Static and dynamic sliding mode control schemes for a permanent magnet stepper motor

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In this paper, the sliding mode control of a permanent magnet (PM) stepper motor is addressed from the perspective of differentially flat systems. Flat systems naturally allow for de-coupled linearization directly leading to static and dynamic discontinuous feedback control alternatives. Implementation results of the proposed sliding mode control schemes on an experimental set-up are given to illustrate the developments.

1. Introduction

Due to their inherent robustness properties and conceptual simplicity, discontinuous feedback control schemes of the sliding mode type have enjoyed deserved attention from researchers and practitioners. The fundamental results of sliding mode control are found in the many books already available on the subject. The interested reader is referred to the books by Emelyanov (1967), Itkis (1978), Utkin (1978, 1992), Slotine and Li (1991) and Zinober (1994). In the following paragraph, we provide a brief review of some of the contributions to the design of sliding mode controllers for non-linear multivariable systems.

Sliding mode control of non-linear multivariable systems has been addressed from different viewpoints. The method of the Hierarchy of Controls, introduced in Utkin (1978) addressed the design problem from a single-input single-output viewpoint by concentrating, at each stage of the design process, in the creation of a sliding regime on a particular sliding surface related to the scalar input under consideration. Sliding mode existence on such a particular surface was guaranteed by considering the rest of the control inputs in accordance with their relative position within an arbitrarily established hierarchy. They were regarded either as bounded perturbations, in their pursuit of their individual sliding mode creation attempts, or as equivalent (average) control inputs already causing partial sliding modes. This method was successfully applied to the control of robotic manipulators in the work of Young (1978). Utkin and Young (1978) proposed another method for multivariable sliding mode control design, constrained to the case of linear time-invariant systems. This method is applicable to systems *in regular canonical form* and it

is based on utilizing a subset of the state vector components as pseudo-control inputs in a linear optimal regulator control problem. The solution to the optimal control problem actually represents the desired sliding surface with prescribed optimality features. A rather complete solution to the design problem for linear time-invariant systems was given, from a geometric viewpoint, by El Ghezawi et al. (1983). The case of regulation of robotic manipulators renewed the interest in multivariable non-linear sliding mode controller design. In this context important contributions were given in the works by Slotine and Sastry (1983). The differential geometric approach to the design problem received the attention of Bartolini and Zolezzi (1986) and Sira-Ramirez (1988). A rather informative tutorial on sliding mode control in which the multivariable aspects of sliding mode control were clearly treated was given by DeCarlo et al. (1988). The work of Kwatny and Kim (1990) has provided a rather complete picture of the geometry of the multivariable sliding mode control problem. A different approach to the multivariable sliding mode control problem for linear systems was given by Fliess and Sira-Ramirez (1993). In this approach, module theory is used in a special manner to formulate and uncover the fundamental differential algebraic nature of the problem. Sira-Ramirez (1996) addressed the problem of sliding mode control of multivariable non-linear systems, from the perspective of linear differential algebra, for a special class of linearizable systems. Recently, Young et al. (1999) offered an assessment of the chattering phenomenon and provided a frame of reference for future sliding mode control research.

On another front of research, flat systems were first introduced by Fliess et al. (1922 a, b) and further developed and characterized in Fliess et al. (1995). Practical examples of some mechanical systems, such as the truck and the trailer, the jumping robot, and the crane were presented in Fliess et al. (1995). In Pomet et al. (1992), connections of flat systems with non-exact Brunovský canonical forms were established. Levine et al. (1996) used the flatness property of the magnetic levitation model of a beam to design a non-linear control scheme for its positioning. Martin and Rouchon (1996) and

Chelouah *et al.* (1996) proposed a sampling control strategy for flat systems; this strategy was applied to induction motors. Rothfuss *et al.* (1996) exploited the flatness of a chemical reactor model to design a linearizing quasi-static feedback controller for it.

Flat systems enjoy the property of possessing a finite set of differentially independent outputs called linearizing outputs. The fundamental characteristics which identify such linearizing outputs are three:

(1) the number of linearizing outputs is identical to the number of inputs of the system;

(2) all variables in the system, including the control input variables, can be written, exclusively, in terms of differential functions of such linearizing outputs; and

(3) the linearizing outputs can be expressed as differential functions of the system state vector components. This includes the possibility of having explicit dependence of the linearizing outputs on the control inputs and a finite number of their time derivatives.

Flat systems are thus control systems which are linearizable to controllable linear systems by means of endogenous feedback, i.e. one that does not require external variables to the system to be completely defined.

In this article we propose the design of sliding mode controllers for a permanent magnet stepper motor. The design approach is based on the developed theory of differentially flat systems. An advantageous combination of sliding mode control strategies and the natural de-coupled linearization properties, exhibited by differentially flat systems, may be conveniently exploited for stabilization and tracking problems. As a consequence, a multivariable robust feedback regulator design technique arises for non-linear controllable systems. The possibilities of designing static or dynamic sliding mode controllers largely becomes a matter of choice dictated by the physical attributes inherent to the nature of each control input variable to the system.

The paper is organized as follows. Section 2 contains a brief overview of stepper motors as well as the non-linear model of the PM stepper motor; in addition the stepper motor is shown to be a flat system. Section 3 deals with the design of a static sliding mode controller for a permanent magnet stepper motor. Section 4 deals with the design of a dynamic sliding mode controller for the stepper motor. Section 5 discusses the implementation results of the proposed control schemes; a brief comparison of the two control schemes is also provided. Section 6 contains the conclusion.

2. The stepper motor system

2.1. *A brief overview of stepper motors*

In recent years, the rapid growth of digital electronics has indirectly influenced the development of the stepper motor technology. This is the case because the 'digital machine properties' of the stepper motor allow it to be easily interfaced with any digital controller. Stepper motors are now widely used in numerous motion-control applications such as: robots, printers, process control systems, index table for automatic assembly machines, etc.

Compared to the conventional methods of using dc motors for motion control, stepper motors offer the following advantages (Kenjo and Sugawara 1994)

- Stepper motors are highly reliable. This is the case because stepper motors do not use brushes which are susceptible to mechanical failure.

- In a stepper motor, there is reduced need to dissipate heat from the rotor because there are no rotor windings.

- Stepper motors have high torque to inertia ratios.

- Since the rotor of the stepper motor is only supported by two bearings at the end of the housing, friction is low in a stepper motor.

- Stepper motors can be easily interfaced with digital systems.

Some disadvantages of stepper motor are:

- Generally, when operating in open loop mode, the response of a stepper motor has high overshoot and long settling time, especially when driving high inertia loads.

- Fixed angle of motion when the stepper motor is operated in open loop mode (i.e. micro-stepping is not possible when the stepper is operating in open loop mode).

Over the years, many control algorithms that can be used to improve the performance of stepper motors have been examined. Zribi and Chiasson (1991) developed an exact feedback linearization control method for controlling a permanent magnet stepper motor. In this method, the exact knowledge of the dynamics of the stepper motor system is required. Experiments using the exact linearization controller were carried out in Aiello *et al.* (1991). Since the dynamics of the stepper motor system may not be fully known, other methods of designing controllers were developed. Bodson and Chiasson (1989) and Bodson *et al.* (1993) presented a control scheme based on feedback linearization with adaptive rules that can be used to estimate the parameters of the system. Speagle and Dawson (1993) developed an

adaptive tracking controller for stepper motors; the proposed controller yields global uniform asymptotic stability results for the motor position tracking error and compensates for parametric uncertainties in the motor/load model. Most of the above mentioned schemes require full state feedback. However, in actual applications, this may not be possible or economical. Few controllers were therefore developed using partial state feedback. Speagle *et al.* (1993) developed a robust tracking control that does not require current measurement. When this controller is used, the system is able to achieve global uniform ultimate bounded stability results for the motor position tracking error. Burg *et al.* (1994) developed a global exponential position tracking controller using output feedback. However, this method requires the knowledge of the exact dynamics of the stepper motor.

