Regulation of non-minimum phase outputs: a flatness based approach

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À Ioan Landau,
en hommage amical pour
son soixantième anniversaire.

Summary. [1] We present a flatness based control synthesis for non-minimum phase outputs of linear and nonlinear systems. Three concrete examples are illustrating our approach.

Introduction

We want to regulate a non-minimum phase controllable and observable SISO system, with input u and output y, given by the transfer function $\frac{P(s)}{Q(s)}$, where

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- $P(s), Q(s) \in \mathbf{R}[s]$ are coprime,
- some of the transmission zeros, i.e., some of the zeros of $P(s)$, have non-negative real parts.

By Bézout's theorem there exists $A, B \in \mathbf{R}[s]$ such that $AP + BQ = 1$. Introduce a new system variable z by

$$z = A(\frac{d}{dt})y + B(\frac{d}{dt})u \tag{11.1}$$

The transfer function of the system with input u and output z is $\frac{1}{Q(s)}$. The quantities u, y and z satisfy

$$\begin{array}{rcl} u & = & Q(\frac{d}{dt})z \\ y & = & P(\frac{d}{dt})z \end{array} \tag{11.2}$$

Equations (11.2) tell us that u and y are expressed as linear combinations of z and its derivatives up to some finite order. According to (11.1), z is, conversely, expressed as a linear combination of u and y and their derivatives up to some finite order. Those properties are reminiscent to *flatness* (cf. [19, 22]) and z is called a *flat output*. We will exploit those properties for our non-minimum phase tracking (see [1] and [7] for other flatness based control strategies of linear systems). Over a finite time interval $T_1 \leq t \leq T_2$, a suitable open loop strategy for $u(t)$, $y(t)$ and $z(t)$ may be given by taking polynomial time functions which satisfy (11.1)-(11.2). This polynomial specification is far from being the only possible one. On an infinite time interval $t \geq T$, one could choose finite Fourier series for tracking some periodic reference trajectory. The feedback stabilization around the reference trajectory may be achieved by classic techniques. Robustness could be tackled as in [1]. Many of the well-known performance limitations related to non-minimum phase systems (see, e.g., [40]) are thus bypassed.

The zero-dynamics, which was introduced by Byrnes and Isidori [5] (see, also, [26, 36] and the references therein), allows checking the non-minimum phase character of input-output nonlinear systems. The control of such non-minimum phase systems has already been the subject of a rich literature (see, e.g., [10, 24, 29, 32, 37, 50, 52][2]).

Here we attack the problem from the point of view of flatness, which has already been utilized in [32, 37]. A (*differentially*) *flat* system [19, 22] is equivalent to a controllable linear system via a special type of dynamic feedback, called *endogenous*. It is best understood without making any distinction between the system variables:

1. Every system variable may be expressed as a function of the components of a finite set $z = (z_1, \ldots, z_m)$ and of a finite number of their time-derivatives.

[2] The thesis of van Nieuwstadt [37] contains an interesting bibliographical analysis.

2. Every component of z may be expressed as a function of the system variables and of a finite number of their time-derivatives.
3. The components of z are differentially independent, i.e., they are not related by any differential relation.

The fictitious output z is called a *flat output*[3]. In all known examples it may be chosen with a clear engineering and physical meaning. More formally, the notion of flat output may be introduced for linear systems using the language of modules (see [3, 13, 14, 17, 19]) and for nonlinear systems via differential fields (see [12, 14, 19]). This last approach could have been replaced by the differential geometry of infinite jets and prolongations (see [21, 22, 38, 39] and the references therein).

Three concrete examples are examined and simulations are provided. The first one is an experimental flexible structure given by a transfer function [8, 9]. The second one, which has already been studied by one of the authors [46, 47], is a dc-to-dc power converter, which is static state-feedback linearizable. The last one is a PVTOL aircraft [24] which is not static state-feedback linearizable but flat, (see [32, 33] and [22]). In all those three cases we have given a stabilizing feedback around the reference trajectory. Note that for the power converter this is a passivity based output feedback [48].

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1. Linear systems

1.1 Modules

Let k be a field and $k[\frac{d}{dt}]$ be the commutative principal ideal ring of polynomials of the form $\sum_{finite} a_\alpha \frac{d^\alpha}{dt^\alpha}$, $a_\alpha \in k$. Let M be a $k[\frac{d}{dt}]$-module. An element $m \in M$ is said to be *torsion* if, and only if, there exists a polynomial $\pi \in k[\frac{d}{dt}]$, $\pi \neq 0$, such that $\pi m = 0$. The set tM of all torsion elements of M is a submodule of M; it is said to be *trivial* if, and only if, $tM = \{0\}$. A $k[\frac{d}{dt}]$-module is said to be *torsion* if, and only if, all its elements are torsion; it is said to be *torsion-free* if, and only if, tM is trivial.

A finitely generated $k[\frac{d}{dt}]$-module M is said to be *free* if, and only if, there exists a *basis*, i.e., a finite set $b = (b_1, \ldots, b_m)$ such that

– any element of M depends $k[\frac{d}{dt}]$-linearly on b,
– the components of b are $k[\frac{d}{dt}]$-linearly independent.

