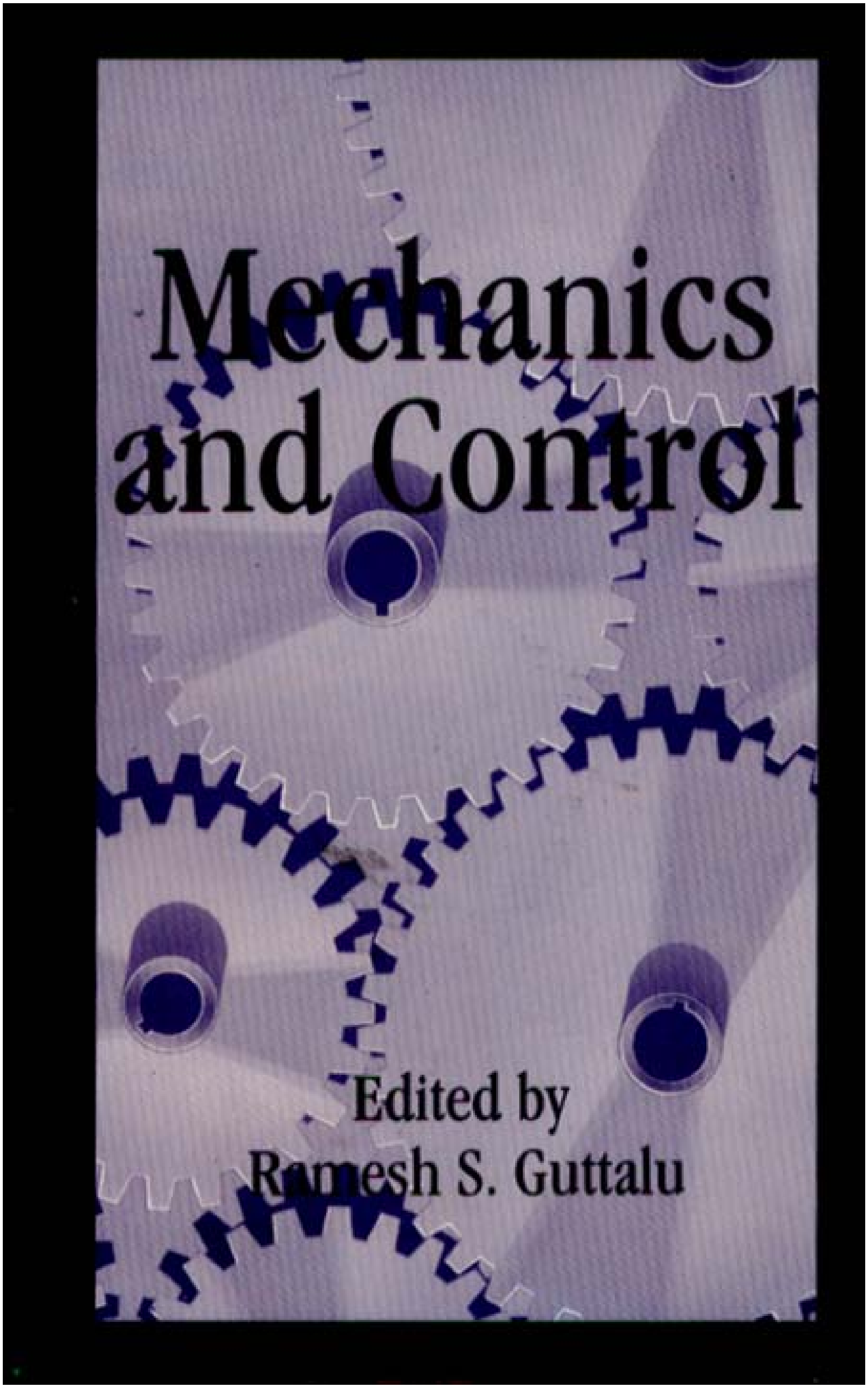
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DYNAMICAL PULSE-WIDTH-MODULATION CONTROL OF RIGID AND FLEXIBLE MANIPULATORS

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ABSTRACT

Due to the associated chattering effects, Pulse-Width-Modulation (PWM) control strategies have been of limited theoretical interest in the investigations about regulation of mechanical systems, such as robotic manipulators. This article proposes dynamical PWM control as a means of circumventing the inconveniences caused by the discontinuities associated to the PWM control strategies and still retain the classical robustness features associated to this practical discontinuous control technique. As examples, the results are applied to, both, rigid and flexible joint single link robotic manipulators.