2.2. *Model of the PM stepper motor*

Consider the following model of a permanent magnet stepper motor (see Zribi and Chiasson 1991)

$$
\left.\begin{aligned}
\frac{di_a}{dt} &= \frac{1}{L}(v_a - Ri_a + K_m\omega\sin(N_r\theta)) \\[2mm]
\frac{di_b}{dt} &= \frac{1}{L}(v_b - Ri_b - K_m\omega\cos(N_r\theta)) \\[2mm]
\frac{d\omega}{dt} &= \frac{1}{J}(-K_m i_a\sin(N_r\theta) + K_m i_b\cos(N_r\theta) - B\omega - \tau_L) \\[2mm]
\frac{d\theta}{dt} &= \omega
\end{aligned}\right\}
$$

$$(1)$$

where i_a is the current in winding A, i_b is the current in winding B, θ is the angular displacement of the shaft of the motor, ω is the angular velocity of the shaft of the motor, v_a is the voltage across the windings of phase A, and v_b is the voltage across the windings of phase B. Also, N_r is the number of rotor teeth, J is the rotor and load inertia, B is the viscous friction coefficient, L and R are the inductance and the resistance of each phase winding, K_m is the motor torque (back-emf) constant, and τ_L is the load torque.

Remark 1: The design of the sliding mode controllers will be performed without taking the load torque into account. However, it should be mentioned that it is a straightforward matter to design an observer to estimate the load torque of the motor; the interested reader is referred to the work of Zribi and Chiasson (1991).

Equations (1) which are used to described the stepper motor model, are highly non-linear. A non-linear transformation, known as the Direct-Quadrature (DQ)

transformation (see Zribi and Chiasson 1991) can be used to transform these equations into a form which is more suitable for designing non-linear controllers. This transformation changes the frame of reference from the fixed phase axes to axes which are moving with the rotor. The DQ transformation is defined as

$$
\begin{bmatrix} i_d \\ i_q \\ \omega \\ \theta \end{bmatrix} =
\begin{bmatrix}
\cos(N_r\theta) & \sin(N_r\theta) & 0 & 0 \\
-\sin(N_r\theta) & \cos(N_r\theta) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} i_a \\ i_b \\ \omega \\ \theta \end{bmatrix}
\tag{2}
$$

In addition, we define the new inputs as

$$
\begin{bmatrix} v_d \\ v_q \end{bmatrix} =
\begin{bmatrix}
\cos(N_r\theta) & \sin(N_r\theta) \\
-\sin(N_r\theta) & \cos(N_r\theta)
\end{bmatrix}
\begin{bmatrix} v_a \\ v_b \end{bmatrix}
\tag{3}
$$

where v_d is the direct voltage, v_q is the quadrature voltage, i_d is the direct current, and i_q is the quadrature current. In terms of these new inputs and new state variables, the transformed model of the PM stepper motor, without taking the load torque into account, can be written as

$$
\left.\begin{aligned}
\frac{di_d}{dt} &= \frac{1}{L}(v_d - Ri_d + N_r\omega L i_q) \\[2mm]
\frac{di_q}{dt} &= \frac{1}{L}(v_q - Ri_q - N_r\omega i_d - K_m\omega) \\[2mm]
\frac{d\omega}{dt} &= \frac{1}{J}(K_m i_q - B\omega) \\[2mm]
\frac{d\theta}{dt} &= \omega
\end{aligned}\right\}
\tag{4}
$$

One obvious advantage of using the DQ transformation is that the cosine and sine functions in (1) have been eliminated. Let

$$
x_1 = i_d, \quad x_2 = i_q, \quad x_3 = \omega, \quad x = [x_1\ x_2\ x_3\ x_4]^T
$$

$$
k_1 = \frac{R}{L}, \quad k_2 = \frac{K_m}{L}, \quad k_3 = \frac{K_m}{J}, \quad k_4 = \frac{B}{J}
$$

$$
k_5 = N_r, \quad u_1 = \frac{v_d}{L}, \quad u_2 = \frac{v_q}{L}
$$

Then, equations (4) can be written as

$$
\left.\begin{aligned}
\dot{x}_1 &= -k_1 x_1 + k_5 x_2 x_3 + u_1 = f_1 + u_1 \\[2mm]
\dot{x}_2 &= -k_1 x_2 - k_5 x_1 x_3 - k_2 x_3 + u_2 = f_2 + u_2 \\[2mm]
\dot{x}_3 &= k_3 x_2 - k_4 x_3 \\[2mm]
\dot{x}_4 &= x_3
\end{aligned}\right\}
\tag{5}
$$

where

$$f_1 = -k_1 x_1 + k_5 x_2 x_3 \qquad (6)$$

$$f_2 = -k_1 x_2 - k_5 x_1 x_3 - k_2 x_3 \qquad (7)$$

and, x_1 is the direct current, x_2 is the quadrature current, x_3 is the angular velocity of the shaft of the motor, and x_4 is the angular position of the shaft of the motor. It should be noted that the model of the stepper motor, after the change of variables, can be written only with differential polynomials.

It is desired that the rotor angular position tracks a given constant reference position θ_d. It is also desired that the direct current tracks a given constant reference direct current, I_{dd}. Therefore, the linearizing as well as the controlled outputs of the system are such that

$$z = y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \qquad (8)$$

In the following, we consider the design of a multivariable sliding mode controller for a permanent magnet stepper motor. The system is first shown to be differentially flat with linearizing outputs given by physically meaningful variables constituted by a combination of the currents in phases A and B of the motor, and the rotor angular position of the shaft of the motor. Static and dynamic sliding mode controllers which stabilize the system to the required equilibrium point are then designed.

2.3. *Flatness of the PM stepper motor*

Within the differential algebraic approach to non-linear control systems, extensively developed by Fliess *et al.* (1992 a, b, 1995) a control system is defined in the following manner.

Definition 1: A system is a finitely generated differential field extension D/k, where k is the ground field, here taken to be the field of real numbers. An input is a finite set $u = \{u_1, \ldots, u_m\}$ of D such that the differential field extension $D/k\langle u \rangle$ is differentially algebraic. The control thus qualifies to be a transcendence basis of D/k.

The following definition gives a formal definition of differentially flat systems.

Definition 2: Consider a system D/k. This system is said to be differentially flat if there exists a differential transcendence basis $y = \{y_1, \ldots, y_m\}$ of a finitely generated differential field extension $\hat{D}/k$ such that the field extensions $\hat{D}/k\langle y \rangle$ and $\hat{D}/D$ are non-differentially algebraic.

The differential transcendence basis $y = \{y_1, \ldots, y_m\}$ constitutes a set of linearizing outputs of the system. These output variables depend on the state of the system

and, possibly, on the control inputs u and a finite number of their time derivatives.

It is found that the stepper motor system is a differentially flat system with linearizing output coordinates given by $y_1 = x_1$ and $y_2 = x_4$. This is the case because all variables in the system can be written as differential functions of the linearizing outputs such that,

$$\left. \begin{array}{l} x_1 = y_1 \\[2mm] x_2 = \dfrac{1}{k_3}(\dot{x}_3 + k_4 x_3) = \dfrac{1}{k_3}(\ddot{y}_2 + k_4 \dot{y}_2) \\[2mm] x_3 = \dot{y}_2 \\[2mm] x_4 = y_2 \end{array} \right\} \qquad (9)$$

In addition, the control actions can be expressed using the linearizing output coordinates such that

$$u_1 = \dot{y}_1 + k_1 y_1 - \frac{k_5}{k_3}(\ddot{y}_2 + k_4 \dot{y}_2)\dot{y}_2 \qquad (10)$$

$$u_2 = \frac{1}{k_3}(y_2^{(3)} + k_4 \ddot{y}_2) + \frac{k_1}{k_3}(\ddot{y}_2 + k_4 \dot{y}_2) + k_5 y_1 \dot{y}_2 + k_2 \dot{y}_2 \qquad (11)$$

Since the control inputs to the system u_1 and u_2 are differential functions of the linearizing outputs $y_1 = x_1$ and $y_2 = x_4$, then one may impose on the highest derivatives of such linearizing output components a particular linear relation involving only smaller order derivatives of the same output component. One immediately obtains the required linearizing controller expression in terms of the involved linearizing outputs. This process may implicitly entitle either dynamic or static feedback control. However, the fact remains that the definition of the control inputs does not require any variable which is external to the system.

From the expressions of u_1 and u_2 of the stepper motor system, it follows that the linearized equations for the system are simply given by

$$\dot{y}_1 = v_1 \qquad y_2^{(3)} = v_2 \qquad (12)$$

In the following section, a multivariable sliding mode controller which asymptotically regulates the output variables towards their desired equilibrium positions is proposed.

