=0$$

$$e_L^{(3)}+\beta_3\ddot{e}_L+\beta_2\dot{e}_L+\beta_1 e_L=0$$

The result follows.

□

The previous proposition is also valid when in the previously specified sliding mode controller, the following "high gain" approximation is used in place of the involved *signum* function:

$$\mathrm{sign}(s_F)\sim\frac{s_F}{|s_F|+\epsilon_F},\quad \mathrm{sign}(s_L)\sim\frac{s_L}{|s_L|+\epsilon_L}\tag{3.5}$$

with ϵ_F and ϵ_L being arbitrary but small positive constants and the notation "$|\cdot|$" denoting *absolute value*.

The substitution (3.5) is known to yield a closed loop response with rapid convergence to zero of the sliding surface coordinates s_F and s_L. However, it is also known that the corresponding controller is not as robust as the original sliding mode controller, with respect to unmodeled input and parameter perturbations.

3.3 Robustness With Respect to Unmodeled External Perturbations.

Consider the following perturbed model of the nonlinear mass-spring system

$$m\ddot{q}_1=k(-2q_1+q_2)+k_p[-q_1^3+(q_2-q_1)^3]+u_1+\xi_1(t)$$

$$m\ddot{q}_2=k(q_1-2q_2+q_3)+k_p[(q_3-q_2)^3-(q_2-q_1)^3]+u_2+\xi_2(t)\tag{3.6}$$

$$m\ddot{q}_3=k(q_2-q_3)+k_p(q_2-q_3)^3+\xi_3(t)$$

where $\xi_1(t)$, $\xi_2(t)$, and $\xi_3(t)$ represent uncertain external perturbations which are known to be absolutely bounded by known constants.

$$\sup|\xi_i(t)|\le X_i,\quad i=1,2,3.\tag{3.7}$$

An interesting feature of the flatness-based sliding mode approach lies in the fact that, in general, the effect of the additive uncertainties affecting the system will be "matched" with respect to the control input channels (i.e., they will belong to the range space of the control input matrix). The effect of the uncertain signals may be traced all the way to the closed-loop system equations for the tracking errors. These are given by

$$\ddot{e}_F+\lambda\dot{e}_F=\xi_1(t)+W_F\,\mathrm{sign}(s_F)$$

$$eL^{(4)}+\beta_3 e_L^{(3)}+\beta_2\ddot{e}_L+\beta_1\dot{e}_L=\xi_2(t)+\xi_3(t)+W_L\,\mathrm{sign}(s_L)\tag{3.8}$$

It is well known in sliding mode control theory that, by choosing $W_F>X_1$, and $W_L>X_1+X_2$, a sliding motion is guaranteed to exist on the sliding surfaces $s_F=0$ and $s_L=0$ for any realization of the perturbation inputs (see Utkin [4]).

We have then the following proposition:

Proposition 3.2. Under the assumptions (3.7) *on the perturbation signals affecting the nonlinear perturbed system*, (3.6). *The feedback controller*: (3.3), (3.4), *with* s_F *and* s_L, *given by* (3.1), *yields a globally asymptotically stable closed loop tracking error dynamics*,

$$\lim_{t\to\infty} e_F(t)=0,\quad \lim_{t\to\infty} e_L(t)=0$$

provided the sliding mode controller gains W_F *and* W_L *satisfy*,

$$W_F>X_1,\quad W_L>X_1+X_2.$$

4 Simulation Results

Simulations were performed on a nonlinear vibration mechanical system characterized by the following set of realistic parameters identified from an *ECP 210/210a Rectilinear Control System* workbench.

$$m=0.50\,[\mathrm{Kg}],\quad k=217.0\,[\mathrm{N/m}],\quad k_p=63.5\,[\mathrm{N/m^3}]$$