INTRODUCTION

The design of PWM control policies for the regulation of dynamical systems is generally based on approximating an average (i.e., smooth) desired behavior by means of high-frequency discontinuous control signals which, somehow, emulate the effects of the designed average feedback control inputs. In spite of its well known robustness features, the PWM control approach induces undesirable bang-bang control inputs causing noticeable high frequency discontinuities, or "chattering", in the controlled responses. This kind of discontinuous behavior severely limits, for a large class of mechanical and electro-mechanical systems, the applicability of PWM stabilizing feedback control policies, as well as that of some other closely related discontinuous feedback control techniques (such as Sliding Mode control and Pulse-Frequency-Modulation control). For mechanical systems, in general, and robotic manipulators, in particular, low limits to possible wear and tear effects, and high accuracy of the regulated variables, are important design concerns when choosing the regulating feedback control scheme.

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PWM control constitutes a robust feedback control policy due to its insensitivity to external disturbance inputs, certain certain immunity to model parameter variations, within known bounds, and enhanced performance in spite of the effects of ever present modeling errors. For basic work related to PWM control schemes, in linear and nonlinear dynamical systems, the reader is referred to Polak [1], Skoog [2], Skoog and Blankenship [3], Tsypkin [4], La Cava et al [5], Csaki [6] and also Sira-Ramirez [7]-[12]. A large collection of mechanical nonlinear systems belong to the class of *systems exactly linearizable by state coordinate transformations and static state feedback.* The *generalized state models* (see Fliess [13], [14]) of such systems do not include input signal derivatives. Thus, static, instead of dynamical, controllers naturally arise from the exact linearization procedure introduced in [14]. In this article, wer present a general design method for synthesizing *static*, as well as *dynamical*, PWM feedback control laws stabilizing to a constant equilibrium point the state of any nonlinear dynamical system belonging to the above class. PWM controllers are proposed which effectively stabilize to zero a suitably designed auxiliary scalar output function of the system. The obtained restricted dynamics results, in turn, in a linearization and local asymptotic stabilization of the original system state to its equilibrium value. However, the obtained static PWM controller is shown to include undesirable chattering effects on the generated input signals. A dynamical PWM controller alternative is then proposed, and synthesized, on the basis of *Fliess's generalized controller canonical form* (FGCCF) (sec [14]) of the associated *extended system* (see Nijmeijer and Van der Schaft [15]). Continuous, instead of bang-bang, feedback control input signals are, thus, obtained which robustly stabilize to a constant operating point the closed loop system, while effectively smoothing out the chattering effects on the control signals and regulated state variables.

Next section presents a fundamental result on the PWM control of an elementary scalar dynamical system. It is shown that, based on this scalar result, a general PWM controller design procedure can be proposed for higher order nonlinear controlled plants. This section also reviews FGCCF on which, both, the static, and dynamical, PWM controller design procedures are based. The dynamical PWM control design scheme for linearizable systems, proposed in this section, utilizes the concept of the *extended system.* Two illustrative examples are presented later. The first one is constituted by a purely mechanical system, representing a single link rigid robotic manipulator, on which the performances of a static and a dynamical PWM feedback controller are tested by means of computer simulations. The second example deals with the more delicate issue of using static and dynamical PWM feedback control policies for a single link flexible robotic manipulator.

PWM CONTROL OF NONLINEAR SYSTEMS

PWM Control of a Scalar System

Consider the scalar PWM controlled dynamical system with state variable s, in which the constants a, W and r are all strictly positive quantities.

$$\dot{s} = -as - Wv$$
$$v = PWM_t(s) = \begin{cases} sgn[s(t_k)] & \text{for } t_k \leq t < t_k + \tau[s(t_k)]T \\ 0 & \text{for } t_k + \tau[s(t_k)]T \leq t < t_k + T \end{cases} \tag{1}$$

$$\tau[s(t)] = \begin{cases} 1 & \text{for } |s(t)| > \frac{1}{r} \\ r|s(t)| & \text{for } |s(t)| \leq \frac{1}{r} \end{cases}$$

$$k = 0, 1, 2, \cdots; \quad t_{k+1} = t_k + T.$$

The t_k's represent *regularly* spaced sampling instants. The sampling interval $[t_k, t_{k+1}]$ is known as the *duty cycle* and its width is here represented by the constant T, i.e., $T = t_{k+1} - t_k$. At each sampling instant, t_k, the value of the width of the next sign-modulated, fixed amplitude, control pulse v is determined by the sampled value of the *duty ratio function*, represented by $\tau[s(t_k)]$. The function "sgn" stands for the *signum* function:

$$sgn(s) = \begin{cases} +1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0 \end{cases}$$