3. A static sliding mode controller for the stepper motor

The static sliding mode control design entitles the specification of the auxiliary, endogenous, control input variables v_1 and v_2 in (12) as sliding mode feedback control laws, such that the forced evolution of the linearized variables asymptotically converge towards

their desired positions. Due to the physical nature of the actual control input variables u_1 and u_2, representing voltage signals for which a switching strategy is entirely feasible, we will proceed to specify a static sliding mode controller.

A sliding surface for the tracking of the linearizing output coordinate y_1 towards its equilibrium point I_{dd}, is constituted by the direct current stabilization error given by

$$s_1 = y_1 - I_{dd} = x_1 - I_{dd} \qquad (13)$$

For the regulation of the second linearizing coordinate, $y_2 = x_4$, a sliding surface expression is proposed which depicts a desired second order dynamic response for the controlled angular position $y_2 = x_4$, towards its desired equilibrium point θ_d. We propose that

$$s_2 = \ddot{y}_2 + \alpha_1 \dot{y}_2 (y_2 - \theta_d)$$
$$= k_3 x_2 - k_4 x_3 + \alpha_1 x_3 + \alpha_2 (x_4 - \theta_d) \qquad (14)$$

where α_1 and α_2 are positive design parameters.

We impose the following sliding mode controlled dynamics on the evolution of the sliding surface coordinate functions s_1 and s_2

$$\dot{s}_1 = -W_1 \operatorname{sgn}(s_1) \qquad (15)$$

$$\dot{s}_2 = -W_2 \operatorname{sgn}(s_2) \qquad (16)$$

where W_1 and W_2 are positive design parameters, and 'sgn' denotes the signum function. Using (13) and (14), the sliding mode dynamics in (15) and (16) yield the following required dynamics of the linearizing output coordinates y_1 and y_2,

$$\dot{y}_1 = -W_1 \operatorname{sgn}(s_1) \qquad (17)$$

$$y_2^{(3)} + \alpha_1 \ddot{y}_2 + \alpha_2 \dot{y}_2 = -W_2 \operatorname{sgn}(s_2) \qquad (18)$$

Note that these equations guarantee sufficient regularity for y_1 and y_2. Using (12)–(18), one immediately obtains the auxiliary control inputs v_1 and v_2 as

$$v_1 = -W_1 \operatorname{sgn}(s_1) \qquad (19)$$

$$v_2 = -\alpha_1 \ddot{y}_2 - \alpha_2 \dot{y}_2 - W_2 \operatorname{sgn}(s_2) \qquad (20)$$

Using (10), (11), (17) and (18) and because of the flatness of the system, one can immediately compute the required actual control inputs u_1 and u_2

$$u_1 = -W_1 \operatorname{sgn}(s_1) + k_1 y_1 - \frac{k_5}{k_3} y_2 (\ddot{y}_2 + k_4 \dot{y}_2) \qquad (21)$$

$$u_2 = \frac{1}{k_3}[(k_1 + k_4 - \alpha_1)\ddot{y}_2 + (k_1 k_4 - \alpha_2)\dot{y}_2$$
$$- W_2 \operatorname{sgn}(s_2)] + k_5 y_1 \dot{y}_2 + k_2 \dot{y}_2 \qquad (22)$$

The obtained static sliding mode control inputs can be expressed in terms of the original state variables of the system (5) by simply substituting the linearizing coordinates y_1 and y_2, and their time derivatives, in terms of the original state variables. After some straightforward algebraic manipulations, we obtain the following expressions for the static discontinuous feedback controllers

$$u_1 = -W_1 \operatorname{sgn}(s_1) + k_1 x_1 - k_5 x_2 x_3 \qquad (23)$$

$$u_2 = k_1 x_2 + k_5 x_1 x_3 + k_2 x_3 + k_4 x_2 - \alpha_1 x_2$$
$$- \frac{1}{k_3}[k_4^2 x_3 - \alpha_1 k_4 x_3 + \alpha_2 x_3 + W_2 \operatorname{sgn}(s_2)] \qquad (24)$$

The previous analysis allows us to state the following theorem. Let

$$s_1 = x_1 - I_{dd}$$

$$s_2 = k_3 x_2 - k_4 x_3 + \alpha_1 x_3 + \alpha_2 (x_4 - \theta_d)$$

Theorem 1: *The following discontinuous static feedback controller when applied to the stepper motor system* (5)

$$u_1 = -W_1 \operatorname{sgn}(s_1) + k_1 x_1 - k_5 x_2 x_3$$

$$u_2 = k_1 x_2 + k_5 x_1 x_3 + k_2 x_3 + k_4 x_2 - \alpha_1 x_2$$

$$- \frac{1}{k_3}[k_4^2 x_3 - \alpha_1 k_4 x_3 + \alpha_2 x_3 + W_2 \operatorname{sgn}(s_2)]$$

asymptotically stabilizes the outputs of the system to their desired values.

Proof: Taking the derivative of (13) and (14) with respect to time and using (5), (7), (23) and (24), one would obtain

$$\dot{s}_1 = -W_i \operatorname{sgn}(s_i) \qquad (i = 1, 2) \qquad (25)$$

The trajectories associated with the unforced discontinuous dynamics (25) exhibit a finite time reachability to zero from any given initial condition $s_{i(0)}$ ($i = 1, 2$) provided that the constant gains W_i ($i = 1, 2$) are chosen to be strictly positive. Thus, the closed loop dynamics in (25) corresponding to $i = 1$, guarantees finite time reachability of the desired motor current equilibrium value I_{dd}. The closed loop dynamics in (25) corresponding to $i = 2$, implies a second order controlled response for the angular position x_4. Since s_2 is driven to zero in finite time, the linearizing output $y_2 = x_4$ is governed, after such a finite amount of time, by the second order dynamics $\ddot{y}_2 + \alpha_1 \dot{y}_2 + \alpha_2 (y_2 - \theta_d) = 0$. The output y_2 will then converge to its desired value θ_d because α_1 and α_2 are positive. Hence the static sliding mode controller guarantees the asymptotic convergence of the outputs to their desired values. □

The next section illustrates the design of a dynamic sliding mode controller for the PM stepper motor.

4. A dynamic sliding mode controller for the stepper motor

From simulation results, chattering is observed when the sliding mode controller designed in the previous section is used. The chattering is due to the basic assumption in variable structure control that control can be switched from one value to another at any moment and with almost zero time delay. However, in practical systems, it is not easy to achieve such switching control (Young *et al.* 1999). This is the case because of two reasons. Firstly, there are time delays due to computations of the control actions. Secondly, there are physical limitations on the actuators used in the plants. It should also be mentioned that at steady state, chattering might cause oscillations about the desired equilibrium point of the system; these high frequency oscillations may excite the unmodelled high frequency dynamics of the system. In order to reduce the chattering induced on the linearizing coordinates y_1 and y_2 by the bang-bang nature of the proposed static sliding mode control signals u_1 and u_2, we propose a dynamic sliding mode control scheme. It should be noted that dynamic sliding mode controllers have been used previously to attenuate chattering; for example see the work by Sira-Ramirez *et al.* (1994).

Using (5)–(8), we can write

$$\ddot{y}_1 = -k_1(f_1 + u_1) + k_5(f_2 + u_2)x_3$$
$$+ k_5(k_3 x_2 - k_4 x_3)x_2 + \dot{u}_1$$
$$= f_3 + \dot{u}_1 \qquad (26)$$

where

$$f_3 = -k_1(f_1 + u_1) + k_5(f_2 + u_2)x_3 + k_5(k_3 x_2 - k_4 x_3)x_2 \qquad (27)$$

Similarly

$$y_2^{(4)} = -(k_1 k_3 + k_3 k_4)(f_4 + u_2) - k_3 k_5(f_1 + u_1)x_3$$
$$- (k_3 k_5 x_1 + k_2 k_3 - k_4^2)(k_3 x_2 - k_4 x_3) + k_3 \dot{u}_2$$
$$= f_4 + k_3 \dot{u}_2 \qquad (28)$$

where

$$f_4 = -(k_1 k_3 + k_3 k_4)(f_2 + u_2) - k_3 k_5(f_1 + u_1)x_3$$
$$- (k_3 k_5 x_1 + k_2 k_3 - k_4^2)(k_3 x_2 - k_4 x_3) \qquad (29)$$

A sliding surface for the tracking of the linearizing output coordinate y_1 towards its equilibrium point I_{dd}, is chosen as

$$\sigma_1 = \dot{y}_1 + \lambda'(y_1 - I_{dd})$$
$$= -k_1 x_1 + k_5 x_2 x_3 + u_1 + \lambda'(x_1 - I_{dd}) \qquad (30)$$

where I_{dd} is the desired value of I_d and λ' is a positive design parameter.