4.1 Control Objectives and Trajectory Planning. The controlled maneuvers were specified as follows: We let the system freely oscillate before a certain time, T_{Ci}. At this moment, we engage the feedback control actions, u_1 and u_2, by means of a "clutch," smoothly increasing the controls amplitudes from zero to its maximum value during a (small) time interval, $[T_{Ci}, T_{Cf}]$. The flat outputs references trajectories are planned so that they have constant nonzero reference equilibrium values, $\bar{F}$ and $\bar{L}$, for all times prior to a certain time T_1, i.e., in the interval, $(-\infty, T_1]$. The control engaging interval is necessarily contained in the infinite interval, i.e., $[T_{Ci}, T_{Cf}]\subset(-\infty, T_1]$. The clutched controllers are thus engaged to achieve, right after time T_{Cf}, asymptotic stabilization of the flat outputs towards the set of specified constant nonzero equilibrium values, $\bar{F}$ and $\bar{L}$. The first stage of the stabilization process, started at T_{Ci}, should not last beyond the time instant $T_1>T_{Cf}$ (see Fig. 2).

At time T_1, the final stabilization maneuver of the flat outputs toward zero is started. The controller proceeds to drive the flat outputs F and L to follow a sufficiently smooth, time-polynomial, trajectory connecting the achieved constant equilibria, $\bar{F}$ and $\bar{L}$, with the final rest value of zero for both flat output displacements. This last maneuver is specified to take place in the closed time

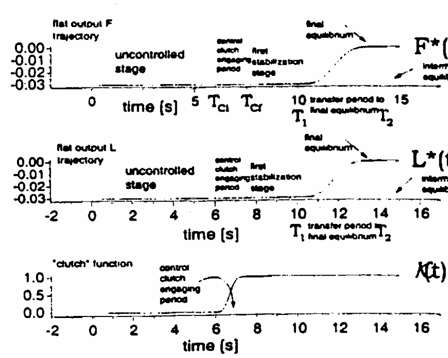

Fig. 2 A two stage stabilization process, via trajectory tracking with clutched control actions

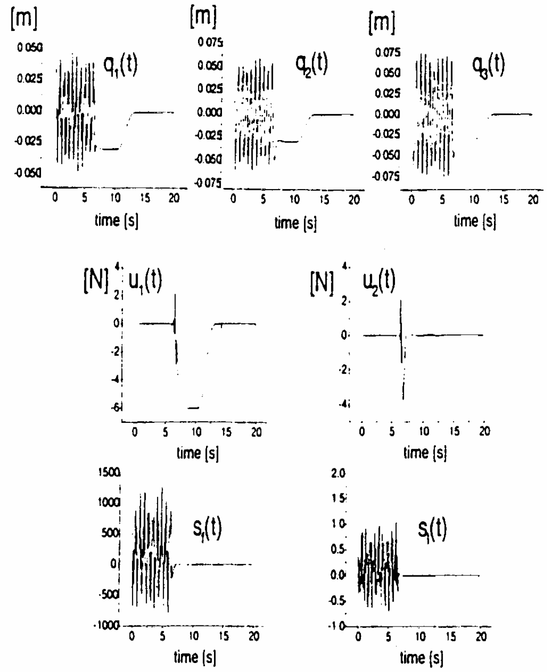

Fig. 3 Responses of sliding mode controlled mechanical vibration system

interval $[T_1, T_2]$. Evidently, the specified control objectives result in a final steady state equilibrium at zero of all the system displacements.

Figure 3 shows the closed-loop responses of the system's position variables q_1, q_2, and q_3, to the designed controller meeting the specified objectives. The figure also shows the applied input forces, $u_1(t)$, $u_2(t)$, and the corresponding sliding surfaces evolutions, $s_F(t)$, $s_L(t)$.