The pulse width, $\tau[s(t_k)]T$, saturates to the value of the duty cycle, T, i.e., $\tau[s(t_k)]$ saturates to 1, as long as the value of the controlled state s is larger, in absolute value, than a given prespecified threshold $1/r$ (see Figure 1). When the absolute value of the state, s, of the scalar system (1) is driven below the value $1/r$, the duty ratio, $\tau[s(t_k)]$, also starts decreasing, in a linear fashion, with respect to $|s|$. When the scalar system

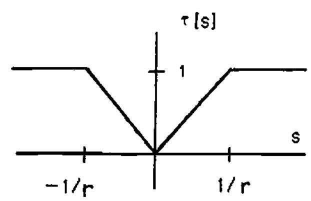

Figure 1. Duty Ratio function for PWM controlled Scalar System.

is, eventually, at rest in $s = 0$, no control pulses are then applied to the system. The basic idea behind the above discontinuous PWM control law (1) is that large errors in the scalar state s should require larger corrective pulse widths during the fixed inter-sampling periods. Small errors, on the other hand, should be driven to zero, on the basis of the adopted sampling frequency, with corrective pulse widths decreasing to zero until stabilization. In a sense, PWM policies are "proportional" feedback policies subject to saturation. The following theorem establishes a sufficient condition for the asymptotic stability to zero of the PWM controlled system (1). The same theorem is also found in [1] and also in [2]. In reference [2] it is further established that the proposed convergence condition is also necessary. We furnish, however, a different proof of the sufficiency result.

Theorem 1: The PWM controlled system (1) is asymptotically stable to $s = 0$, if

$$rW < a \tanh\left(\frac{aT}{2}\right) \tag{2}$$

Proof: Due to the piecewise constant nature of the control inputs and the linearity of the underlying continuous system (1), it suffices to study the stability of the discretized version of (1) at the sampling instants. An exact discretization of the PWM controlled system (1) thus yields

$$s(t_k+T) = e^{-aT}s(t_k) - \frac{We^{-aT}}{a}\left(e^{a\tau[s(t_k)]T}-1\right)\ sgn[s(t_k)] \tag{3}$$

Suppose the initial condition $s(t_0)$ is chosen deep into the region $|s| > 1/r$. The evolution of the sampled values of $s(t)$ obey, according to (1):

$$s(t_{k+1}) = e^{-aT}\left\{s(t_k) - \frac{W}{a}(e^{aT}-1)\ sgn[s(t_k)]\right\} \tag{4}$$

The absolute value of the incremental step, $\Delta s(t_k) := s(t_{k+1}) - s(t_k)$, is readily obtained from (4), from where it is easily found that

$$\begin{aligned} |\Delta s(t_k)| &= \left(1-e^{-aT}\right)\left|\left(s(t_k)+\frac{W}{a}\right)\right| \\ &< \left(1-e^{-aT}\right)\left[|s(t_k)|+\frac{W}{a}\right] \end{aligned} \tag{5}$$

The condition $|\Delta s(t_k)| < 2/r$ is sufficient to ensure that the value of $s(t_k)$ will be eventually found within the bounded region $|s| < 1/r$, irrespectively of the initial condition value $s(t_0)$ given in $|s| > 1/r$. Sufficiency is clear from the facts that, in the region $|s| > 1/r$, the $|\Delta s(t_k)|$ decrease at each step and that one must guarantee that $s(t_k)$ does not persistently "jump" over the band $|s| < 1/r$, thus falling into a possible limit cycle behavior. ¿From (5) and the fact that $|(t_k)| > 1/r$, the condition $|\Delta s(t_k)| > 1/r$, the condition $|\Delta s(t_k)| < 2/r$ is guaranteed if we let

$$\left(1-e^{-aT}\right)\left[1+\frac{rW}{a}\cdot\right] < 2 \tag{6}$$

which is just the expression (2), after some straightforward manipulations.