The *rank* of this free module is m.

Here are some standard properties of finitely generated modules over principal ideal rings (see, e.g., [30]).

[3] If m independent channels have been distinguished, this last statement is equivalent to saying that z possesses m components.

1. A finitely generated $k[\frac{d}{dt}]$-module M may be written
$$M \simeq tM \oplus \mathcal{F} \tag{11.3}$$
where tM is the torsion submodule and $\mathcal{F} \simeq M/tM$ is a free module. The *rank* of M is, by definition, the rank of $\mathcal{F}$.
2. For a finitely generated $k[\frac{d}{dt}]$-module M, the following two properties are equivalent:
   – M is torsion,
   – the dimension $\dim_k M$ of M as a k-vector space is finite.
3. Any submodule of a finitely generated (free) $k[\frac{d}{dt}]$-module is a finitely generated (free) $k[\frac{d}{dt}]$-module. Any quotient module of a finitely generated $k[\frac{d}{dt}]$-module is a finitely generated $k[\frac{d}{dt}]$-module.
4. For a finitely generated $k[\frac{d}{dt}]$-module M, the following two properties are equivalent:
   – M is torsion-free,
   – M is free.

Remark 1.1. All modules considered in the sequel will be finitely generated $k[\frac{d}{dt}]$-modules.

Let M be a module. The derivation $\frac{d}{dt}$ defines an endomorphism of the torsion submodule tM, which may be viewed as a k-linear endomorphism τ of the finite-dimensional k-vector space tM. A *Smith zero* [3] of M is an eigenvalue of τ over the algebraic closure $\bar{k}$ of k.

Notation. Write $[S]$ the submodule spanned by a subset S of M.

1.2 Systems

A *k-linear system* Λ is a module. A *k-linear dynamics* is a k-linear system Λ with an *input*, i.e., with a finite subset $u = (u_1, \dots, u_m)$ such that the quotient module $\Lambda/[u]$ is torsion. The input u is assumed to be free, i.e., the submodule $[u]$ is free of rank m. Then, the rank of Λ is equal to m. A *k-linear input-output system* is a k-linear dynamics Λ with an *output*, i.e., with a finite subset $y = (y_1, \dots, y_p)$ of Λ.

There exists a short exact sequence
$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{F} \longrightarrow \Lambda \longrightarrow 0$$
The module $\mathcal{F}$ is free. The free module $\mathcal{N}$, which is called sometimes the *module of relations*, should be viewed as a *system of equations* defining Λ.

Example 1.1. Consider the system of equations

$$\sum_{\kappa=1}^{\mu} a_{\iota\kappa} w_\kappa = 0 \tag{11.4}$$

where $a_{\iota\kappa} \in k[\frac{d}{dt}]$, $\iota = 1, \ldots, \nu$. The unknowns are $w_1, \ldots, w_\mu$. Let $\mathcal{F}$ be the free module spanned by $f_1, \ldots, f_\mu$. Let $\mathcal{N} \subseteq \mathcal{F}$ be the submodule spanned by $\sum_{\kappa=1}^{\mu} a_{\iota\kappa} f_\kappa$. The module corresponding to (11.4) is $\mathcal{F}/\mathcal{N}$.

1.3 Controllability and observability

A k-linear system Λ is called *controllable* if, and only if, the module Λ is free[4]. Any basis of Λ, which may be viewed as a fictitious output, is called a *flat*, or *basic, output*.

Example 1.2. Consider the classic state-variable representation

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + B \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \tag{11.5}$$

where $A \in k^{n\times n}$, $B \in k^{n\times m}$. The control variables $u = (u_1, \ldots, u_m)$ are assumed to be independent. It follows from [13] that (11.5) is controllable, i.e., that $\mathrm{rk}(B, AB, \ldots, A^{n-1}B) = n$, if, and only if, the corresponding module Λ is free. As a matter of fact, the torsion submodule $t\Lambda$ in the decomposition (11.3) corresponds to the Kalman uncontrollable subspace.

Example 1.3. Assume that (11.5) is controllable. There exists a static state feedback which transforms it into the famous *Brunovský canonical form* (see, e.g., [27]; see also [16, 18] for a module-theoretic derivation which comprises the time-varying case) which reads $z^{(\nu_i)} = v_i$, $i = 1, \ldots, m$, where the v_i's are the new control variables and the ν_i's the *controllability*, or *Kronecker, indices*.

The next property is clear.

Proposition 1.1. *The set $z = (z_1, \ldots, z_m)$ is a flat output.*

Example 1.4. Consider the input-output system

$$\mathcal{A}\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = \mathcal{B}\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \tag{11.6}$$

where $\mathcal{A} \in k[\frac{d}{dt}]^{p\times p}$, $\det \mathcal{A} \neq 0$, $\mathcal{B} \in k[\frac{d}{dt}]^{p\times m}$. It is known that (11.6) is controllable if, and only if, $\mathcal{A}$ and $\mathcal{B}$ are left coprime (see, e.g., [25] and [3, 17, 18]). Denote by Λ its corresponding module.

[4] See [15] for the connection with Willems' behavioral approach [51].