For the regulation of the second linearizing coordinate, $y_2 = x_4$, a sliding surface expression is proposed which depicts a desired third order dynamic response for the controlled angular position $y_2 = x_4$, towards its desired equilibrium point $x_4 = \theta_d$. We choose

$$\sigma_2 = y_2^{(3)} + \alpha_1' \ddot{y}_2 + \alpha_2' \dot{y}_2 + \alpha_3'(y_2 - \theta_d)$$
$$= -k_1 k_3 x_2 - k_3 k_5 x_1 x_3 - k_2 k_3 x_3 + k_3 u_2$$
$$+ (\alpha_1' - k_4)(k_3 x_2 - k_4 x_3) + \alpha_2' x_3 + \alpha_3'(x_4 - \theta_d)$$
$$\qquad (31)$$

where θ_d is the desired value of $\theta = x_4$; the constants α_1', α_2' and α_3' are chosen such that the polynomial $s^3 + \alpha_1' s^2 + \alpha_2' s + \alpha_3'$ is Hurwitz. We impose the following sliding mode controlled dynamics on the evolution of the sliding surface coordinate functions σ_1 and σ_2

$$\dot{\sigma}_1 = -W_1' \operatorname{sgn}(\sigma_1) \qquad (32)$$

$$\dot{\sigma}_2 = -W_2' \operatorname{sgn}(\sigma_2) \qquad (33)$$

where W_1' and W_2' are positive design parameters.

Using (26)–(33) and after some manipulations, we obtain

$$\dot{u}_1 = -f_3 - \lambda'(f_1 + u_1) - W_1' \operatorname{sgn}(\sigma_1) \qquad (34)$$

$$\dot{u}_2 = \frac{1}{k_3}[-f_4 - \alpha_1'(k_3 f_2 - k_3 k_4 x_2 + k_4^2 x_3 + k_3 u_2)$$
$$- \alpha_2'(k_3 x_2 - k_4 x_3) - \alpha_3' x_3 - W_2' \operatorname{sgn}(\sigma_2)] \qquad (35)$$

The equations in (34) and (35) constitute the equations for a dynamic sliding mode feedback controller. Such a controller is then characterized by the solutions of the underlying differential equations for the control inputs u_1 and u_2. Indeed, the previous equations may be immediately rewritten as time-varying non-linear ordinary differential equations, with discontinuous right-hand side, for the original control inputs u_1 and u_2.

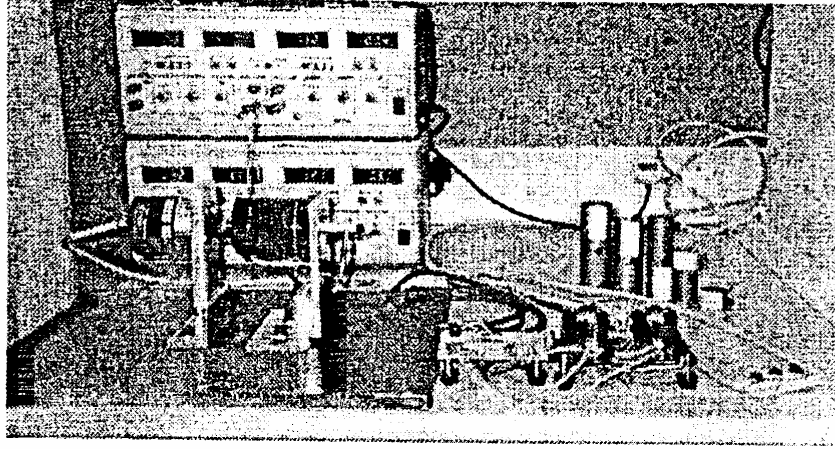The above analysis allows us to state the following theorem. Let

Figure 1. Overall system.

$$\sigma_1 = -k_1 x_1 + k_5 x_2 x_3 + u_1 + \lambda'(x_1 - I_{dd})$$

$$\sigma_2 = -k_1 k_3 x_2 - k_3 k_5 x_1 x_3 - k_2 k_3 x_3 + k_3 u_2$$

$$+ (\alpha_1' - k_4)(k_3 x_2 - k_4 x_3) + \alpha_2' x_3 + \alpha_3'(x_4 - \theta_d)$$

$$f_1 = -k_1 x_1 + k_5 x_2 x_3$$

$$f_2 = -k_1 x_2 - k_5 x_1 x_3 - k_2 x_3$$

$$f_3 = -k_1(f_1 + u_1) + k_5(f_2 + u_2)x_3 + k_5(k_3 x_2 - k_4 x_3)x_2$$

$$f_4 = -(k_1 k_3 + k_3 k_4)(f_2 + u_2) - k_3 k_5(f_1 + u_1)x_3$$

$$- (k_3 k_5 x_1 + k_2 k_3 - k_4^2)(k_3 x_2 - k_4 x_3)$$

Theorem 2: *The following dynamic feedback controller when applied to the stepper motor system* (5)

$$\dot{u}_1 = -f_3 - \lambda'(f_1 + u_1) - W_1' \operatorname{sgn}(\sigma_1)$$

$$\dot{u}_2 = \frac{1}{k_3}[-f_4 - \alpha_1'(k_3 f_2 - k_3 k_4 x_2 + k_4^2 x_3 + k_3 u_2)$$

$$- \alpha_2'(k_3 x_2 - k_4 x_3) - \alpha_3' x_3 - W_2' \operatorname{sgn}(\sigma_2)]$$

asymptotically stabilizes the outputs of the system to their desired values.

Proof: Taking the derivative of (30) and (31) with respect to time and using (5), (7), (34) and (35), one would obtain

$$\dot{\sigma}_i = -W_i' \operatorname{sgn}(\sigma_i) \qquad (i = 1, 2) \qquad (36)$$

The trajectories associated with the unforced discontinuous dynamics (36) exhibit a finite time reachability to zero from any given initial condition $\sigma_i(0)$ ($i = 1, 2$) provided that the constant gains W_i' ($i = 1, 2$) are chosen to be strictly positive. Since σ_1 and σ_2 converge to zero in

finite time, we will obtain after such a finite time $\dot{y}_1 = -\lambda'(y_1 - I_{dd})$ and $y_2^{(3)} = -\alpha_1' \ddot{y}_2 - \alpha_2' \dot{y}_2 - \alpha_3'(y_2 - \theta_d)$. Because of the choice of λ', α_1', α_2', α_3', we are guaranteed that $y_1 = I_d$ will converge to I_{dd}, and $y_2 = \theta$ will converge to θ_d in finite time. Hence the dynamic sliding mode controller guarantees the asymptotic convergence of the outputs to their desired values.

□

5. Implementation results of the sliding mode controllers

The designed static and dynamic sliding mode controllers are implemented using an experimental set-up depicted in figure 1. The system consists of a PM stepper motor, different loads to be attached to the shaft of the motor, drive circuitry, two current sensors, an optical encoder and a controller board. The two controllers are implemented using a digital signal processor (DSP). The algorithms for the two controllers are written as C programs and then compiled into assembly codes. These codes are then downloaded into the DSP (which resides on a controller board) from the computer.

The block diagram representation of the overall system is shown in figure 2.

The motor used in the implementation is a unipolar hybrid permanent magnet stepper motor (Vexta, model PH268-22B). The motor has a shaft at both of its ends. Therefore, the attachment of different loads and an optical encoder to the shaft is simple. A driver circuit is required to interface between the controller board and the PM stepper motor. This is the case because the peripheral sub-systems of the controller board have very low current capabilities. An optical encoder is used to
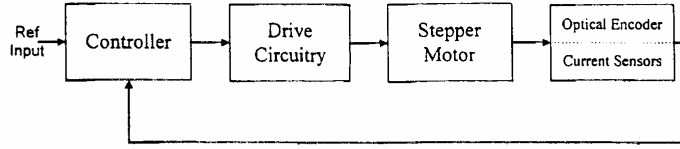
Figure 2. Block diagram of the overall system.

measure the position of the PM stepper motor shaft. Two current sensors are also used to measure the currents supplied to the two coils of the PM stepper motor. The measured data is then fed back to the controller board for processing. An observer is used to estimate the angular velocity of the shaft of the motor; see the work of Chiasson and Novotnal (1993). The brain of the system is a digital signal processor (the DS1102 floating point processor board which is based on Texas Instrument TMS320C31 floating point Digital Signal Processor).

