International Journal of Bifurcation and Chaos, Vol. 11, No. 5 (2001) 1381–1395

SYNCHRONIZATION OF CHAOTIC SYSTEMS: A GENERALIZED HAMILTONIAN SYSTEMS APPROACH*

HEBERTT SIRA-RAMÍREZ†
Departamento de Ingeniería Eléctrica,
Sección de Mecatrónica,
CINVESTAV-IPN, Avenida IPN No. 2508,
Colonia San Pedro Zacatenco, A.P. 14740,
7300 México, D.F., México

CÉSAR CRUZ-HERNÁNDEZ‡
Departamento de Electrónica y Telecomunicaciones,
Centro de Investigación Científica y de Educación Superior de Ensenada,
CICESE, Km. 107, Carretera Tijuana-Ensenada,
22860 Ensenada, Baja California, México

Received May 23, 2000; Revised July 25, 2000

A reapproach to chaotic systems synchronization is presented from the perspective of passivity-based state observer design in the context of Generalized Hamiltonian systems including dissipation and destabilizing vector fields. The synchronization and lack of synchronization of several well-studied chaotic systems is reexplained in these terms.

1. Introduction

Synchronization of chaotic systems has received a lot of attention from mathematicians, physicists and control engineers in the last decade. Three special issues [1997a, 1993, 1997b] of major journals have been devoted to the problem of chaos, in general, and synchronization and control of chaotic systems, in particular. Aside from several edited books on the subject (see e.g. [Ott *et al.*, 1994; Fradkov & Pogromsky, 1998]), a staggering collection of references has been collected by Chen [1997]. The enormous interest in the topic of synchronization arises from the possibilities of encoding, or masking, messages using as analog "carriers" the chaotic signal generated as a state, or as an output, of a chaotic system, called the "transmitter". The effectively random nature of the carrier signal, additively, or multiplicatively modulated by the masked message signal, makes it, to say the least, "dis-encouraging" to attempt the decoding of the message from the intercepted signal (see [Cuomo *et al.*, 1993]).

For the decoding or unmasking process to be reliable, a second chaotic system, called the "receiver", is proposed which is (1) "synchronized"

*This research was supported by the Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, (CINVESTAV-IPN), of Mexico, by the Consejo Nacional de Ciencia y Tecnologia (CONACYT) of México, under Research Grant No. 32681-A and by the Centro de Investigación Científica y de Educación Superior de Ensenada (CICESE) of Baja California, México, under Research Grant No. 31874-A.
†E-mail: hsira@mail.cinvestav.mx
‡E-mail: ccruz@cicese.mx

with the transmitter chaotic behavior. Synchronization means that, under the assumption of no masked signal transmission, the receiver state trajectory asymptotically tracks that of the transmitter. The receiver has as an external *input* signal either a particular state variable or, in general, an *output* function, of the transmitter system and (2) the receiver is designed in such a way that, when it is externally excited by the transmitter's chaotic emitted signal, containing the masked message, the zero synchronization error condition is not lost, at least for the corresponding synchronized state used as the carrier. In other words, certain "robustness" is exhibited by the receiver in its generation of the synchronized signal when excited by inputs which do not entirely coincide with the original external exciting carrier signal. The final detection stage simply consists in subtracting, or filtering out, the transmitted signal (comprising the synchronized state signal and the masked message) from the locally generated synchronized state. In the understanding that the generated receiver state is still synchronized with the transmitted carrier (i.e. they robustly coincide), the message is immediately recovered by the elementary detection process.

The synchronization problem is of a rather similar nature to that of designing a nonlinear observer for the transmitter system, as already remarked in [Nijmeijer & Mareels, 1997]. However, saddle but important limitations must be taken into account for designing a meaningful receiver system which is capable of tracking the transmitter's state and robustly sustaining the addition of a masked signal input after synchronization has taken place. For a passivity-based adaptive approach to synchronization the reader is referred to the interesting articles by Fradkov and Markov [1997] and by Pogromsky [1998].

In this article, we are only concerned with the synchronization issue from the perspective of Generalized Hamiltonian systems including nonconservative terms. It turns out that the great majority of chaotic systems can be placed in such a Generalized Hamiltonian canonical form, from where the reconstructibility of the state vector, from a defined output signal, may be assessed from the observability or, in its absence, the detectability of a pair of *constant* matrices. The Generalized Hamiltonian structure of most known chaotic systems allows one to clearly decide on the nature of the synchronizing (output) signal on the basis of the system dissipation and conservative energy managing structure and a need for elimination, at the receiver end, of the locally, or globally, destabilizing vector field.

Section 2 contains a brief introduction to Generalized Hamiltonian Systems, and gives two familiar examples of chaotic systems. Section 3 is devoted to the observer construction for a special class of Generalized Hamiltonian systems. The proposed class comprises nearly all of the best known chaotic systems addressed in the literature. Section 4 analyses the synchronization problem, from the perspective of the obtained results, for a collection of standard chaotic system examples. The last section is devoted to some conclusions and suggestions for further work.

2. Generalized Hamiltonian Systems

Consider a smooth nonlinear system, given in the following "Generalized Hamiltonian" canonical form,

$$\dot{x} = \mathcal{J}(x)\frac{\partial H}{\partial x} + \mathcal{S}(x)\frac{\partial H}{\partial x}, \quad x \in R^n \qquad (1)$$

where $H(x)$ denotes a smooth energy function which is globally positive definite in R^n. The column *gradient vector* of H, denoted by $\partial H/\partial x$, is assumed to exist everywhere. We frequently use *quadratic* energy functions of the form

$$H(x) = \frac{1}{2}x^T\mathcal{M}x \qquad (2)$$

with $\mathcal{M}$ begin a symmetric, positive definite, constant matrix. In such a case, $\partial H/\partial x = \mathcal{M}x$. The square matrices, $\mathcal{J}(x)$ and $\mathcal{S}(x)$, entering the expression in (1) satisfy, for all $x \in R^n$, the following properties, which clearly depict the *energy managing* structure of the system,

$$\mathcal{J}(x) + \mathcal{J}^T(x) = 0, \quad \mathcal{S}(x) = \mathcal{S}^T(x) \qquad (3)$$