Proposition 1.2. *The output $y = (y_1, \ldots, y_p)$ is flat if, and only if, the following two conditions are satisfied:*
1) the matrices $\mathcal{A}$ and $\mathcal{B}$ are left coprime;
2) the system is square, i.e., $m = p$, and the matrix $\mathcal{B}$ is unimodular.

Proof. The first condition ensures the controllability of (11.6) and is equivalent to the freeness of Λ. The second condition is equivalent to $u_i \in [y]$, $i = 1, \ldots, m$.

A k-linear system Λ with input $u = (u_1, \ldots, u_m)$ and output $y = (y_1, \ldots, y_p)$ is called *observable* if, and only if, $[u, y] = \Lambda$ (see [13], where it is shown that this intrinsic definition coincides with the classic one for systems given by the Kalman state-variable representation).

1.4 Equivalence and endogenous feedback

Two systems Λ_ι, $\iota = 1, 2$, are said to be *equivalent* if, and only if, the modules Λ_1 and Λ_2 are isomorphic. In other words any variable of one system may be expressed as a $k[\frac{d}{dt}]$-linear combination of the variables of the other: those expressions define an *endogenous feedback* (compare with [19, 22]).

Example 1.5. The controllable dynamics $x_\iota^{(n_\iota)} = u_\iota$ corresponds to the rank 1 free module Λ_ι with basis x_ι, $\iota = 1, 2$. The isomorphism $\Lambda_1 \to \Lambda_2$, $x_1 \longmapsto x_2$, shows the equivalence of the two dynamics. Note that, if $n_1 \neq n_2$, u_2 is not the image of u_1 under this isomorphism.

1.5 Transmission zeros

Take a system Λ with input $u = (u_1, \ldots, u_m)$ and output $y = (y_1, \ldots, y_p)$. Let $\varpi : \Lambda \to \Lambda/t\Lambda$ be the canonical epimorphism (the free module $\Lambda/t\Lambda$ is the *transfer module* [45]). Set $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_m)$, $\overline{y} = (\overline{y}_1, \ldots, \overline{y}_p)$, where $\overline{u}_i = \varpi u_i$, $i = 1, \ldots, m$, $\overline{y}_j = \varpi y_j$, $j = 1, \ldots, p$.

Lemma 1.1. *The restriction of ι to $[u]$ induces an isomorphism $[u] \to [\overline{u}]$.*

Proof. It follows at once from $[u] \cap t\Lambda = \{0\}$ and $\ker \varpi = t\Lambda$.

Let $\mathcal{T}$ be the quotient module $[\overline{u}, \overline{y}]/[\overline{y}]$. The *transmission zeros* of Λ (see [3]) are the Smith zeros of $\mathcal{T}$.

When k is a subfield of the field $\mathbb{C}$ of complex numbers, the system Λ is said to be *minimum phase* if, and only if, the real parts of the transmission zeros are strictly negative.

Remark 1.2. Assume that the input-output system Λ is controllable, i.e., that the module Λ is free. Then $\mathcal{T} = [u, y]/[y]$.

Example 1.6. Take again (11.6), which is assumed to be controllable, i.e., $\mathcal{A}$ and $\mathcal{B}$ are left coprime. The transmission zeros of (11.6) are the zeros of the invariant factors of $\mathcal{B}(s)$, where s stands for $\frac{d}{dt}$.

The notion of equivalence is independent of any distinction between the system variables. We now introduce an equivalence for input-output systems, which preserves the output. Consider a system Λ with input u and output y and another system Σ with the same output y and a new input v. These two systems are said to be *equivalent* (*by endogenous feedback*) if, and only if, the following conditions are satisfied:

1. there exists an isomorphism $\alpha : \Lambda \rightarrow \Sigma$,
2. the restriction of α to $[u, y]$ defines an isomorphism between $[u, y]$ and $[v, y]$.

The next result follows at once.

Proposition 1.3. *The transmission zeros are invariant under equivalence.*

Remark 1.3. Two input-output systems which are equivalent by static state feedback are not necessarily equivalent in the above sense as static state feedback does not necessarily preserves observability. This is why transmission zeros are not preserved by static state feedback equivalence (see, e.g., [11]).

1.6 Controllable and observable SISO systems

To a controllable and observable system Λ with a single input u and a single output y corresponds a rank 1 free module Λ such that $\Lambda = [u, y]$. Let z be a basis of Λ (any other basis is of the form αz, $\alpha \in k, \alpha \neq 0$). Set $u = Qz, y = Pz; P, Q \in k[\frac{d}{dt}]$.

Proposition 1.4. *The polynomials P and Q are coprime. The transmission zeros of Λ are the zeros of $P(s)$ over $\overline{k}$, where the variable s stands for $\frac{d}{dt}$.*

Proof. Bézout's theorem shows that $[u, y] = [z]$ $(= \Lambda)$ if, and only if, P and Q are coprime. From $Qy = Pu$ and from the property of the transmission zeros of (11.6), we see that the transmission zeros of Λ are the zeros of $P(s)$ over $\overline{k}$.

The transfer function of the input-output system Λ is $\frac{P(s)}{Q(s)}$ (see [17] for a module-theoretic approach to the Laplace transform and to transfer matrices, which comprises the time-varying case). The next property is clear.