Suppose now that the initial state, $s(t_0)$, of the scalar system (1), is found in the region $|s| < 1/r$. The exact discretization of the PWM controlled system is now given by

$$s(t_{k+1}) = e^{-aT}s(t_k) - \frac{We^{-aT}}{a}\left(e^{ar|s(t_k)|T}-1\right)\ sgn[s(t_k)] \tag{7}$$

The absolute values of the incremental steps $\Delta s(t_k) = s(t_k+T) - s(t_k)$ are thus given by

$$\begin{aligned} |\Delta s(t_k)| &= \left(1-e^{-aT}\right)\left|s(t_k)+\frac{W}{a}\frac{\left(e^{a(r|s(t_k)|-1)T}-e^{-aT}\right)}{(1-e^{-aT})}\right| \\ &< \left(1-e^{-aT}\right)\left[|s(t_k)|+\frac{W}{a}\frac{\left(e^{a(r|s(t_k)|-1)T}-e^{-aT}\right)}{(1-e^{-aT})}\right] \end{aligned} \tag{8}$$

A sufficient condition for asymptotic stability of (7) to zero is given by $|\Delta s(t_k)| < 2|s(t_k)|$. Notice however that from the fact that $r|s(t_k)| < 1$, (8) implies that

$$|\Delta s(t_k)| < \left(1-e^{-aT}\right)\left[|s(t_k)|+\frac{W}{a}\right]$$

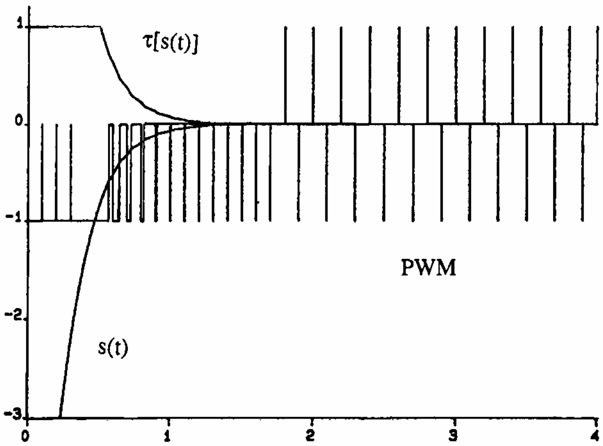

Figure 2. State Response, duty ratio function and PWM control signal for first order PWM controlled scalar system.

Hence, the above convergence condition is fully guaranteed if we let

$$|\Delta s(t_k)| \ < \left(1-e^{-aT}\right)\left[|s(t_k)|+\frac{W}{a}\right] \ < 2|s(t_k)| \tag{9}$$

i.e., from the second inequality one has

$$\frac{W}{a} \ < \ \frac{2}{(1-e^{-aT})}|s(t_k)| - |s(t_k)| = |s(t_k)| \tanh\left(\frac{aT}{2}\right)$$

which after multiplication of by r, and the fact that $r|s(t_k)| < 1$, results in:

$$\frac{rW}{a} < r|s(t_k)| \tanh\left(\frac{aT}{2}\right) < \tanh\left(\frac{aT}{2}\right)$$

The result follows.

A computer simulation of system (1) is shown below in Figure 2. This figure depicts the state s(t), the PWM control input signal v(t) and the duty ratio function $\tau(s(t))$. The values of the involved constants were chosen as: $a = 5.4, r = 0.01, T = 0.1, W = 10$. Since $rW/a = 0.0185 < \tanh(0.27) = 0.2636$, asymptotic stability of s(t) to zero is guaranteed by Theorem 1.

Fliess' Generalized Controller Canonical Form

Consider the analytical n-dimensional state variable representation of a nonlinear system:

$$\dot{x} = F(x,u) \tag{10}$$

It is assumed that the nonlinear system (8) exhibits a constant equilibrium point of interest characterized by $F(X(U),U) = 0$. We refer to this equilibrium point as

$(X(U), U)$. Associated to the system (10) is FGCCF given by (see [14] for further details)

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= c(z, u, \dot{u}, \cdots, u^{(\alpha)}) \end{aligned} \tag{11}$$

Implicit in this representation is the assumption of the existence of an element, $z_1 = h(x)$, called the *differential primitive element,* which generates the *generalized state representation* (11) of system (10). Under these circumstances, system (10) is transformed into system (11) by means of an invertible input-dependent state coordinate transformation of the form

$$z = \Phi(x, u, \dot{u}, \cdots, u^{(\alpha-1)}) \tag{12}$$

given by

$$\Phi(x, u, \dot{u}, \cdots, u^{(\alpha-1)}) = [h(x), \dot{h}(x), \cdots, h^{(n-1)}(x, u, \dot{u}, \cdots, u^{(\alpha-1)})]' \tag{13}$$