The parameters of the system are needed for implementation purposes. The least square estimation technique (see Blauch *et al.* 1993) was used to estimate the motor torque constant, the inertia constant and the coefficient of viscous friction. The other parameters are either measurable or are supplied by the manufacturer. It is found that these parameters are such: $R = 19.1388 \, \Omega$, $L = 40 \, mH$, $K_m = 0.1349 \, Nm/A$, $J = 4.1295 \times 10^{-4} \, kg \, m^2$, $B = 0.0013 \, Nm/rad/s$ and $N_r = 50$.

The objective of the implementation is for the shaft of the motor to rotate by one step (one step $= 1.8° = 0.031 \, 42$ radians) as fast as possible and with the least amount of overshoot (critically damped response). Such a behaviour is a typical representation of fast and accurate positioning for robotic applications.

To simulate the different loads experienced by the stepper motor, different weights are attached to the shaft of the motor by an extension arm of about 6 cm. It is found that the maximum load the motor can drive is 1060 g. For this implementation, several loads are used to simulate different loads on the motor. The results shown in this paper correspond to loads of 105 g and 880 g.

The design parameters for the static controller are chosen such that $W_1 = 1000$, $W_2 = 7 \times 10^5$, $\alpha_1 = 550$ and $\alpha_2 = 7.5 \times 10^4$. The design parameters for the dynamic controller are chosen such that $W_1' = 2000$, $W_2' = 5.5 \times 10^7$, $\lambda' = 480$ and $\alpha_1' = 1250$, $\alpha_2' = 47 \times 10^4$, $\alpha_3' = 5.2 \times 10^7$. Obviously the choice of these parameters is not unique. However, because of the hardware limitations and the required specifications the parameters have been chosen as given above.

The implementation results are given in figures 3–6. It can be seen from the figures that the state variables

converge towards their corresponding equilibrium points when the static and the dynamic sliding mode control schemes are used. Figures 3 and 5 depict the implementation results when the static sliding mode controller is used with loads of 105 g and 880 g. It can been seen from these two figures that the angular position of the shaft of the motor converges towards its desired value ($1.8° = 0.031 \, 42$ radians) in less than 0.1 s when the load is 105 g and in less than 0.2 s when the load is 880 g. As expected, the settling time is longer for heavier loads. In addition, the response of the angular position of the shaft of the motor is critically damped in both cases. The currents in coils A and B, the voltages v_a, v_b, as well as the sliding surfaces are also shown in figures 3 and 5. It should be noted that because of the hardware constraints, the input voltages v_a and v_b are restricted to the range of 0–12 V. The control inputs v_a and v_b are seen to exhibit a discontinuous behaviour of the bang-bang type caused by the existence, in finite time, of a sliding regime on the intersection of the proposed stabilizing sliding surfaces $s_1 = 0$ *and* $s_2 = 0$. The controlled angular position trajectory x_4, is seen to be quite smooth, due to the fact that a third order integration separates the bang-bang control input u_2 from the angular position x_4. However, chattering is quite strong on the regulated motor currents, since only one order of integration separates the bang-bang control input u_1 from the regulated variable x_1. Figures 4 and 6 depict the implementation results when the dynamic sliding mode controller is used with loads of 105 g and 880 g. It can be seen from these two figures that the angular position of the shaft of the motor converges towards its desired value ($1.8° = 0.031 \, 42$ radians) in less than 0.15 s when the load is 105 g and in less than 0.2 s when the load is 880 g. In addition, the response of the angular position of the shaft of the motor is critically damped in both cases. The currents in coils A and B, the voltages v_a, v_b, as well as the sliding surfaces are shown in figures 4 and 6. The control inputs v_a and v_b, the currents and the sliding surfaces are seen to exhibit substantially less chattering than in the static sliding mode control case. Note that all the state variables asymptotically converge to the desired equilibrium points.