The vector field $\mathcal{J}(x)\partial H/\partial x$ exhibits the *conservative* part of the system and it is also referred to as the *workless* part, or *work-less forces* of the system. The matrix $\mathcal{S}(x)$ is, in general, a symmetric matrix depicting the *working* or *nonconservative* part of the system. For certain systems, the symmetric matrix $\mathcal{S}(x)$ is *negative definite* or *negative semi-definite*. In such cases, the vector field is addressed to as the *dissipative* part of the system. If, on the other hand, $\mathcal{S}(x)$ is positive definite, positive semi-definite, or indefinite, it clearly represents, respectively, the global, semi-global and local *destabilizing* part of the system. In the last case, we can always (although nonuniquely) decompose such an indefinite symmetric matrix into the sum of a symmetric negative semi-definite matrix $\mathcal{R}(x)$ and a symmetric positive semi-definite matrix $\mathcal{N}(x)$. If we denote by $L\phi H(x)$ the directional (Lie) derivative of $H(x)$ with respect to a vector field $\phi(x)$, then the previously identified vector fields satisfy the following properties, from where the adopted terminology is fully justified:

$$\begin{aligned} L_{\mathcal{J}(x)\frac{\partial H}{\partial x}} H(x) &= \frac{\partial H}{\partial x^T}\mathcal{J}(x)\frac{\partial H}{\partial x} = 0 \\ L_{\mathcal{R}(x)\frac{\partial H}{\partial x}} H(x) &= \frac{\partial H}{\partial x^T}\mathcal{R}(x)\frac{\partial H}{\partial x} \le 0 \\ L_{\mathcal{N}(x)\frac{\partial H}{\partial x}} H(x) &= \frac{\partial H}{\partial x^T}\mathcal{N}(x)\frac{\partial H}{\partial x} = \begin{cases} \ge 0 \text{ in all of } R^n \text{ or} \\ \text{indefinite} \end{cases} \end{aligned} \tag{4}$$

Sometimes, specially in the context of observer design, we will write a system's set of equations in the special form

$$\dot{x} = \mathcal{J}(x)\frac{\partial H}{\partial x} + \mathcal{S}(x)\frac{\partial H}{\partial x} + \mathcal{F}(x) \tag{5}$$

where $\mathcal{F}(x)$ represents a *locally destabilizing* vector field and $\mathcal{S}(x)$ is a symmetric matrix, not necessarily of definite sign. Evidently, the form (5) can be reduced, under mild assumptions, to the form (1) for any given vector field $\mathcal{F}(x)$.

However, many physical systems are already in Generalized Hamiltonian canonical form, as the following examples illustrate.

Example 1. Consider the Duffing system, extensively treated in the literature (see e.g. [Fradkov & Markov, 1997])

$$\ddot{x} + p\dot{x} + qx + x^3 = 0, \quad p, q > 0 \tag{6}$$

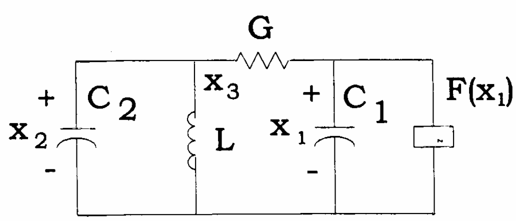

Fig. 1. Chua's circuit.

Let $x_1 = x$ and $x_2 = \dot{x}$. We associate the following Hamiltonian energy function $H(x, \dot{x}) = 1/2(x^2 + (1/q)\dot{x}^2) = 1/2(x_1^2 + (1/q)x_2^2)$ with the system. The gradient vector of the energy function is clearly given by $\partial H/\partial x = [x_1 \;\; (1/q)x_2]^T$ with $\mathcal{M} = \text{diag}[1, 1/q]$. The Duffing system is rewritten in Generalized Hamiltonian form as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & q + \frac{1}{2}x_1^2 \\ -q - \frac{1}{2}x_1^2 & 0 \end{bmatrix} \frac{\partial H}{\partial x} \\ &\quad + \begin{bmatrix} 0 & -\frac{1}{2}x_1^2 \\ -\frac{1}{2}x_1^2 & -pq \end{bmatrix} \frac{\partial H}{\partial x} \end{aligned} \tag{7}$$

Example 2. Consider Chua's circuit [Chua & Wu, 1993] shown in Fig. 1. This circuit is described by the following set of differential equations

$$\begin{aligned} C_1\dot{x}_1 &= G(x_2 - x_1) - F(x_1) \\ C_2\dot{x}_2 &= G(x_1 - x_2) + x_3 \\ L\dot{x}_3 &= -x_2 \end{aligned} \tag{8}$$

where $F(x_1)$ is a voltage-dependent nonlinear resistance of the form

$$F(x_1) = ax_1 + \frac{1}{2}(b-a)(|1+x_1| - |1-x_1|), \; a, \; b < 0$$

clearly playing the role of a *negative* resistor.

Consider, as a Hamiltonian energy function, the total stored energy in the circuit, given by

$$H(x) = \frac{1}{2}[C_1x_1^2 + C_2x_2^2 + Lx_3^2] \tag{9}$$

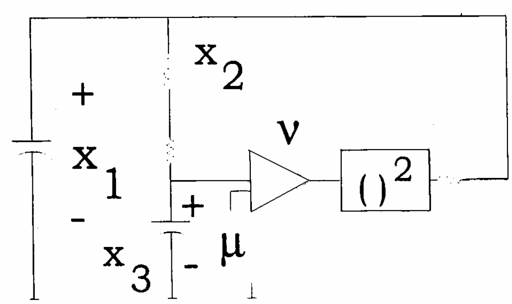

Fig. 2. The Mitscke–Flüggen optical bistable chaotic system.

whose gradient vector is readily obtained as

$$\frac{\partial H}{\partial x} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} C_1 x_1 \\ C_2 x_2 \\ L x_3 \end{bmatrix} \tag{10}$$

The system may be written in Generalized Hamiltonian Canonical form, with a destabilizing vector field, as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{LC_2} \\ 0 & -\frac{1}{LC_2} & 0 \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} -\frac{G}{C_1^2} & \frac{G}{C_1 C_2} & 0 \\ \frac{G}{C_1 C_2} & -\frac{G}{C_2^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} -\frac{1}{C_1} F(x_1) \\ 0 \\ 0 \end{bmatrix} \tag{11}$$

where the dissipation structure matrix $\mathcal{R}$ is seen to be only negative semi-definite and the nonlinear *negative resistance* element, characterized by, the nonlinear function $-F(x_1)$, is clearly seen to play the role of a globally destabilizing vector field.

3. Nonlinear Observer Design for a Class of Systems in Generalized Hamiltonian Form

We consider a special class of Generalized Hamiltonian systems with destabilizing vector fields and linear output map, y, given by

$$\dot{x} = \mathcal{J}(y)\frac{\partial H}{\partial x} + (\mathcal{I} + \mathcal{S})\frac{\partial H}{\partial x} + \mathcal{F}(y), \quad x \in R^n$$
$$y = \mathcal{C}\frac{\partial H}{\partial x}, \quad y \in R^m \tag{12}$$

where $\mathcal{S}$ is a constant symmetric matrix, not necessarily of definite sign. The matrix $\mathcal{I}$ is a constant skew symmetric matrix. The vector variable y is referred to as the system *output*. The matrix $\mathcal{C}$ is a constant matrix.

We denote the *estimate* of the state vector x by ξ, and consider the Hamiltonian energy function $H(\xi)$ to be the particularization of H in terms of ξ. Similarly, we denote by η the estimated output, computed in terms of the estimated state ξ. The gradient vector $\partial H(\xi)/\partial \xi$ is, naturally, of the form $\mathcal{M}\xi$ with $\mathcal{M}$ being a, constant, symmetric positive definite matrix.