Proposition 1.5. *The system output y is a flat output if, and only if, the numerator of the transfer function is a non-zero constant.*

2. Application to a linear SISO system

2.1 Description of the System

The following transfer function [8, 9], which has been obtained by identification,

$$\frac{0.27s^2 - 0.1187s + 6.1041}{s^4 + 0.2765s^3 + 6.1041s^2}$$

describes an experimental flexible structure used for studying elastic vibrations. It consists of two freely rotating disks connected by a thin shaft as shown in Figure 11.1.

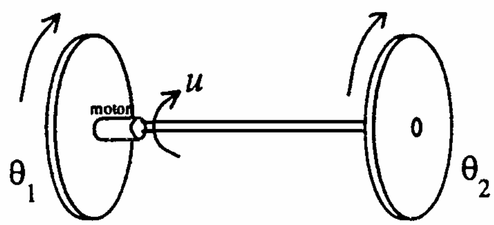

Fig. 11.1. Flexible system example: Schematic diagram

The control is the torque u provided by a dc motor. The output $y = \theta_2$ is the farthest away angular rotation, which is measured by potentiometers. The transmission zeros $0.2198 \pm 4.7497\sqrt{-1}$ are unstable and y is non-minimum phase.

2.2 Open loop strategy

According to (11.1) and (11.2), the flat output z satisfies the following relations

$$\begin{aligned} y &= \left[a_1 \frac{d^2}{dt^2} + a_2 \frac{d}{dt} + a_3\right] z \\ u &= \left[\frac{d^4}{dt^4} + b_1 \frac{d^3}{dt^3} + b_2 \frac{d^2}{dt^2}\right] z \\ z &= \left[c_1 \frac{d}{dt} + c_2\right] u + \left[c_3 \frac{d^3}{dt^3} c_4 \frac{d^2}{dt^2} + c_5 \frac{d}{dt} + c_6\right] y \end{aligned}$$

(where $a_1 = 0.27, a_2 = -0.1187, a_3 = 6.1041, b_1 = 0.2765, b_2 = 6.1041, c_1 = 0.0001635, c_2 = 0.002497, c_3 = -0.0006058, c_4 = -0.00968, c_5 = 0.00319, c_6 = 0.16382$)

We are driving y from an operating point $y(T_1) = \overline{y}_1$ to another one $y(T_2) = \overline{y}_2$, for $T_1 < T_2$. Set $z_d(T_i) = \overline{y}_i / a_3$, $i = 1, 2$. The time function $z_d(t)$, $T_1 \leq t \leq T_2$ is defined as a suitable polynomial in t, of degree 9. It yields functions $y(t)$, $u(t)$, $T_1 \leq t \leq T_2$, which are, at least, continuous. The behavior of y, u and z is illustrated in Figure 11.2.

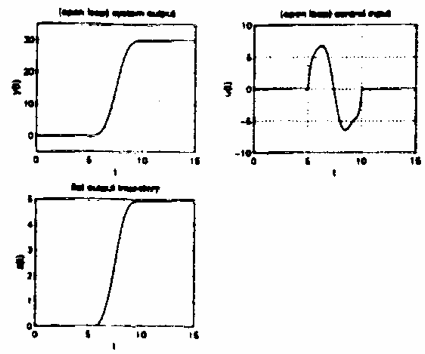

Fig. 11.2. Open loop control

2.3 Time scaling

Apply, as already often done for flat systems (see, e.g., [19, 20] and [1]), a time-scaling which is here linear, i.e., $t = \frac{\tau}{\alpha}$, where $\alpha > 0$ is a constant. This results in

$$\begin{aligned} u &= \left[\alpha^4 \frac{d^4}{d\tau^4} + \alpha^3 b_1 \frac{d^3}{d\tau^3} + \alpha^2 b_2 \frac{d^2}{d\tau^2}\right] z \\ y &= \left[\alpha^2 a_1 \frac{d^2}{d\tau^2} + \alpha a_2 \frac{d}{d\tau} + a_3\right] z \end{aligned}$$

For $0 < \alpha < 1$, we can reduce the magnitude of u if necessary (compare with [8, 9]). Figure 11.3 shows y, u and z for $\alpha = 0.5$.