For the simulations shown in Fig. 3, we have chosen a Hurwitz polynomial $r(p)$, for the closed-loop characteristic polynomial of $e_L(t)$ given by the product: $r(p)=(p+b)(p^2+2\xi\omega_n p+\omega_n^2)$ with b, ξ, $\omega_n>0$. The controller design parameters were set to be,

$$\xi=0.8, \quad b=18, \quad w_n=20, \quad \lambda=5, \quad W_F=5, \quad W_L=20.$$

$$\epsilon_F=\epsilon_L=0.05$$

According to the described control objectives we specified the flat output trajectories as follows:

$$F^*(t)=\begin{cases} \bar{F} & \text{for } t\leq T_1 \\ \bar{F}[1-\psi_F(t,T_1,T_2)] & \text{for } T_1<t<T_2 \\ 0 & \text{for } t\geq T_2 \end{cases}$$

$$L^*(t)=\begin{cases} \bar{F} & \text{for } t\leq T_1 \\ \bar{F}[1-\psi_L(t,T_1,T_2)] & \text{for } T_1<t<T_2 \\ 0 & \text{for } t\geq T_2 \end{cases}$$

with $\psi_L(t,T_1,T_2)$ and $\psi_F(t,T_1,T_2)$ being sufficiently differentiable time functions satisfying $\psi_{L,F}(T_1,T_1,T_2)=0$ and $\psi_{L,F}(T_2,T_1,T_2)=1$. For the simulations we used polynomial splines of the Bézier type in order to have a sufficiently smooth transfer maneuver between the imposed temporary equilibrium value of the corresponding flat output and zero. For simplicity, the polynomial splines, $\psi_L(t,T_1,T_2)$ and $\psi_F(t,T_1,T_2)$, may be chosen to be identical.

The "clutches" engaging the designed controllers into full action were modeled as time varying factors, $\mathcal{K}_1(t)$ and $\mathcal{K}_2(t)$, multiplying the expressions of the feedback control inputs as $\mathcal{K}_1 u_1$ and $\mathcal{K}_2 u_2$. The clutches were also specified using polynomial splines of the Bézier type, which smoothly interpolated between the values of 0 and 1.

$$\mathcal{K}_{ci}=\begin{cases} 0 & \text{for } t\leq T_{Ci} \\ \psi_{ci}(t,T_{Ci},T_{Cf}) & \text{for } T_{Ci}<t<T_{Cf};\ i=1,2. \\ 1 & \text{for } t\geq T_{Cf} \end{cases}$$

with $\psi_{ci}(T_{Ci},T_{Ci},T_{Cf})=0$ and $\psi_{ci}(T_{Cf},T_{Ci},T_{Cf})=1$, for $i=1,2$. For simplicity, the time functions, $\psi_{ci}(t,T_{Ci},T_{Cf})$, $i=1,2$, were set to be identical for the two control inputs, and given by the following polynomial spline interpolating between 0 and 1.

$$\psi(t,T_{Ci},T_{Cf})=\left(\frac{t-T_{Ci}}{T_{Cf}-T_{Ci}}\right)^3\left[r_1-r_2\left(\frac{t-T_{Ci}}{T_{Cf}-T_{Ci}}\right)+r_3\left(\frac{t-T_{Ci}}{T_{Cf}-T_{Cf}}\right)^2-r_4\left(\frac{t-T_{Ci}}{T_{Cf}-T_{Ci}}\right)^3\right]$$

The constants $r_1,\ldots,r_4$ were suitably chosen to guarantee smooth departures and arrivals i.e., with enough time derivatives being equal to zero at the instants, T_{Ci} and T_{Cf}.

In the simulations shown in Fig. 3, we set: $T_{Ci}=6[s]$, $T_{Cf}=7.5[s]$, $T_1=10[s]$, $T_2=14[s]$. The intermediate equilibrium values of the flat outputs were chosen to be $\bar{F}=\bar{L}=-0.03[m]$.