A dynamic nonlinear state observer for the system (12) is readily obtained as

$$\dot{\xi} = \mathcal{J}(y)\frac{\partial H}{\partial \xi} + (\mathcal{I} + \mathcal{S})\frac{\partial H}{\partial \xi} + \mathcal{F}(y) + K(y - \eta)$$
$$\eta = \mathcal{C}\frac{\partial H}{\partial \xi} \tag{13}$$

where K is a constant vector, known as the *observer gain*.

The state estimation error, defined as $e = x - \xi$ and the output estimation error, defined as $e_y = y - \eta$, are governed by

$$\dot{e} = \mathcal{J}(y)\frac{\partial H}{\partial e} + [\mathcal{I} + \mathcal{S} - \mathcal{K}\mathcal{C}]\frac{\partial H}{\partial e}, \quad e \in R^n$$
$$e_y = \mathcal{C}\frac{\partial H}{\partial e}, \quad e_y \in R^m \tag{14}$$

where the vector, $\partial H/\partial e$ actually stands, with some abuse of notation, for the gradient vector of the *modified* energy function, $\partial H(e)/\partial e = \partial H/\partial x - \partial H/\partial \xi = \mathcal{M}(x - \xi) = \mathcal{M}e$. Below, we set, when needed, $\mathcal{I} + \mathcal{S} = \mathcal{W}$.

We recall the basic definitions of *detectability* and *observability* in linear systems.

Definition 1. Given a pair of constant matrices $(\mathcal{C}, \mathcal{A})$, respectively of dimensions $m \times n$ and $n \times n$. The pair is said to be detectable if the matrix

$$\begin{bmatrix} \mathcal{C} \\ sI - \mathcal{A} \end{bmatrix} \tag{15}$$

has full rank n for all values of s in the open right half of the complex plane. The system is said to be observable if the above matrix is full rank for all values of s in the complex plane.

If the pair of matrices $(\mathcal{C}, \mathcal{W})$ (resp. $(\mathcal{C}, \mathcal{S})$) is either *observable*, or *detectable*, it is well known, from linear systems theory, that there exists a constant vector $\mathcal{K}$ such that all, or at least the *observable*, eigenvalues of the matrix $\mathcal{W} - \mathcal{K}\mathcal{C}$ (resp. $(\mathcal{C}, \mathcal{S})$)

are placeable, modulo symmetry with respect to the real line, at prespecified locations of the open left half of the complex plane. The distinction made above regarding *observable eigenvalues* means that some eigenvalues of $(\mathcal{C}, \mathcal{W})$ (resp. $(\mathcal{C}, \mathcal{S})$) may be *fixed* and cannot be influenced by any value of $\mathcal{K}$. In the case of a *detectable* pair, those fixed unobservable eigenvalues already exhibit negative real parts. If the pair of matrices $(\mathcal{C}, \mathcal{W})$, (resp. $(\mathcal{C}, \mathcal{S})$) is *observable* it means that, modulo the mentioned symmetry, *all* eigenvalues of $\mathcal{W} - \mathcal{KC}$ (resp. $(\mathcal{C}, \mathcal{S})$) can be placed at will in the left half of the complex plane by suitable choice of the matrix $\mathcal{K}$. As a consequence, the matrix $(\mathcal{W} - \mathcal{KC})^T$ also exhibits eigenvalues with negative real parts. This also implies that the sum

$$[\mathcal{W} - \mathcal{KC}] + [\mathcal{W} - \mathcal{KC}]^T = [\mathcal{S} - \mathcal{KC}] + [\mathcal{S} - \mathcal{KC}]^T = 2\left[\mathcal{S} - \frac{1}{2}(\mathcal{KC} + \mathcal{C}^T\mathcal{K}^T)\right]$$

is a symmetric matrix with negative (real) eigenvalues.

Notice that the matrix $\mathcal{W} - \mathcal{KC}$ is a square matrix, with no particular structure. We can *always* trivially replace such a matrix by the following sum

$$\mathcal{W} - \mathcal{KC} = \left\{\mathcal{S} - \frac{1}{2}[\mathcal{KC} + \mathcal{C}^T\mathcal{K}^T]\right\} + \left\{\mathcal{I} - \frac{1}{2}[\mathcal{KC} - \mathcal{C}^T\mathcal{K}^T]\right\} \quad (16)$$

The first two summands clearly conform a symmetric negative definite matrix while the second two summands conform a skew-symmetric matrix.

The state estimation error system may then be written in the following form

$$\dot{e} = \left[\mathcal{J}(y) + \mathcal{I} - \frac{1}{2}(\mathcal{KC} - \mathcal{C}^T\mathcal{K}^T)\right]\frac{\partial H}{\partial e} + \left[\mathcal{S} - \frac{1}{2}(\mathcal{KC} + \mathcal{C}^T\mathcal{K}^T)\right]\frac{\partial H}{\partial e}$$

Then, taking as a modified Hamiltonian energy function the positive definite function $H(e)$, it is readily found that the time derivative of this function, along the trajectories of the observation error system, satisfies

$$\dot{H}(e) = \frac{\partial H(e)}{\partial e^T}\dot{e} = \frac{\partial H(e)}{\partial e^T}\left[\mathcal{S} - \frac{1}{2}(\mathcal{KC} + \mathcal{C}^T\mathcal{K}^T)\right]\frac{\partial H(e)}{\partial e} \leq 0 \quad (17)$$

with $\dot{H}(e) = 0$ if and only if $e = 0$. In fact, it is not difficult to show that the stability of the error space origin $e = 0$ is *exponentially* asymptotically stable for an energy function of the form $H(e) = (1/2)e^T\mathcal{M}e$. In this case we have

$$\dot{H}(e) = e^T\mathcal{M}^T\left[\mathcal{S} - \frac{1}{2}(\mathcal{KC} + \mathcal{C}^T\mathcal{K}^T)\right]\mathcal{M}e \leq -\frac{1}{2}\alpha e^T\mathcal{M}e = -\alpha H(e) \quad (18)$$

with α being a suitable scalar constant. We have then proven the following result.

Theorem 3.1. *The state x of the nonlinear system (12) can be globally, exponentially, asymptotically estimated by the state ξ of an observer of the form (13), if the pair of matrices $(\mathcal{C}, \mathcal{W})$, or the pair $(\mathcal{C}, \mathcal{S})$, is either observable or, at least, detectable.*

An observability condition on either of the pairs $(\mathcal{C}, \mathcal{W})$, or $(\mathcal{C}, \mathcal{S})$, is clearly a *sufficient* but not necessary condition for asymptotic state reconstruction. The following simple example readily demonstrates this issue.

Example 3. The pair of matrices

$$\mathcal{S} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathcal{C} = [0 \quad 1]$$

constitutes a nonobservable, although it is a detectable pair. Nevertheless, setting $\mathcal{K} = 0$ already renders the sum, $2[\mathcal{S} - (1/2)(\mathcal{KC} + \mathcal{C}^T\mathcal{K}^T)] = 2\mathcal{S}$, as a *negative definite* matrix.

A necessary and sufficient condition for global asymptotic stability to zero of the state estimation error is given by the following theorem.