The responses of the angular position of the shaft of the motor when the static and dynamic sliding mode controllers are used are depicted in the same plot; the
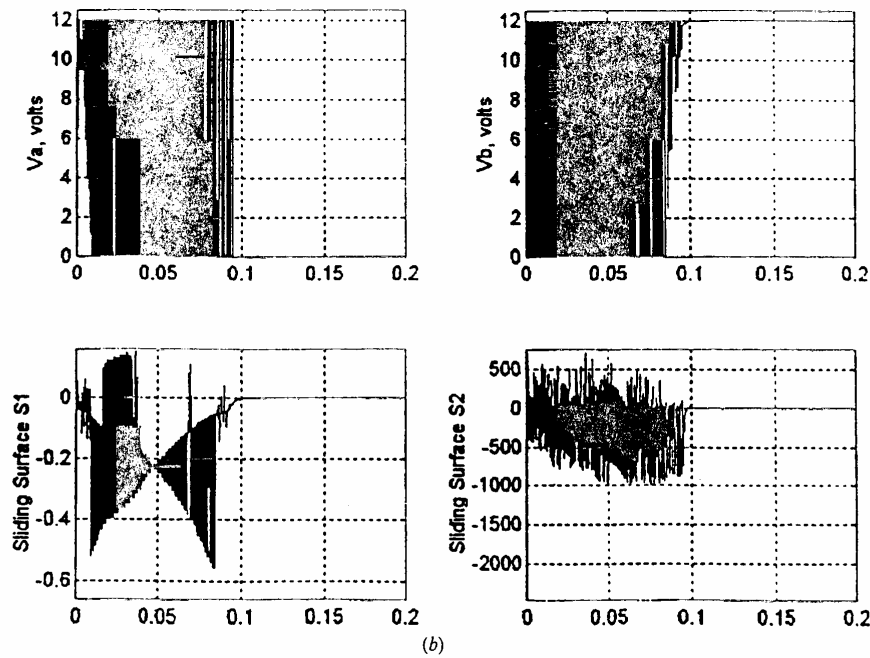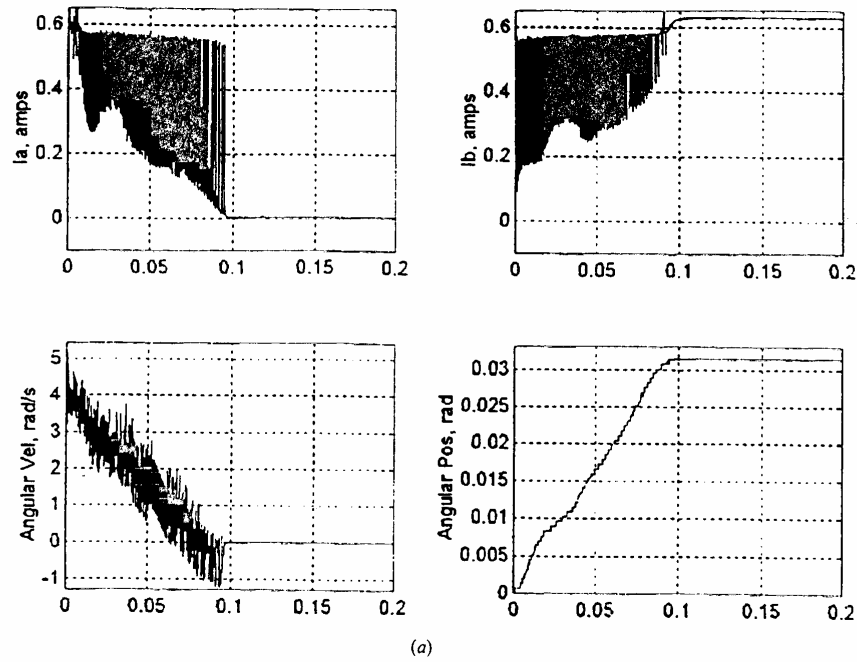
Figure 3. (*a*) Actual responses of the motor when the static sliding mode controller is used (with a load of 105 g). (*b*) Actual switching surfaces and actual static sliding mode controllers (with a load of 105 g).
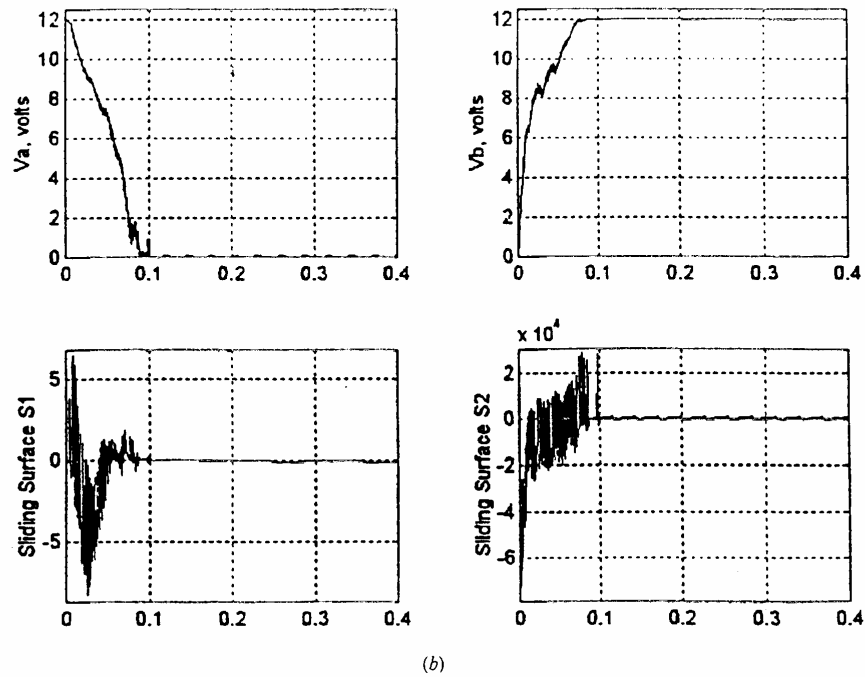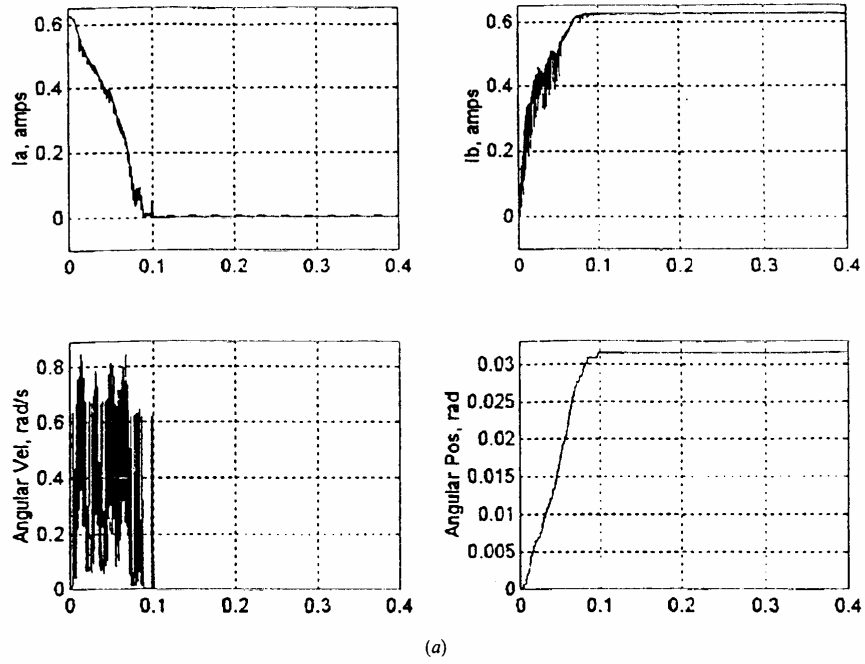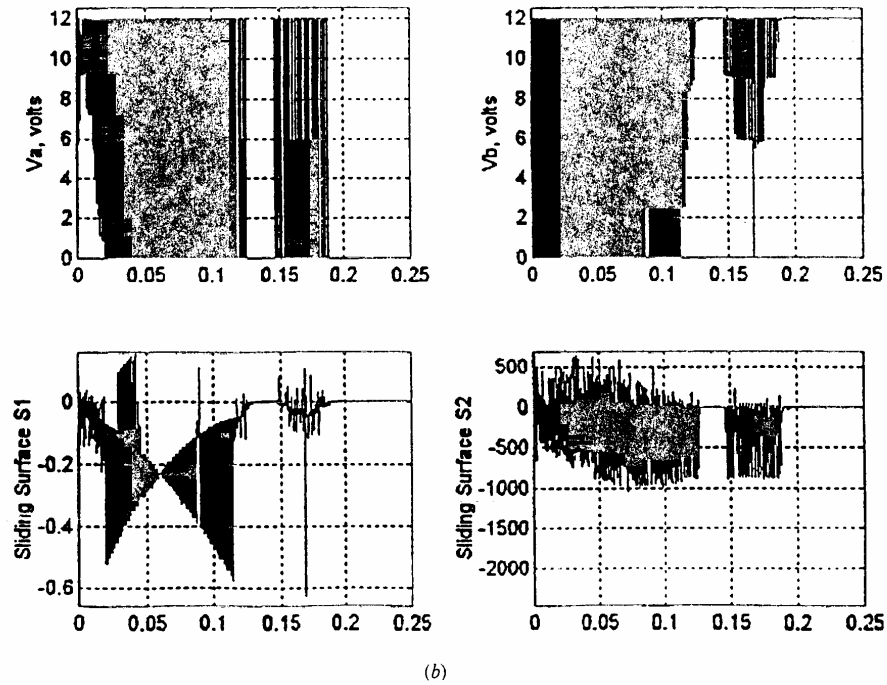
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Figure 4. (a) Actual responses of the motor when the dynamic sliding mode controller is used (with a load of 105 g). (b) Actual switching surfaces and actual dynamic sliding mode controllers (with a load of 105 g).
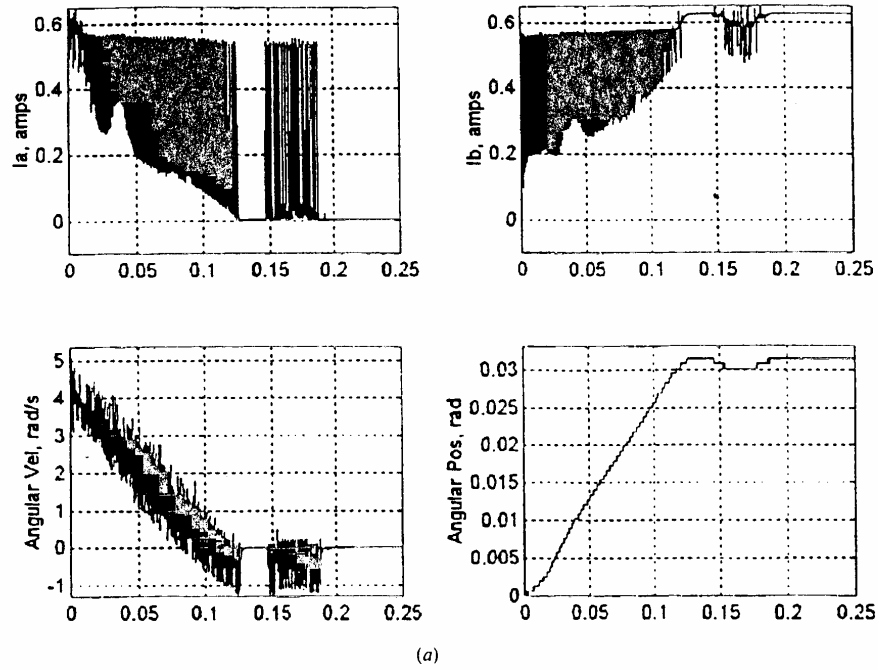
(a)

(b)