4.2 Robustness Test. In order to test the robustness of the control scheme with respect to sustained unmodeled oscillatory perturbations (such as those obtained from an eccentric actuator) we used the developed sliding mode controller on the following perturbed version of the nonlinear mechanical system:

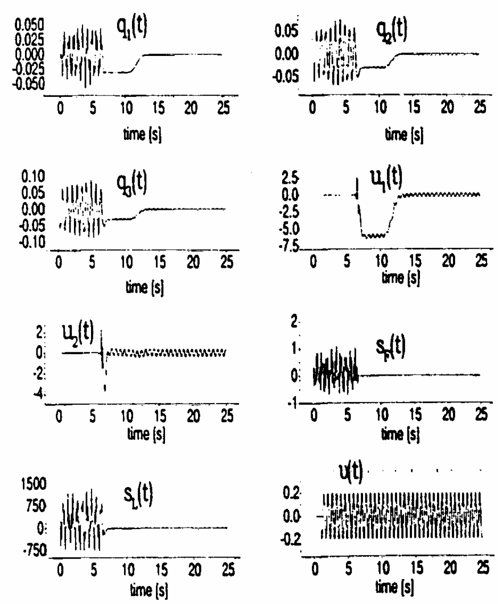

Fig. 4 Responses of sliding mode controlled perturbed mechanical vibration system

$$m\ddot{q}_2 = k(q_1 - 2q_2 + q_3) + k_p[(q_3 - q_2)^3 - (q_2 - q_1)^3] + u_2 + \nu(t) \tag{4.1}$$

$$m\ddot{q}_3 = k(q_2 - q_3) + k_p(q_2 - q_3)^3 + 0.2\nu(t)$$

with $\nu(t)$ being an unmodeled sinusoidal perturbation, of significant amplitude and a reasonable high frequency, affecting the dynamics of the flat output, F, and the dynamics of the unactuated (third) mass. This perturbation was set to be:

$$\nu(t) = 0.2 \sin(\omega t)$$

whose amplitude is approximately 3.2 percent of the maximum obtained amplitude value of the two acting forces and ω was set to be 10 [rad/s].

Figure 4 shows the performance of the designed controller to the significant unmodeled perturbations.

5 Conclusions

In this article, we have developed a multivariable sliding mode feedback control scheme for the regulation of a realistic undamped, underactuated, nonlinear mechanical vibration system. The sliding mode controller is designed on the basis of the flatness property exhibited by the nonlinear system. The feedback stabilization problem was approached by solving a trajectory tracking problem which proposes a two stage tracking process for achieving the final stabilization. First, the controller accomplishes an on line regulation of the freely oscillating system toward an intermediate, and rather convenient, set of constant equilibrium values for the system's flat output. Second, the initial stabilization stage is followed by a smooth planned transfer to the final rest equilibrium position, located at zero, of the flat outputs. In order to avoid excessive values, or amplitude saturations, of the applied control input forces, a "clutch mechanism" was also provided for the smooth engaging of the designed feedback control actions during the first stabilization stage.

The main advantage of the approach lies in the suitable combination of the flatness property and sliding mode control. Flatness allows for the identification of a special set of physically meaningful output variables, capable of parameterizing all system variables, including the inputs. The flat outputs are completely controllable in the sense that their dynamics leave no room for undesirable transient or permanent effects of the zero dynamics. The flatness property greatly facilitates the design task and naturally allows for a trajectory tracking approach to solve the stabilization problem. As an added bonus, the flatness of the system reduces the problem to controlling the two flat outputs under an external perturbation matching property. This guards all possible effects of ummodeled input perturbations through the non-actuated channel in the system. Sliding mode control, on the other hand, is a paradigmatic robust and simple feedback control scheme which provides a certain degree of robustness with respect to external signals and system parameter perturbations. A disadvantage of the approach, which can certainly be circumvented, lies in the use of state dependent expressions for the higher order derivatives of the flat outputs.

The demonstrated robustness property was also subject to a performance tests which included unmodeled persistent oscillatory perturbations. The designed controller managed to satisfactorily correct for the effects of these perturbations on the desired mass positions.

Even though the full feedback controller expression is rather complex, the simulated performance of the system is rather encouraging as to attempt actual experimental implementation.

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