Theorem 3.2. *The state x of the nonlinear system (12) can be globally, exponentially, asymptotically estimated, by the state ξ of the observer (13) if and only if there exists a constant matrix $\mathcal{K}$ such that the symmetric matrix*

$$[\mathcal{W} - \mathcal{KC}] + [\mathcal{W} - \mathcal{KC}]^T = [\mathcal{S} - \mathcal{KC}] + [\mathcal{S} - \mathcal{KC}]^T = 2\left[\mathcal{S} - \frac{1}{2}(\mathcal{KC} + \mathcal{C}^T\mathcal{K}^T)\right]$$

is negative definite.

4. Applications to Synchronization of Chaotic Circuits

In the context of synchronization, a key observation, provided by the special form (12) of the Generalized Hamiltonian canonical form, is that the destabilizing vector field $\mathcal{F}(y)$ already fixes the output signal y that needs to be transmitted towards the receiver (observer). Thus we propose:

1. Given a nonlinear chaotic system, we write it in Generalized Canonical form and proceed to identify the destabilizing, or locally destabilizing vector field $\mathcal{F}(x)$, by ascribing, as much as it is possible, the nonlinear state dependent terms in $\mathcal{S}(x)$ and $\mathcal{J}(x)$ to the destabilizing vector field. If the destabilizing vector field is actually a nonlinear injection into R^n of a *single* nonlinear map of the state, say $y = h(x)$, then this output vector should be taken as the set of signals to be transmitted towards the receiver. One should also try to obtain a *linear function* of the state for h in the form $\mathcal{C}\partial H/\partial x$. If this is not possible, one must resort to state coordinate transformations and possibly to output coordinate transformations. This topic is not pursued in this article.
2. Once the output y of the system has been decided, and the system is placed in the form (12), then one should proceed to check the observability, or the detectability, of the obtained pair $(\mathcal{S}, \mathcal{C})$. If at least one of the properties is verified, then the receiver is readily designed as in (13). If the pair $(\mathcal{S}, \mathcal{C})$ is not detectable, one should add the skew symmetric matrix $\mathcal{I}$ to $\mathcal{S}$, to form $\mathcal{W} = \mathcal{S} + \mathcal{I}$ and proceed to check if the pair of matrices $(\mathcal{W}, \mathcal{C})$ is either detectable or observable. In such a case the receiver is the same as in (13). If still we do not have detectability or observability of the mentioned constant pairs, then, before resorting to nonlinear observer theory results, one should check whether or not a constant matrix $\mathcal{K}$ exists such that the symmetric matrix $[\mathcal{S} - (\mathcal{K}\mathcal{C} + \mathcal{C}^T\mathcal{K}^T)]$ is negative definite or semi-definite.
3. Sometimes, the matrix $\mathcal{S}$ is already negative definite and the error system is sufficiently dissipative without addition of an output reconstruction error injection term using the matrix $\mathcal{K}$. However, if the negative real part of the observable eigenvalues of the matrix $\mathcal{S}$ need enhancement, to guarantee a faster synchronization, then an output error injection term should be considered through the constant matrix $\mathcal{K}$, as specified by (13).

4.1. *The Lorenz system*

Consider the Lorenz system [Lorenz, 1963]

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= rx_1 - x_2 - x_1x_3 \\ \dot{x}_3 &= x_1x_2 - bx_3 \end{aligned} \tag{19}$$

The system can be easily written in Generalized Hamiltonian form, taking as the Hamiltonian energy function the scalar function

$$H(x) = \frac{1}{2}\left[\frac{1}{\sigma}x_1^2 + x_2^2 + x_3^2\right] \tag{20}$$

This yields, according to the previous procedure,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & \frac{1}{2}\sigma & 0 \\ -\frac{1}{2}\sigma & 0 & -x_1 \\ 0 & x_1 & 0 \end{bmatrix} \frac{\partial H}{\partial x} \\ &\quad + \begin{bmatrix} -\sigma^2 & \frac{1}{2}\sigma & 0 \\ \frac{1}{2}\sigma & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} 0 \\ rx_1 \\ 0 \end{bmatrix} \end{aligned} \tag{21}$$

The output signal to be transmitted should be the state $y = x_1 = [\sigma \ \ 0 \ \ 0]\partial H/\partial x$. The matrices $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{I}$, are given by

$$\begin{aligned} \mathcal{C} = [\sigma \ \ 0 \ \ 0], \quad \mathcal{S} &= \begin{bmatrix} -\sigma^2 & \frac{1}{2}\sigma & 0 \\ \frac{1}{2}\sigma & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}, \\ \mathcal{I} &= \begin{bmatrix} 0 & \frac{1}{2}\sigma & 0 \\ -\frac{1}{2}\sigma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{22}$$

The pair of matrices $(\mathcal{C}, \mathcal{S})$ already constitutes a pair of detectable, but nonobservable, matrices. Even though the addition of the matrix $\mathcal{I}$ to $\mathcal{S}$

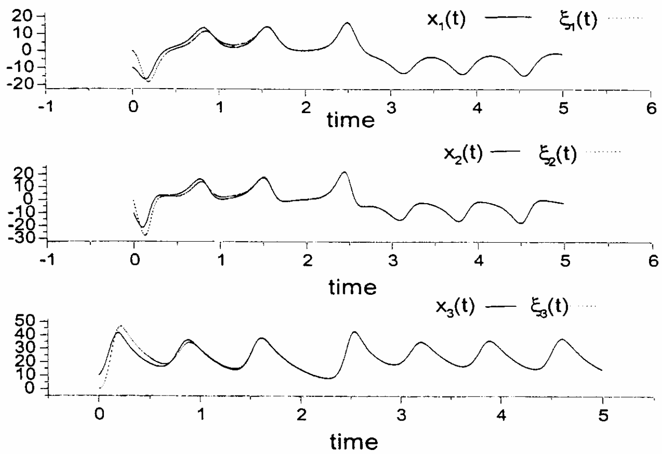

Fig. 3. Lorenz system trajectories and synchronized receiver trajectories.

does not improve the lack of observability, the pair $(\mathcal{C}, \mathcal{W}) = (\mathcal{C}, \mathcal{S} + \mathcal{I})$ remains, nevertheless, detectable. In this case, the dissipative structure of the system is fully "damped" due to the negative definiteness of the matrix $\mathcal{S}$. Then, there is no need for an output estimation error injection for complementing, or enhancing, the system's natural dissipative structure. The receptor is designed as

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\sigma & 0 \\ -\frac{1}{2}\sigma & 0 & -y \\ 0 & y & 0 \end{bmatrix} \frac{\partial H}{\partial \xi} + \begin{bmatrix} -\sigma^2 & \frac{1}{2}\sigma & 0 \\ \frac{1}{2}\sigma & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \frac{\partial H}{\partial \xi} + \begin{bmatrix} 0 \\ ry \\ 0 \end{bmatrix} \tag{23}$$

and the synchronization error is therefore governed by the globally asymptotically stable system

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\sigma & 0 \\ -\frac{1}{2}\sigma & 0 & -y \\ 0 & y & 0 \end{bmatrix} \frac{\partial H}{\partial e} + \begin{bmatrix} -\sigma^2 & \frac{1}{2}\sigma & 0 \\ \frac{1}{2}\sigma & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \frac{\partial H}{\partial e} \tag{24}$$