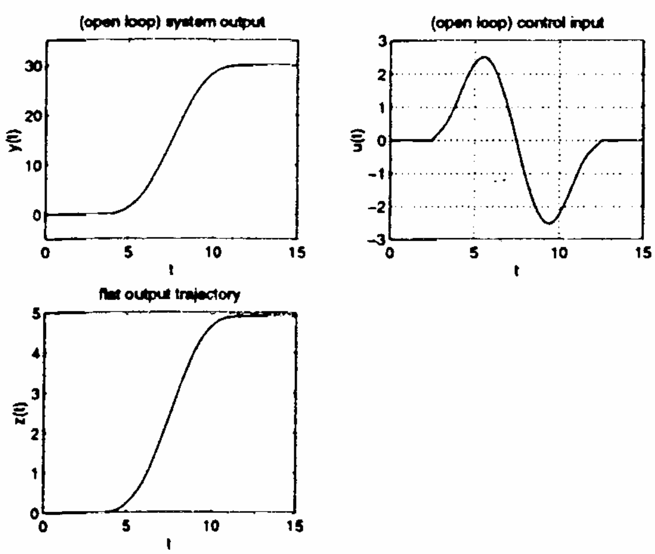

Fig. 11.3. Open loop control under time scaling of the reference trajectory

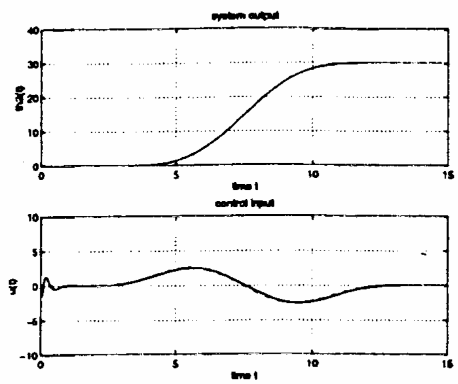

Fig. 11.4. Close loop control

2.4 Feedback stabilization

In Figure 11.4 we present simulations of a closed-loop control strategy using a standard observer-controller scheme for a controllable and observable state space representation:

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & x_4 \\ \dot{x}_4 & = & -b_2 x_3 - b_1 x_4 \\ y & = & a_3 x_1 + a_2 x_2 + a_1 x_3 \end{array}$$

3. Nonlinear systems

3.1 Differential fields

All fields have characteristic zero. An *(ordinary) differential field* K (see, e.g. [4, 28])[5] is a field which is equipped with a single derivation $\frac{d}{dt}$ = "$\dot{}$" such that

$$\forall a \in K, \ \frac{da}{dt} = \dot{a} \in K$$
$$\forall a, b \in K, \ \frac{d}{dt}(a+b) = \dot{a} + \dot{b}, \ \frac{d}{dt}(ab) = \dot{a}b + a\dot{b}$$

A differential field extension L/K is given by two differential fields K, L such that $K \subseteq L$. An element in L is *differentially algebraic* over K if, and only if, it satisfies an algebraic differential equation with coefficients in K. An element in L is *differentially transcendental* over K if, and only if, it is not differentially algebraic over K. The extension L/K is differentially algebraic if, and only if, any element of L is differentially algebraic over K; L/K is *differentially transcendental* if, and only if, it is not differentially algebraic.

[5] See [14, 19] for a detailed review.

A set $\xi = \{\xi_i \mid i \in I\}$ of elements in L is said to be *differentially algebraically (in)dependent* over K if, and only if, the set of derivatives of arbitrary orders $\{\xi_i^{(\nu_i)} \mid i \in I, \nu_i = 0, 1, 2, \dots\}$ is algebraically (in)dependent over K. An independent set which is maximal with respect to inclusion is called a *differential transcendence basis* of L/K. Two such bases have the same cardinality which is the *differential transcendence degree* of L/K; it is written diff tr d^0 L/K.

Notation. The differential subfield of L generated by K and the set ξ is denoted $K\langle\xi\rangle$.

A differentially transcendental field extension L/K is said to be *pure* if, and only if, there exists a differential transcendence basis ξ such that $L = K\langle\xi\rangle$.

Remark 3.1. From now on all differential field extensions will be finitely generated.

3.2 Systems

Let k be a given differential ground field. A k-system is a differential field extension K/k [12, 14, 19]. A k-dynamics is a k-system K/k equipped with an *input*, i.e., with a finite set $u = (u_1, \dots, u_m)$ such that $K/k\langle u\rangle$ is differentially algebraic. From now on the input is assumed to be *independent*, i.e., $k\langle u\rangle/k$ is a pure differentially transcendental extension and diff tr d^0 $k\langle u\rangle = m$. An *output* is a finite set $y = (y_1, \dots, y_p)$ of elements in K/k.

3.3 Equivalence, endogenous feedback, differential flatness

Two systems K_ι/k, $\iota = 1, 2$, are *k-equivalent* [19] if, and only if, there exist differential fields M_ι such that M_ι/K_ι is algebraic and the two differential extensions M_1/k and M_2/k are differentially k-isomorphic. In other words, any variable of one of the systems may be expressed as an algebraic function of the variables of the other one and of their derivatives up to some finite order. Those expressions define an *endogenous feedback* between the two systems [19].

A system K/k is (*differentially*) *flat* [19] if, and only if, K/k is k-equivalent to a pure differentially transcendental extension of k. A differential transcendence basis $z = (z_1, \dots, z_m)$ of K/k such that $K/k\langle z\rangle$ is (non-differentially) algebraic, which may be viewed as a fictitious output, is called a *flat*, or *linearizing, output*. Recall that a flat system is equivalent to a controllable linear system.

3.4 Residual dynamics

Consider a system K/k with input u and output y. Its *residual dynamics* is the differential field extension $K/k\langle y\rangle$; its *input residual dynamics* is $k\langle u, y\rangle/k\langle y\rangle$

(compare with the notion of zero dynamics [26, 36]). When the system K/k is flat, it might also be useful to consider the extension $k\langle u, z\rangle/k\langle z\rangle$, where z is a flat output.