Figure 5. (a) Actual responses of the motor when the static sliding mode controller is used (with a load of 880 g). (b) Actual switching surfaces and actual static sliding mode controllers (with a load of 880 g).
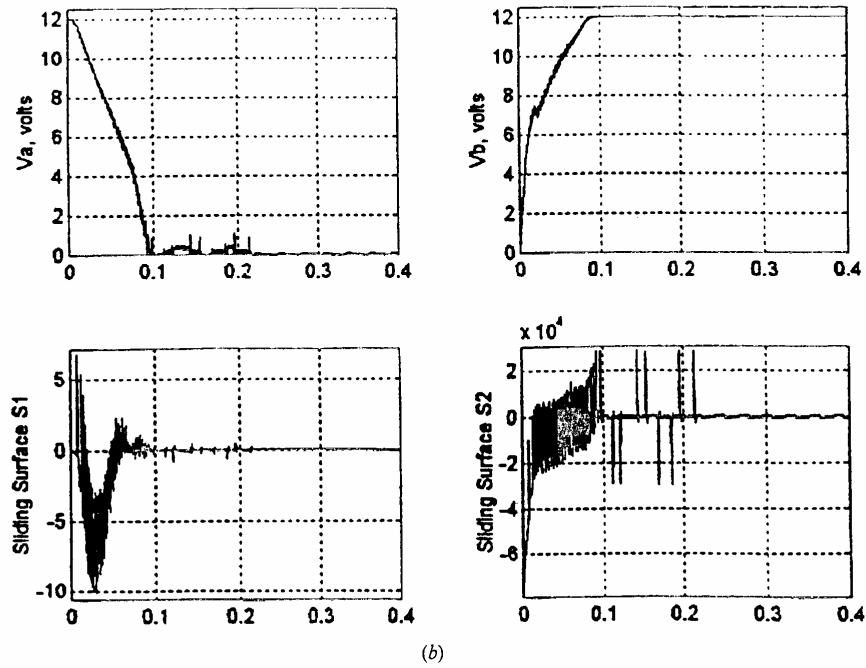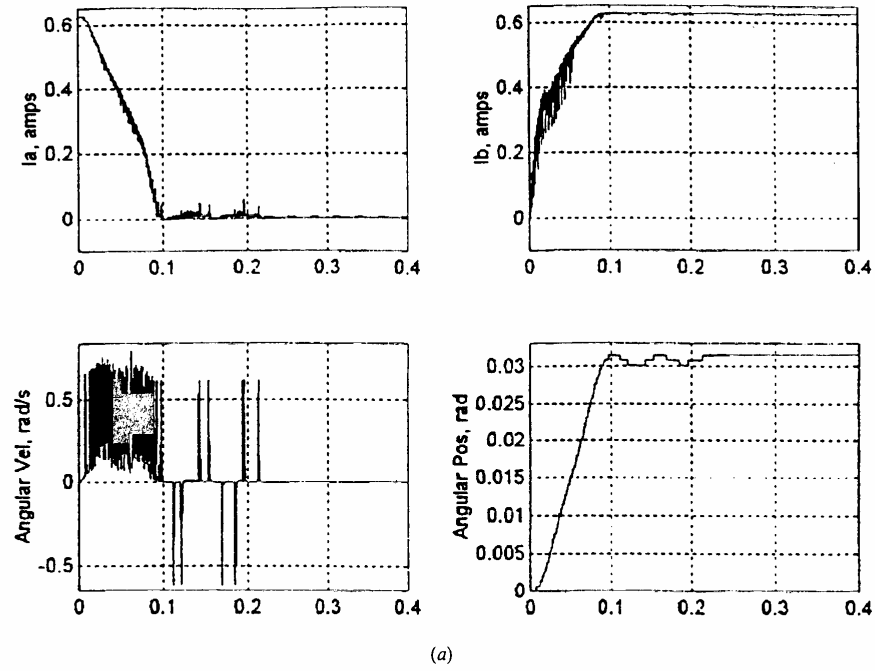
(a)

(b)

Figure 6. (a) Actual responses of the motor when the dynamic sliding mode controller is used (with a load of 880 g). (b) Actual switching surfaces and actual dynamic sliding mode controllers (with a load of 880 g).
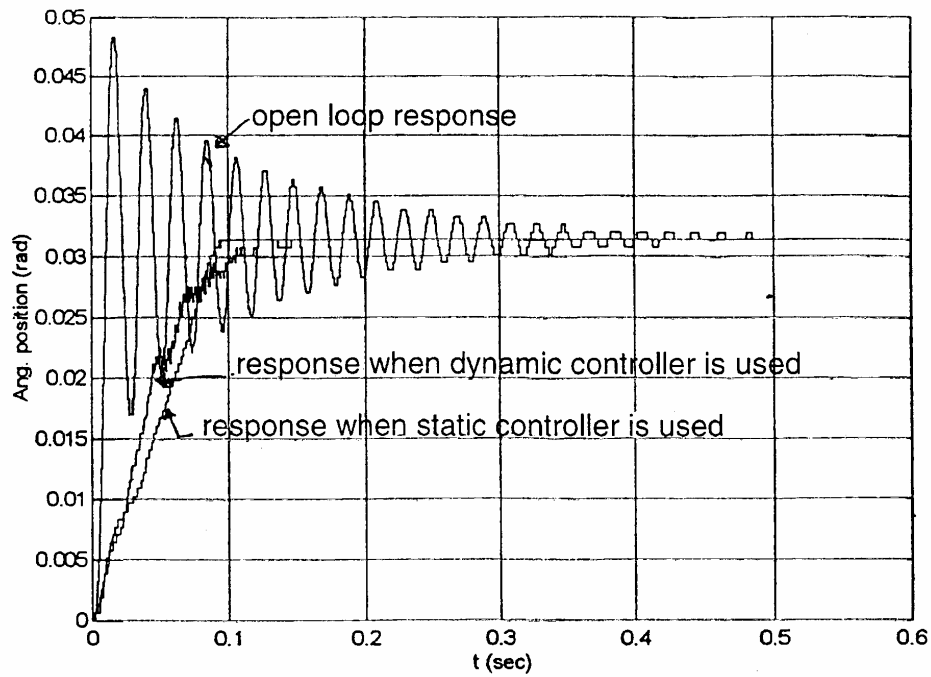
Figure 7.   Comparison of the responses of angular position of the shaft of the motor with load of 105 g.
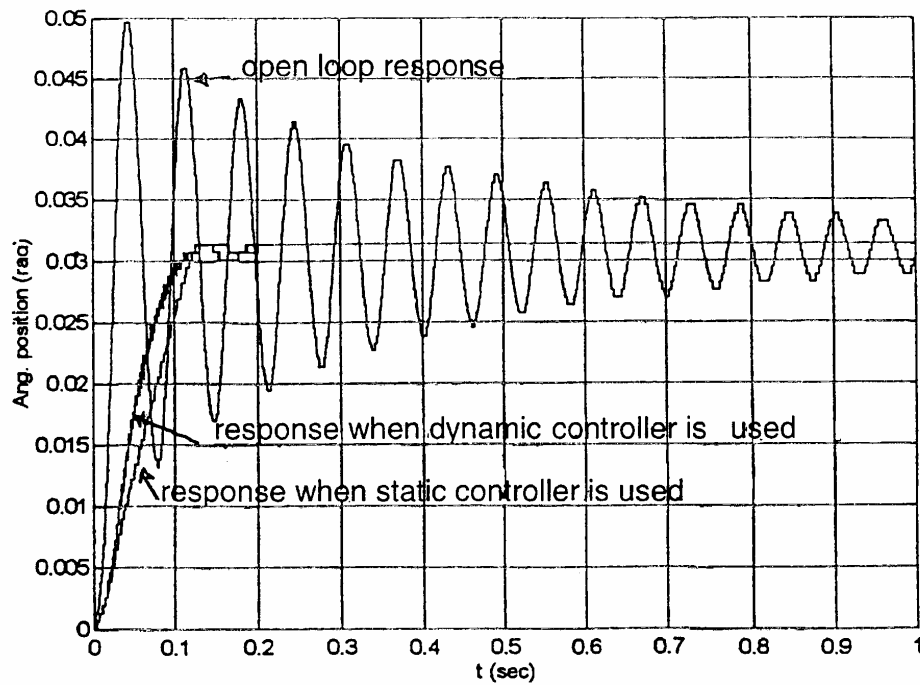

Figure 8.   Comparison of the responses of angular position of the shaft of the motor with load of 880 g.

open loop response is also shown in that plot. Figure 7 shows the plots when the load is 105 g; figure 8 shows the plots when the load is 880 g. From these two figures, it can been seen that closed loop plots are far superior to the open loop plots. In addition, the response of the shaft of the motor, when the dynamic sliding mode controller is used, is slightly better than the one when the static sliding mode controller is used. In summary, it can be concluded that the closed loop operation of the PM stepper motor system with the static and the dynamic sliding mode controllers is found to be working very well under different loading conditions.