If the negative real part of the observable eigenvalues, related to the constant dissipation structure matrix of the error dynamics, must be enhanced, one can still use the above observer but now including an output reconstruction error injection term. The resulting observer is given by

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\sigma & 0 \\ -\frac{1}{2}\sigma & 0 & -y \\ 0 & y & 0 \end{bmatrix} \frac{\partial H}{\partial \xi} + \begin{bmatrix} -\sigma^2 & \frac{1}{2}\sigma & 0 \\ \frac{1}{2}\sigma & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \frac{\partial H}{\partial \xi} + \begin{bmatrix} 0 \\ ry \\ 0 \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} e_1$$

The asymptotically stable reconstruction error dynamics is then governed by

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\sigma(1+K_2) & \frac{1}{2}\sigma K_3 \\ -\frac{1}{2}\sigma(1+K_2) & \frac{1}{2}\sigma K_3 & -y \\ -\frac{1}{2}\sigma K_3 & y & 0 \end{bmatrix} \frac{\partial H}{\partial e} + \begin{bmatrix} -\sigma(\sigma+K_1) & \frac{1}{2}\sigma(1-K_2) & -\frac{1}{2}\sigma K_3 \\ \frac{1}{2}\sigma(1-K_2) & -1 & 0 \\ -\frac{1}{2}\sigma K_3 & 0 & -b \end{bmatrix} \frac{\partial H}{\partial e} \tag{25}$$

One may now specify the values of K_1, K_2 and K_3 in order to guarantee a faster asymptotic stability to zero of the state reconstruction error trajectories. Figure 3 shows the simulations of the Lorenz system and the receiver's state tracking abilities for large initial deviations. The system parameters were set to be

$$\sigma = 10, \quad r = 28, \quad b = \frac{8}{3}, \quad K_1 = K_2 = K_3 = 0$$

4.2. *Chen's chaotic attractor*

Consider now Chen's chaotic attractor. This system is described by the following set of differential equations

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= (c-a)x_1 - x_1x_3 + cx_2 \\ \dot{x}_3 &= x_1x_2 - bx_3 \end{aligned} \tag{26}$$

Taking as a Hamiltonian energy function the scalar function

$$H(x) = \frac{1}{2}[x_1^2 + x_2^2 + x_3^2] \tag{27}$$

we write the system in Generalized Hamiltonian Canonical form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & a-\frac{c}{2} & 0 \\ -a+\frac{c}{2} & 0 & -x_1 \\ 0 & x_1 & 0 \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} -a & \frac{c}{2} & 0 \\ \frac{c}{2} & c & 0 \\ 0 & 0 & -b \end{bmatrix} \frac{\partial H}{\partial x} \tag{28}$$

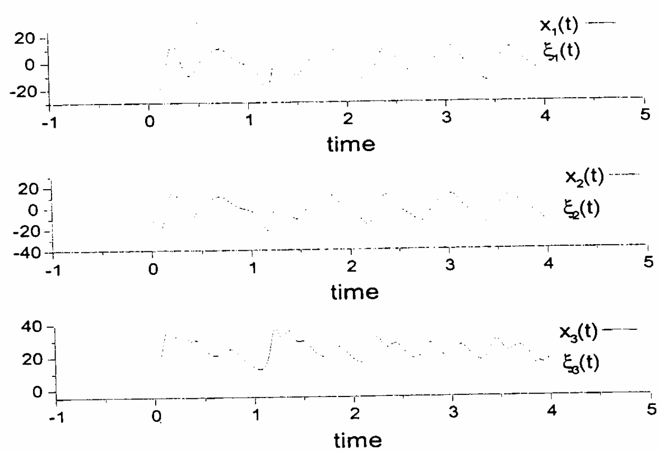

Fig. 4. Chen's chaotic system trajectories and synchronized receiver trajectories.

Choosing the output as $y = x_1$ one obtains,

$$\mathcal{C} = [1\ \ 0\ \ 0]\,, \quad \mathcal{S} = \begin{bmatrix} -a & \frac{c}{2} & 0 \\ \frac{c}{2} & c & 0 \\ 0 & 0 & -b \end{bmatrix}, \quad \mathcal{I} = \begin{bmatrix} 0 & a - \frac{c}{2} & 0 \\ -a + \frac{c}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{29}$$

The pair of matrices $(\mathcal{C}, \mathcal{S})$ already constitute a detectable, but not observable, pair. The addition to $\mathcal{S}$ of the matrix $\mathcal{I}$ does not improve the lack of observability. In this case, clearly, the unstable nature of the observable eigenvalues of $\mathcal{S}$ requires the introduction of damping through the output error injection map and proceed to place the eigenvalues of the observable part of the dissipative structure of the reconstruction error in suitable (asymptotically) stable locations in the complex plane. This results in the receiver,

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} 0 & a - \frac{c}{2} & 0 \\ -a + \frac{c}{2} & 0 & -x_1 \\ 0 & x_1 & 0 \end{bmatrix} \frac{\partial H}{\partial \xi} + \begin{bmatrix} -a & \frac{c}{2} & 0 \\ \frac{c}{2} & c & 0 \\ 0 & 0 & -b \end{bmatrix} \frac{\partial H}{\partial \xi} + \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} (x_1 - \xi_1) \tag{30}$$

The synchronization error, corresponding to this receiver, is found to be

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & a - \frac{c - K_2}{2} & \frac{K_3}{2} \\ -a + \frac{c - K_2}{2} & 0 & -x_1 \\ -\frac{K_3}{2} & x_1 & 0 \end{bmatrix} \frac{\partial H}{\partial e} + \begin{bmatrix} -K_1 - a & \frac{c - K_2}{2} & -\frac{K_3}{2} \\ \frac{c - K_2}{2} & c & 0 \\ -\frac{K_3}{2} & 0 & -b \end{bmatrix} \frac{\partial H}{\partial e} \tag{31}$$

We may now prescribe K_1, K_2 and K_3 in order to ensure asymptotic stability to zero of the synchronization error. This is achieved by setting $K_1 > c - a$, $K_2 > c/2 + 2(a + K_1)$. We may set $K_3 = 0$ since it has no influence on the observable eigenvalues of the nonconservative structure of the system.