Take two systems K/k and L/k with respective inputs u and v but with the output y. They are said to be *k-equivalent* (*by endogenous feedback*) if, and only if, the next two conditions are satisfied:

1. There exists a differential k-isomorphism $\alpha : \mathcal{M}/k \to \mathcal{N}/k$, where the differential fields $\mathcal{M}$ and $\mathcal{N}$ are algebraic extensions of K and L.
2. There exists differential fields M and N, which are finite algebraic extensions of $k\langle u, y\rangle$ and $k\langle v, y\rangle$, such that the restriction of α to M induces a differential k-isomorphism between M/k and N/k.

The next result is clear.

Proposition 3.1. *Take two k-equivalent input-output systems with the same output y. Then their (input) residual dynamics are $k\langle y\rangle$-equivalent.*

4. A dc-to-dc power converter

4.1 Description and flatness

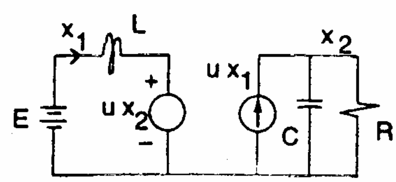

Fig. 11.5. "Boost" converter circuit

The *average* behavior of a pulse-width-modulated "boost" converter circuit (see Figure 11.5) may be, according to [35], defined by

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{L}ux_2 + \frac{E}{L} \\ \dot{x}_2 &= \frac{1}{C}ux_1 - \frac{1}{RC}x_2 \end{aligned} \qquad (11.7)$$

where x_1 is the average inductor current, x_2 is the average output capacitor voltage and u is the *duty ratio function* taking values in the closed interval $[0, 1]$ and acting as an input variable. It is known [47] that the output x_1 is minimum phase, while x_2 is not.

Taking as an output $\zeta = x_1$, we propose the following observer for the output feedback implementation of the control law (11.9).

$$\begin{aligned} \dot{\hat{x}}_1 &= -\frac{1}{L}u\hat{x}_2 + \frac{E}{L} + \frac{R_1}{L}(\zeta - \hat{\zeta}) \\ \dot{\hat{x}}_2 &= \frac{1}{C}u\hat{x}_1 - \frac{1}{RC}\hat{x}_2 \\ \hat{\zeta} &= \hat{x}_1 \end{aligned}$$

with $R_1 > 0$ being an observer design parameter.

The state estimation error components, $\epsilon_1 = \zeta - \hat{\zeta} = x_1 - \hat{x}_1$, $\epsilon_2 = x_2 - \hat{x}_2$, evolve according to the following *controlled* dynamics

$$\begin{aligned} \dot{\epsilon}_1 &= -\frac{1}{L}u\epsilon_2 - \frac{R_1}{L}\epsilon_1 \\ \dot{\epsilon}_2 &= \frac{1}{C}u\epsilon_1 - \frac{1}{RC}\epsilon_2 \end{aligned}$$

If one takes as a Lyapunov function candidate the positive definite function $V(\epsilon) = \frac{1}{2}(L\epsilon_1^2 + C\epsilon_2^2)$, one easily verifies that the time derivative of $V(\epsilon)$, along the controlled trajectories of the state estimation error vector ϵ, is not only independent of u, but it is also negative definite,

$$\dot{V}(\epsilon) = -R_1\epsilon_1^2 - \frac{1}{R}\epsilon_2^2 \leq -\alpha V(\epsilon)$$

for $\alpha = min\{R_1, 1/R\}/max\{L, C\} > 0$. i.e., the quadratic function $V(\epsilon)$ *exponentially* converges to zero. As a consequence, the components of the estimation error vector e also exponentially converge to zero.

The output feedback controller is now given by

$$\begin{aligned} u \;=\; & \frac{LRC}{[2Lx_1 + ERC]\,\hat{x}_2}\left[\frac{E^2}{L} + \frac{2}{R^2C}\hat{x}_2^2 - \ddot{y}\ (t) + 2\xi\omega_n\left(Ex_1 - \frac{\hat{x}_2^2}{R} - \dot{y}^*(t)\right)\right. \\ & \left. + \omega_n^2\left(\frac{1}{2}Lx_1^2 + \frac{1}{2}C\hat{x}_2^2 - y^*(t)\right)\right] \end{aligned}$$

4.4 Simulation results

Two voltage transitions were performed. First, from arbitrary initial conditions, it was set as a control objective the stabilization of the voltage x_2 around a constant equilibrium value $x_2 = V_{d1} = 30$ V. From the reached equilibrium, a second transfer, starting at time $T_1 = 0.07$ s., was enforced to reach, at time $T_2 = 0.12$ s., a new equilibrium value given by, $x_2(T_2) = V_{d2} = 60$ V. The observer design parameter was chosen to be $R_1 = 0.01$. The simulation results shown in Figure 11.6 correspond to the following set of converter parameter values,

$$L = 20 \text{ mH} \;\; ; \;\; C = 20\ \mu\text{F} \;\; ; \;\; R = 30\ \Omega \;\; ; \;\; E = 15 \text{ V}$$

The controller design parameters ξ and ω_n were chosen to be $\xi = 0.9$, $\omega_n = 200$.