6. Conclusion

In this article we have used multivariable sliding mode controllers to control a PM stepper motor. Sliding mode controller design is shown to be greatly facilitated by resorting to the differential flatness of the system. The method presented constitutes a systematic means of sliding mode controller design, which largely solves the problem of obtaining de-coupled linearizing sliding modes. At the same time the approach is quite permeable to dynamic extensions of the system. This fact and the flatness of the system naturally results in the possibilities of proposing dynamic multivariable sliding mode controllers for non-linear controllable systems. An added bonus of resorting to the flatness of the system is the fact that the obtained sliding mode controllers are entirely endogenous. Implementation results of the developed controllers on an experimental setup illustrate the developed theory.

References

AIELLO, M., REKOWSKI, R., BODSON, M., CHIASSON, J., and SCHUERER, D., 1991, Experimental results of parameter identification and nonlinear control of a PM stepper motor. In S. G. Tzafestas (Ed.) *Microprocessors in Robotic and Manufacturing Systems* (Amsterdam: Kluwer Academic Publishers), pp. 353–368.

BARTOLINI, G., and ZOLEZZI, T., 1986, Control of nonlinear variable structure systems. *Journal of Mathematical Analysis and Applications*, 118, 42–62.

BLAUCH, A. J., BODSON, M., and CHIASSON, J., 1993, High-speed parameter estimation of stepper motors. *IEEE Transactions on Control Systems Technology*, 1, 270–279.

BODSON, M., and CHIASSON, J., 1989, Application of nonlinear control methods to the positioning of a permanent magnet stepper motor. *Proceedings of the 28th Conference on Decision and Control*, Austin, Texas, USA, pp. 531–532.

BODSON, M., CHIASSON, J., NOVOTNAK, R., and REKOWSKI, R., 1993, High-performance nonlinear feedback control of a permanent magnet stepper motor. *IEEE Transactions on Control Systems Technology*, 1, 5–14.

BURG, T., HU, J., DAWSON, D. M., and VEDAGARBHA, P., 1994, A global exponential position tracking controller for a permanent magnet stepper motor via output feedback. *Proceedings of the 3rd IEEE Conference on Control Applications*, Glasgow, Scotland, pp. 213–218.

CHELOUAH, A., DELALEAU, E., MARTIN, P., and ROUCHON, P., 1996, Differential flatness and control of induction motors. *Proceedings of the Symposium on Control, Optimization and Supervision—Computational Engineering in Systems Applications*, IMACS Multi-conference, Lille, France, pp. 80–85.

CHIASSON, J., and NOVOTNAL, R. T., 1993, Nonlinear speed observer for the PM stepper motor. *IEEE Transactions on Automatic Control*, 38, 1584–1588.

DECARLO, R., ZAK, S., and MATHEWS, G., 1988, Variable structure of nonlinear multivariable systems: a tutorial. *Proceedings of the IEEE*, 76, 212–232.

EL-GHEZAWI, O. M. E., ZINOBER, A. S. I., and BILLINGS, S. A., 1983, Analysis and design of variable structure systems using a geometric approach. *International Journal of Control*, 38, 657–671.

EMELYANOV, S. V., 1967, *Variable Structure Control Systems* (Moscow: Nauka).

FLIESS, M., and SIRA-RAMIREZ, H., 1993, A module theoretic approach to sliding mode control of linear systems. *Proceedings of the IEEE Conference on Decision and Control*, San Antonio, Texas, USA, pp. 1322–1323.

FLIESS, M., LEVINE, J., MARTIN, P., and ROUCHON, P., 1992a, Sur les systemes lineaires differentiellement plats. *C. R. Acad. Sci. Paris*, 316, 619–624.

FLIESS, M., LEVINE, J., MARTIN, P., and ROUCHON, P., 1992b, On differentially flat nonlinear systems. In M. Fliess (Ed.) *Nonlinear Control Systems Design* (Oxford: Pergamon Press), pp. 408–412.

FLIESS, M., LEVINE, J., MARTIN, P., and ROUCHON, P., 1995, Flatness and defect of nonlinear systems: Introductory theory and examples. *International Journal of Control*, 61, 1327–1362.

ITKIS, Y., 1978, *Control Systems of Variable Structure* (New York: Wiley).

KENJO, T., and SUGAWARA, A., 1994, *Stepping Motors and their Microprocessor Controls*, Second Edition (Oxford: Clarendon Press).

KWATNY, H., and KIM, H., 1990, Variable structure regulations of partially linearizable dynamics. *Systems and Control Letters*, 15, 67–79.

LEVINE, J., LOTTIN, J., and PONSART, J. C., 1996, A nonlinear approach to the control of magnetic bearings. *IEEE transactions on Control Systems Technology*, 4, 524–544.

MARTIN, P., and ROUCHON, P., 1996, Flatness and sampling control of induction motors. *Proceedings of the 13th Triennial World Congress*, San Francisco, California, USA, pp. 389–394.

POMET, J. B., MOOG, C. H., and ARANDA-BRICAIRE, E., 1992, A non-exact Brunovsky form and dynamic feedback linearization. *Proceedings of the IEEE Conference on Decision and Control*, Tucson, Arizona, USA, pp. 2012–2017.

ROTHFUSS, R., RUDOLPH, J., and ZEITZ, M., 1996, Flatness based control of a nonlinear chemical reactor. *Automatica*, 32, 1433–1439.

SIRA-RAMIREZ, H., 1988, Differential geometric methods in variable structure control. *International Journal of Control*, 48, 1359–1391.

SIRA-RAMIREZ, H., 1996, On the sliding mode control of multivariable nonlinear systems. *International Journal of Control*, 64, 745–765.

SIRA-RAMIREZ, H., ZRIBI, M., and AHMAD, A., 1994, Dynamical sliding mode control approach for vertical flight regulation in helicopters. *IEE proceedings Control Theory and Applications*, 141, 19–24.

SLOTINE, J.-J., and LI, W., 1991, *Applied Nonlinear Control* (New Jersey: Prentice Hall).

SLOTINE, J. J. E., and SASTRY, S. S., 1983, Tracking control of nonlinear systems using sliding surfaces with applications to robot manipulators. *International Journal of Control*, **38**, 465–492.

SPEAGLE, R. C., and DAWSON, D. M., 1993, Adaptive tracking control of a permanent magnet stepper motor driving a mechanical load. *Proceedings of the IEEE Southeastcon*, Charlotte, North Carolina, USA, pp. 115–119.

SPEAGLE, R. C., HU, J., DAWSON, D. M., and QU, Z., 1993, Robust control of a permanent magnet stepper motor without current measurements. *Proceedings of the 2nd IEEE Conference on Control Applications*, Vancouver, Canada, pp. 153–158.

UTKIN, V. I., 1978, *Sliding Modes and Their Applications in Variable Structure Systems* (Moscow: MIR).

UTKIN, V. I., 1992, *Sliding Modes in Control and Optimization* (Berlin: Springer-Verlag).

UTKIN, V. I., and YOUNG, K. K. D., 1978, Methods of constructing discontinuity planes in multidimensional variable structure systems. *Automation and Remote Control*, **39**, 1466–1470.

YOUNG, K. K. D., 1978, Controller design for a manipulator using the theory of variable structure systems. *IEEE Transactions on Systems, Man and Cybernetics*, **8**, 101–109.

YOUNG, K. K. D., UTKIN, V. I., and OZGUNER, U., 1999, A control engineer's guide to sliding mode control. *IEEE Transactions on Control, Systems Technology*, **7**, 328–342.

ZINOBER, A. S. I., 1994, Variable structure and Lyapunov control. *Lecture Notes in Control and Information Sciences*, 193 (New York: Springer-Verlag).

ZRIBI, M., and CHIASSON, J., 1991, Position control of a PM stepper motor by exact linearization. *IEEE Transactions on Automatic Control*, **36**, 620–625.