Figure 4 shows the performance of the designed receiver with the following parameter values for the system and for the constant gains.

$$a = 35,\ b = 3,\ c = 28,\ K_1 = 2,\ K_2 = 100,\ K_3 = 0$$

4.3. *Chua's circuit*

The set of differential equations describing Chua's circuit were placed in Hamiltonian Canonical Form with a destabilizing field in the previous section. These are reproduced here, just for convenience.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{LC_2} \\ 0 & -\frac{1}{LC_2} & 0 \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} -\frac{G}{C_1^2} & \frac{G}{C_1C_2} & 0 \\ \frac{G}{C_1C_2} & -\frac{G}{C_2^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} -\frac{1}{C_1}F(x_1) \\ 0 \\ 0 \end{bmatrix} \tag{32}$$

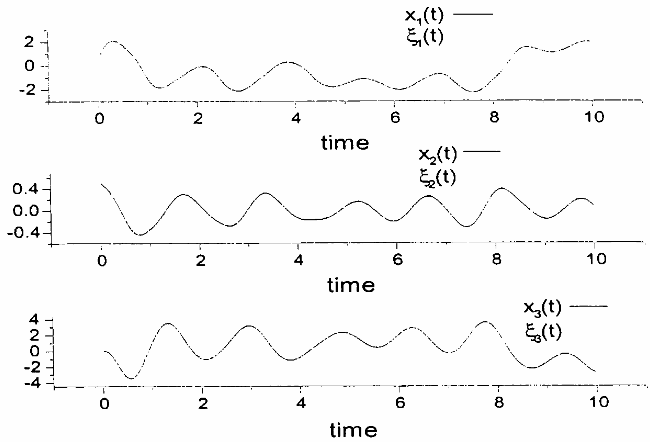

Fig: 5. Chua's chaotic circuit state trajectories and synchronized receiver trajectories.

The destabilizing vector field evidently calls for x_1 to be used as the output, y, of the transmitter. The matrices $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{I}$ are found to be

$$\mathcal{C} = \begin{bmatrix} \frac{1}{C_1} & 0 & 0 \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} -\frac{G}{C_1^2} & \frac{G}{C_1 C_2} & 0 \\ \frac{G}{C_1 C_2} & -\frac{G}{C_2^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{LC_2} \\ 0 & -\frac{1}{LC_2} & 0 \end{bmatrix} \tag{33}$$

The pair $(\mathcal{C}, \mathcal{S})$ is neither observable nor detectable. However, the pair $(\mathcal{C}, \mathcal{W})$ is observable. The system lacks damping in the x_3 variable, and either in the x_1 or the x_2 variable as inferred from the negative semi-definite nature of the dissipation structure matrix, $\mathcal{S}$. If x_1 is used as an output, then the output error injection term can enhance the dissipation in the error state dynamics. The receiver is designed as

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{LC_2} \\ 0 & -\frac{1}{LC_2} & 0 \end{bmatrix} \frac{\partial H}{\partial \xi} + \begin{bmatrix} -\frac{G}{C_1^2} & \frac{G}{C_1 C_2} & 0 \\ \frac{G}{C_1 C_2} & -\frac{G}{C_2^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\partial H}{\partial \xi} + \begin{bmatrix} -\frac{1}{C_1} F(y) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} e_y \tag{34}$$

The choice of K_1, K_2 and K_3 as arbitrary strictly positive constants suffices to guarantee the asymptotic exponential stability to zero of the synchronization error.

The synchronization error dynamics is governed by

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{K_2}{2C_1 C_2} & \frac{K_3}{2LC_1} \\ -\frac{K_2}{2C_1 C_2} & 0 & \frac{1}{LC_2} \\ -\frac{K_3}{2LC_1} & -\frac{1}{LC_2} & 0 \end{bmatrix} \frac{\partial H}{\partial e} + \begin{bmatrix} -\frac{G + C_1 K_1}{C_1^2} & \frac{2G - K_2}{2C_1 C_2} & -\frac{K_3}{2LC_1} \\ \frac{2G - K_2}{2C_1 C_2} & -\frac{G}{C_2^2} & 0 \\ -\frac{K_3}{2LC_1} & 0 & 0 \end{bmatrix} \frac{\partial H}{\partial e} \tag{35}$$

Figure 5 depicts the simulations of Chua's chaotic circuit state trajectories with the corresponding receiver responses. To ease the simulations we resorted to the following normalized version of the circuit (see [Huijberts *et al.*, 1998])

$$\begin{aligned}\dot{x}_1 &= \beta(-x_1 + x_2 - \phi(y)) \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -\gamma x_2\end{aligned} \tag{36}$$

with $\phi(y) = ay + \frac{1}{2}(b-a)[|1+y| - |1-y|]$ and

$$a = -\frac{5}{7}, \quad b = -\frac{8}{7}, \quad \beta = 15.6, \quad \gamma = 27$$

The parameter gains for the receiver were chosen to be

$$K_1 = 2, \quad K_2 = 3, \quad K_3 = 3$$

4.4. *The hysteretic circuit*

Consider the following nonlinear circuit equations treated by Carroll and Pecora [1991]

$$\begin{aligned}\dot{x}_1 &= x_2 + \gamma x_1 + c x_3 \\ \dot{x}_2 &= -\omega x_1 - \delta x_2 \\ \epsilon \dot{x}_3 &= (1 - x_3^2)(s x_1 + x_3) - \beta x_3\end{aligned} \tag{37}$$

The system can be written in Generalized Hamiltonian canonical form with the energy function given by

$$H(x) = \frac{1}{2}[x_1^2 + x_2^2 + \epsilon x_3^2] \tag{38}$$

Indeed,

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3\end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}(1+\omega) & \frac{1}{2\epsilon}(c-s) \\ -\frac{1}{2}(1+\omega) & 0 & 0 \\ -\frac{1}{2\epsilon}(c-s) & 0 & 0 \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} \gamma & \frac{1}{2}(1-\omega) & \frac{1}{2\epsilon}(c+s) \\ \frac{1}{2}(1-\omega) & -\delta & 0 \\ \frac{1}{2\epsilon}(c+s) & 0 & -\frac{1}{\epsilon^2}(\beta-1) \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} 0 \\ 0 \\ -x_3^2(x_3 + s x_1) \end{bmatrix} \tag{39}$$

The destabilizing vector field requires two signals for complete cancellation at the receiver. Namely, the variables, x_1 and x_3. The output is then chosen as the vector $y = [y_1,\ y_2]^T = [x_1,\ \epsilon x_3]^T$. The $\mathcal{C}$ and $\mathcal{S}$ matrices are given by

$$\mathcal{C} = \begin{bmatrix}\mathcal{C}_1 \\ \mathcal{C}_2\end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1\end{bmatrix},$$

$$\mathcal{S} = \begin{bmatrix} \gamma & \frac{1}{2}(1-\omega) & \frac{1}{2\epsilon}(c+s) \\ \frac{1}{2}(1-\omega)\epsilon & -\delta & 0 \\ \frac{1}{2\epsilon}(c+s) & 0 & -\frac{1}{\epsilon^2}(\beta-1) \end{bmatrix} \tag{40}$$

The pair $(\mathcal{C},\ \mathcal{S})$ is observable, and hence detectable. In order to achieve chaotic behavior, β is, in general, a small number, and the $\mathcal{S}$ matrix is therefore of indefinite sign. This means that the required receiver needs to add "multivariable" damping, through an output reconstruction error vector injection. However, one can easily avoid the multivariable pole placement problem by observing that the pair of matrices $(\mathcal{C}_1,\ \mathcal{S})$ is also an observable pair. An injection of the synchronization error $e_1 = x_1 - \xi_1$ suffices to have an asymptotically stable trajectory convergence. The receiver would then be designed, exploiting this last observation, as follows.