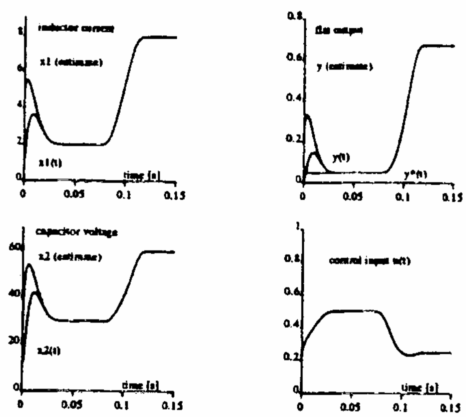

Fig. 11.6. Simulation of output feedback controlled dc-to-dc power converter

5. A PVTOL aircraft

5.1 Description and flatness

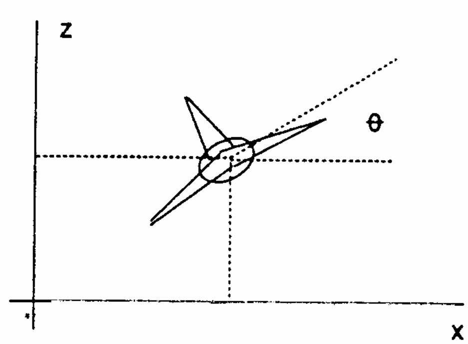

Fig. 11.7. Planar Vertical Take-Off and Landing Aircraft System

Consider, with [24, 32, 33] and [22], the following simplified description of a PVTOL aircraft (see Figure 11.7)

$$\begin{aligned} \ddot{x} &= -u_1 \sin\theta + \epsilon u_2 \cos\theta \\ \ddot{z} &= u_1 \cos\theta + \epsilon u_2 \sin\theta - g \\ \ddot{\theta} &= u_2 \end{aligned} \tag{11.10}$$

where x and z are the horizontal and vertical coordinates of the center of mass of the aircraft, respectively measured along an orthonormal set of fixed horizontal and vertical coordinates. The angle θ is the aircraft's longitudinal axis angular rotation as measured with respect to the fixed horizontal coordinate axis. The controls u_1 and u_2 represent normalized quantities related to the vertical thrust and the angular torque applied around the longitudinal axis of the aircraft respectively. The constant g is the gravity acceleration and ϵ is a fixed constant related to the geometry of the aircraft. The outputs x and z are known to be non-minimum phase [24].

System (11.10) is not static state-feedback linearizable, but differentially flat [22, 32, 33]. A physically motivated flat output is $y = (F, L)$ with $F = x - \epsilon \sin\theta$; $L = z + \epsilon \cos\theta$. Thus, state variables are given by

$$\begin{aligned} x &= F + \epsilon \frac{\ddot{F}}{\sqrt{(\ddot{F})^2 + (\ddot{L}+g)^2}} \\ z &= L - \epsilon \frac{(\ddot{L}+g)}{\sqrt{(\ddot{F})^2 + (\ddot{L}+g)^2}} \\ \theta &= \arctan\left(\frac{\ddot{F}}{\ddot{L}+g}\right) \end{aligned}$$

By setting

$$\varsigma = -\frac{\left(\ddot{F}\right)^2}{\left(\ddot{F}\right)^2 + \left(\ddot{L}+g\right)^2}$$

we obtain the following (linearizing) endogenous feedback

$$\begin{aligned} u_1 &= \varsigma + \epsilon \left(\dot{\theta}\right)^2 \\ u_2 &= \frac{1}{\varsigma}\left(-v_1 \cos\theta - v_2 \sin\theta - 2\dot{\varsigma}\dot{\theta}\right) \\ \ddot{\varsigma} &= -v_1 \sin\theta + v_2 \cos\theta + \varsigma \left(\dot{\theta}\right)^2 \end{aligned}$$

where $v_1 = F^{(4)}$, $v_2 = L^{(4)}$ are new control variables.

5.2 Trajectory planning

It is desired to transfer, in a finite amount of time $\Delta T > 0$, the aircraft position in the x-z plane, from a given fixed initial position, specified by a given set of constant horizontal and constant vertical coordinate values, $\overline{x}_{in}$ and $\overline{z}_{in}$, towards a second constant position represented by the set of coordinates $\overline{x}_f$ and $\overline{z}_f$ with the angular coordinate θ changing from an initial value $\overline{\theta}_{in} = 0$ towards a final value $\overline{\theta}_f = 0$. In [32] the same problem is solved by constructing a bounded trajectory for the internal dynamics, represented by the angular displacement θ, on the basis of the solutions of a sequence of linear ordinary differential equations with suitable initial conditions. This trajectory is in turn translated into a state space trajectory which is then tracked in a conventional manner.