$$\begin{bmatrix}\dot{\xi}_{\square} \\ \dot{\xi}_{\square} \\ \dot{\xi}_{\square}\end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}(1+\omega) & \frac{1}{2\epsilon}(c-s) \\ -\frac{1}{2}(1+\omega)\epsilon & 0 & 0 \\ -\frac{1}{2\epsilon}(c-s) & 0 & 0 \end{bmatrix} \frac{\partial H}{\partial \xi} + \begin{bmatrix} \gamma & \frac{1}{2}(1-\omega) & \frac{1}{2\epsilon}(c+s) \\ \frac{1}{2}(1-\omega)\epsilon & -\delta & 0 \\ \frac{1}{2\epsilon}(c+s) & 0 & -\frac{1}{\epsilon^{\square}}(\beta-1) \end{bmatrix} \frac{\partial H}{\partial \xi} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{\epsilon^{\square}} y_{\square}^{\square} \left(\frac{1}{\epsilon} y_{\square} + s y_{\square}\right) \end{bmatrix} + \begin{bmatrix} K_{\square} \\ K_{\square} \\ K_{\square} \end{bmatrix} [x_{\square} - \xi_{\square}] \tag{41}$$

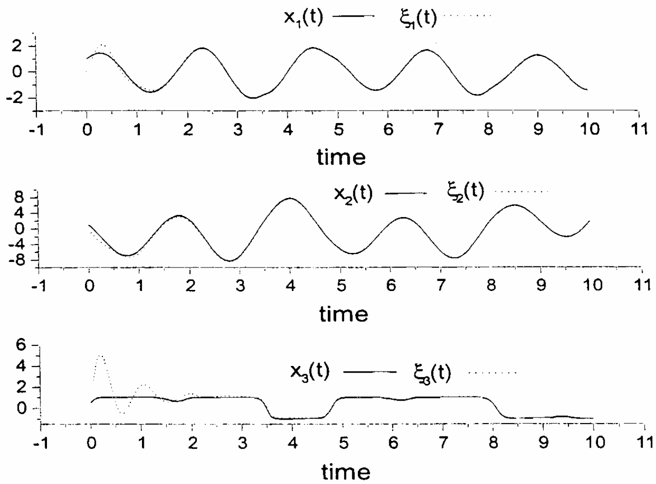

Fig. 6. Hysteretic system state trajectories and synchronized receiver trajectories.

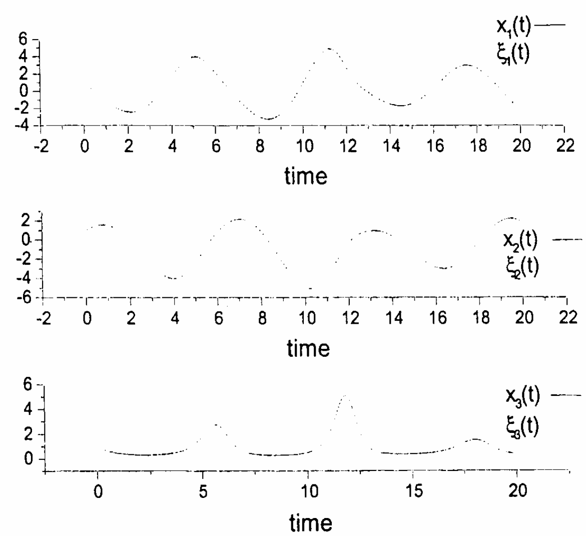

Fig. 7. Rössler chaotic system state trajectories and synchronized receiver trajectories.

Figure 6 shows the performance of the proposed synchronization scheme. The chosen parameters were set, following [Pecora & Carroll, 1991], as

$$\gamma = 0.2, \quad c = 2, \quad \omega = 10, \quad \delta = 0.001, \quad s = 1.667,$$
$$\beta = 0.001, \quad \epsilon = 0.3$$

with receiver parameter gains: $K_1 = 7.198$, $K_2 = -17.988$ and $K_3 = 13.927$.

4.5. *The Rössler system*

Consider the following chaotic system, known as the Rössler system [Pecora & Carroll, 1991]

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3\\ \dot{x}_2 &= x_1 + ax_2\\ \dot{x}_3 &= b + x_3(x_1 - c)\end{aligned} \tag{42}$$

With the energy function $H = (1/2)(x_1^2 + x_2^2 + x_3^2)$ we immediately obtain the system equations in the form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & -1 & -\frac{1}{2}\\ 1 & 0 & 0\\ \frac{1}{2} & 0 & 0\end{bmatrix}\frac{\partial H}{\partial x} + \begin{bmatrix}0 & 0 & -\frac{1}{2}\\ 0 & a & 0\\ -\frac{1}{2} & 0 & -c\end{bmatrix}\frac{\partial H}{\partial x} + \begin{bmatrix}0\\ 0\\ b + x_1x_3\end{bmatrix} \tag{43}$$

The destabilizing field is a function of x_1 and x_3. Thus, the outputs should be taken as $y_1 = x_1$ and $y_2 = x_3$. Notice, however, that the pair of matrices

$$\mathcal{C} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}, \quad \mathcal{W} = \begin{bmatrix}0 & -1 & -1\\ 1 & a & 0\\ 0 & 0 & -c\end{bmatrix} \tag{44}$$

is detectable and observable. This allows us to perform an eigenvalue placement using only injections of the synchronization error $e_1 = y_1 - \xi_1$ and, thus the multivariable pole placement is evaded.

The receiver may then be designed as

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\\ \dot{\xi}_3\end{bmatrix} = \begin{bmatrix}0 & -1 & -\frac{1}{2}\\ 1 & 0 & 0\\ \frac{1}{2} & 0 & 0\end{bmatrix}\frac{\partial H}{\partial \xi} + \begin{bmatrix}0 & 0 & -\frac{1}{2}\\ 0 & a & 0\\ -\frac{1}{2} & 0 & -c\end{bmatrix}\frac{\partial H}{\partial \xi} + \begin{bmatrix}0\\ 0\\ b + y_1y_2\end{bmatrix} + \begin{bmatrix}K_1\\ K_2\\ K_3\end{bmatrix}[y_1 - \xi_1]$$

Figure 7 shows the state trajectories of the Rössler system along with those of the synchronizing system. The parameters for the system, and for the observer gains, used in the simulation were taken as,

$$a = 0.4,\quad b = 2,\quad c = -4,\quad K_1 = 2.4,$$
$$K_2 = -2.1418,\quad K_3 = -1.8182.$$

4.6. *The Mitschke–Flüggen hybrid optical bistable chaotic system*

In [Mitschke & Flüggen, 1984] the circuit shown in Fig. 2 is proposed as an analog electronic model of an hybrid optical bistable system. The circuit equations are given as

$$\begin{aligned}C\frac{dx_1}{dt} &= \frac{1}{R}[-x_1 + \nu^2(x_3 - \mu)^2]\\ L_m\frac{dx_2}{dt} &= -R_mx_2 - x_3 + x_1\\ C_m\frac{dx_3}{dt} &= x_2\end{aligned} \tag{45}$$

where x_1 is the voltage across the capacitor C, x_2 is the current through the inductor and x_3 is the voltage in the second capacitor C_m. The total stored energy in the system can be taken as the positive definite Hamiltonian function

$$H(x) = \frac{1}{2}[Cx_1^2 + L_mx_2^2 + C_mx_3^2] \tag{46}$$

This leads to the following system in Generalized Hamiltonian canonical form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & -\frac{1}{2CL_m} & 0\\ \frac{1}{2CL_m} & 0 & -\frac{1}{L_mC_m}\\ 0 & \frac{1}{L_mC_m} & 0\end{bmatrix}\frac{\partial H}{\partial x} + \begin{bmatrix}-\frac{1}{RC^2} & \frac{1}{2CL_m} & 0\\ \frac{1}{2CL_m} & -\frac{R_m}{Lm^2} & 0\\ 0 & 0 & 0\end{bmatrix}\frac{\partial H}{\partial x} + \begin{bmatrix}\frac{1}{R}\nu^2(x_3 - \mu)^2\\ 0\\ 0\end{bmatrix} \tag{47}$$

The destabilizing presence of x_3 suggests that the output of the transmitter should be the voltage variable $y = x_3$. This implies that the matrices $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{I}$ are given by

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & \frac{1}{C_m} \end{bmatrix},$$

$$\mathcal{S} = \begin{bmatrix} -\frac{1}{RC^2} & \frac{1}{2CL_m} & 0 \\ \frac{1}{2CL_m} & -\frac{R_m}{Lm^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{48}$$

$$\mathcal{I} = \begin{bmatrix} 0 & -\frac{1}{2CL_m} & 0 \\ \frac{1}{2CL_m} & 0 & -\frac{1}{L_mC_m} \\ 0 & \frac{1}{L_mC_m} & 0 \end{bmatrix}$$

The pair of matrices $(\mathcal{C}, \mathcal{S})$ is not observable but it is detectable. However, the pair of matrices, $(\mathcal{C}, \mathcal{W})$, with $\mathcal{W}$ given by

$$\mathcal{W} = \begin{bmatrix} -\frac{1}{RC^2} & 0 & 0 \\ \frac{1}{CL_m} & -\frac{R_m}{Lm^2} & -\frac{1}{L_mC_m} \\ 0 & \frac{1}{L_mC_m} & 0 \end{bmatrix} \tag{49}$$

is found to be observable. In order to add suitable damping to the synchronization error dynamics an ouput reconstruction error injection is needed. A receiver can then be designed as

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC^2} & 0 & 0 \\ \frac{1}{CL_m} & -\frac{R_m}{Lm^2} & -\frac{1}{L_mC_m} \\ 0 & \frac{1}{L_mC_m} & 0 \end{bmatrix} \frac{\partial H}{\partial \xi} + \begin{bmatrix} \frac{1}{R}\nu^2(y-\mu)^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} (y - \xi_3) \tag{50}$$

To guarantee asymptotic stability of the error dynamics, it suffices to choose K_1, K_2, K_3 as arbitrary strictly positive constants.

The synchronization error evolves according to

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC^2} & 0 & -K_1 \\ \frac{1}{CL_m} & -\frac{R_m}{Lm^2} & -\left(\frac{1}{L_mC_m} + K_2\right) \\ 0 & \frac{1}{L_mC_m} & -K_3 \end{bmatrix} \frac{\partial H}{\partial e} \tag{51}$$

5. Conclusions

In this article, we have approached the problem of synchronization of chaotic systems from the perspective of Generalized Hamiltonian systems including dissipation and destabilizing terms. The approach allows to give a simple design procedure for the receiver system and clarifies the issue of deciding on the nature of the output signal to be transmitted. This may be accomplished on the basis of a simple linear detectability or observability test. Several chaotic systems were analyzed from this new perspective and their possibilities for synchronization were either confirmed, in the case of already obtained positive results, or it was explained in those cases where there is a known lack of synchronization.

The Generalized Hamiltonian nature of many chaotic systems definitely helps in the study of robust synchronization, under the addition of masked transmitted signals seen as perturbations of the state reconstruction error dynamics. More importantly, given the clear energy managing structure of Generalized Hamiltonian systems, the approach definitely helps in the study, via passivity-based techniques, of linear and nonlinear feedback control strategies for chaotic systems. These will be the issues of a forthcoming publication.

References

Carroll, T. L. & Pecora, L. [1991] "Synchronizing chaotic circuits," *IEEE Trans. Circuits Syst.* **38**(4), 453–456.

Chen, G. [1997] *Control and Synchronization of Chaos, a Bibliography*, Department of Electrical Engineering, University of Houston, Houston TX (also available via ftp at ftp:uhcop.uh.edu/pub/chaos.tex).

Cuomo, K. M., Oppenheim, A. V. & Strogatz, S. H. [1993] "Synchronization of Lorenz-based chaotic circuits with applications to communications," *IEEE Trans. Circuits Syst. II: Anal. Digit. Sign. Process.* **40**, 626–633.

Fradkov, A. L. & Markov, A. Yu. [1997] "Adaptive synchronization of chaotic systems based on speed gradient method and passification," *IEEE Trans. Circuits Syst. I: Fundamental Th. Appl.* **44**(10), 905–917.

Fradkov, A. L. & Pogromsky, A. Yu. [1998] *Introduction to Control of Oscillations and Chaos*, Series A, Vol. 35 (World Scientific, Singapore).

Huijberts, H. J. C., Nijmeijer, H. & Willems, R. M. A. [1998] "A control perspective on communications using chaotic systems," *Proc. 37th IEEE Conf. Decision and Control*, Tampa, Florida, pp. 1957–1962.

Lorenz, E. N. [1963] "Deterministic nonperiodic flow," *J. Atmosph. Sci.* **20**, 130–141.

Mitschke, F. & Flüggen, N. [1984] "Chaotic behavior of a hybrid optical bistable system without a time delay," *Appl. Phys.* **B35**, 59–64.

Nijmeijer, H. & Mareels, M. Y. [1997] "An observer looks at synchronization," *IEEE Trans. Circuits Syst. I: Fundamental Th. Appl.* **44**(10), 882–890.

Ott, E., Sauer, T. & Yorke, J. A. (eds.) [1994] *Coping with Chaos: Analysis of Chaotic Data and the Exploitation of Chaotic Systems* (Wiley-Interscience, NY).

Pecora, L. M. & Carroll, T. L. [1991] "Driving systems with chaotic signals," *Phys. Rev.* **A44**(4), 2374–2383.

Pogromsky, A. Yu. [1998] "Passivity-based design of synchronizing systems," *Int. J. Bifurcation and Chaos* **8**(2), 295–319.

Special Issue [1993] "Chaos synchronization and control: Theory and applications," *IEEE Trans. Circuits Syst. I: Fundamental Th. Appl.* **40**.

Special Issue [1997a] *Syst. Contr. Lett.* **31**.

Special Issue [1997b] "Chaos synchronization and control: Theory and applications," *IEEE Trans. Circuits Syst. I: Fundamental Th. Appl.* **44**(10).

Wu, C. W. & Chua, L. [1993] "A simple way to synchronize chaotic systems with applications to secure communication systems," *Int. J. Bifurcation and Chaos* **3**(6), 1619–1627.