$$
\begin{aligned}
v_1 &= F^{(4)} = F^{*(4)}(t) + a_3\left(F^{(3)}(t) - F^{*(3)}(t)\right) \\
&\quad + a_2\left(\ddot{F}(t) - \ddot{F}^*(t)\right) + a_1\left(\dot{F}(t) - \dot{F}^*(t)\right) + a_0\left(F(t) - F^*(t)\right) \\
&= F^{*(4)}(t) + a_3\left(-\dot{\varsigma}\sin\theta - \varsigma\dot{\theta}\cos\theta - F^{*(3)}(t)\right) + a_2\left(-\varsigma\sin(\theta) - \ddot{F}^*(t)\right) \\
&\quad + a_1\left(\dot{x} - \epsilon\dot{\theta}\cos\theta - \dot{F}^*(t)\right) + a_0\left(x - \epsilon\sin\theta - F^*(t)\right) \\
v_2 &= L^{(4)} = L^{*(4)}(t) + b_3\left(L^{(3)}(t) - L^{*(3)}(t)\right) \\
&\quad + b_2\left(\ddot{L}(t) - \ddot{L}^*(t)\right) + b_1\left(\dot{L}(t) - \dot{L}^*(t)\right) + b_0\left(L(t) - L^*(t)\right) \\
&= L^{*(4)}(t) + b_3\left(\dot{\varsigma}\cos(\theta) - \varsigma\dot{\theta}\sin(\theta) - L^{*(3)}(t)\right) + b_2\left(\varsigma\cos(\theta) - g - \ddot{L}^*(t)\right) \\
&\quad + b_1\left(\dot{z} + \epsilon\dot{\theta}\sin\theta - \dot{L}^*(t)\right) + b_0\left(z + \epsilon\cos\theta - L^*(t)\right)
\end{aligned}
$$

5.4 Simulation results

A maneuver transfering the PVTOL aircraft center of mass outputs (x, z) from a given initial equilibrium position towards a prescribed second equilibrium position was performed. The initial equilibrium point was set at $(\overline{x}_{in}, \overline{z}_{in}) = (0, 0)$ while the second equilibrium position for the center of mass was set to be located at $(\overline{x}_f, \overline{z}_f) = (1, 1)$. The maneuver was set to smoothly begin at $T_1 = 6$ time units, and it was prescribed to be completed at $T_2 = 14$ time units. The simulation results shown in Figure 11.8 correspond to the following set of (normalized) system parameter values $\epsilon = 0.5$, $g = 1$. The controller design parameters were chosen so that the polynomials $p_F(s)$ and $p_L(s)$ each had four roots located at the point -2, i.e, $a_3 = 8$, $a_2 = 24$, $a_1 = 32$, $a_0 = 16$, $b_3 = 8$, $b_2 = 24$, $b_1 = 32$, $b_0 = 16$.

Conclusion

This communication might be considered as a contribution to predictive control, which until now has been mainly developed for linear systems (see, e.g., [2, 41, 49]). It demonstrates, once again, the power of flatness, which is quite easy to teach to engineers (see, e.g., [31, 42, 44]), for dealing with concrete topics, concerning the motion planning and the stabilization of nonholonomic mechanical systems [19, 20], magnetic bearings [31], chemical reactors [42, 43], electric motors [6, 34], windshield wipers [1], tracking observers [23], etc. Other connections may be found in [33].

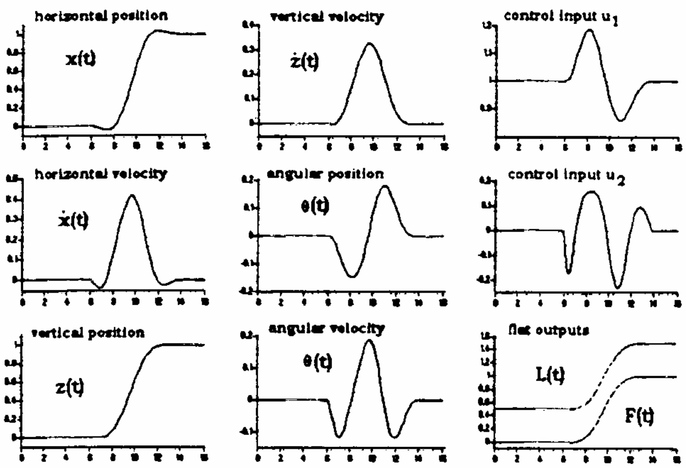

Fig. 11.8. State variables, control inputs and flat outputs in position transfer maneuver for a PVTOL example

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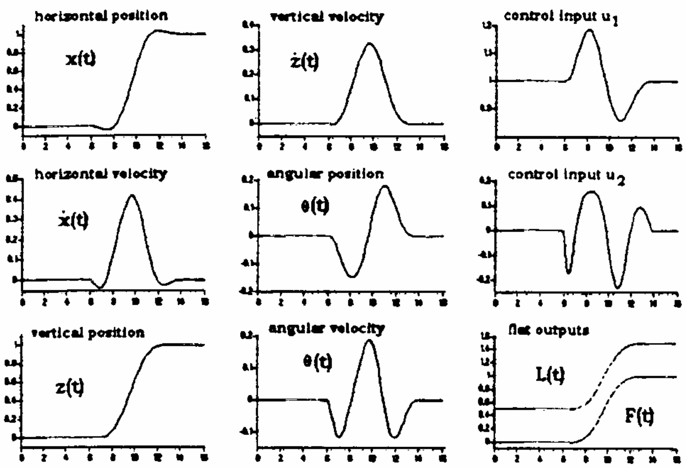

Fig. 11.8. State variables, control inputs and flat outputs in position transfer maneuver for a PVTOL example

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