We assume that the equilibrium point $(X(U), U)$ of (8) is transformed, by (10), into the vector $z = 0$, i.e., $\Phi(X(U), U, 0, \cdots, 0) = 0$. ¿From this assumption, it readily follows that $c(0, U, \cdots, 0) = 0$. Suppose that for a suitably designed feedback control policy the state of (9) is asymptotically driven to zero. The autonomous dynamics described by

$$c(0, u, \dot{u}, \cdots, u^{(\alpha)}) = 0 \tag{14}$$

constitutes the *zero dynamics* (see Fliess [16]). In order to guarantee an overall stable performance of the controlled system, it is of crucial importance to assess the stability of such a dynamics around its possible equilibrium points. Around the equilibrium point of interest $u = U$ we assume that the dynamical system (14) is asymptotically stable, i.e., the system $dx/dt = F(x, u)$ with output $y = h(x)$ is *minimum phase. Non-minimum phase* systems can also be controlled by PWM control policies by means of suitable change of the differential primitive element. This topic, however, will be explored elsewhere (see also Fliess and Messager [17] for related aspects in connection with sliding mode control of linear time invariant systems). Motivated by the fact that for a large class of mechanical and electromechanical systems the integer α, in the corresponding GCCF (11), is equal to zero, we shall concentrate our developments, from now on, on such a particular class of systems. This class corresponds to *systems exactly linearizable by means of state coordinate transformations and static state feedback* i.e., those systems in which the function c, in (11), is of the form $c(z, u) = c(\Phi(x), u)$, and for which the transformation (12) only involves state variables. We further assume that $\partial c/\partial u$ is not identically zero, at least, locally around the equilibrium point.

Static PWM Control for Exactly Linearizable Systems

Let $P_{n-1}(\lambda)$ be an $(n-1)$th order Hurwitz polynomial with constant coefficients:

$$P_{n-1}(\lambda) = \lambda^{n-1} + a_{n-1}\lambda^{n-2} + \cdots + a_2\lambda + a_1 \tag{15}$$

Consider now the following auxiliary output function of the system (11):

$$s(z) = z_n + a_{n-1}z_{n-1} + \cdots + a_2 z_2 + a_1 z_1 \tag{16}$$

If the condition $s = 0$ is achieved by means of suitable controls, the restricted motions of the system (11) satisfy the following asymptotically stable linear time-invariant dynamics:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_{n-1} &= -a_{n-1}z_{n-1} - \cdots - a_2 z_2 - a_1 z_1 \end{aligned} \tag{17}$$

The following proposition is a direct consequence of the preceding considerations and of Theorem 1.

Proposition 1: A system of the form (10) is locally asymptotically stabilizable to the equilibrium point $(U, X(U))$ if the control action u is specified as a PWM control policy given by the solution of the following implicit (algebraic) equation

$$c(\Phi(x), u) = -\sum_{i=1}^{n}(a_{i-1} + aa_i)h^{(i-1)}(x) - W\ PWM_\tau \left[\sum_{i=1}^{n} a_i h^{i-1}(x)\right] \tag{18}$$

where $a_0 = 0$, and $a_n = 1$.

Proof: Imposing on the auxiliary output function $s(z)$, given in (16), the asymptotically stable discontinuous PWM controlled dynamics defined by (1), one immediately obtains, in terms of the transformed coordinates z, an *implicit* nonlinear equation for the required control input u. Rewriting the obtained expression in original state and input coordinates the static controller adopts the form (18). ¿From the assumption that $\partial c/\partial u$ is locally non zero, it follows that (18) can be explicitly solved for u. We denote, in general, the solution for u in an equation of the form: $c(\Phi(x), u) = z_{n+1}$, as $u = g(x, z_{n+1})$, i.e., $c(\Phi(x), g(x, z_{n+1})) \equiv z_{n+1}$ for some given indeterminate z_{n+1}.

As it can be easily seen from (18), the case of exactly linearizable systems results in a *static* PWM controller and, hence, the proposed scheme yields discontinuous control actions. Hence, "bang-bang" feedback control signals are generated in the closed loop system.

Dynamical PWM Control of Nonlinear Systems

Consider now the *extended system*, associated to system (8) (see [15]):

$$\begin{aligned} \dot{x} &= F(x, u) \\ \dot{u} &= v \end{aligned} \tag{19}$$

It is easy to see that if $z_1 = h(x)$ is a differential primitive element for (10), z_1 also qualifies as a differential primitive element for (19). Letting $c(z, u)$ become a new state variable z_{n+1}, it is follows readily that the GCCF of (19) is written as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \end{aligned} \tag{20}$$

$$\dot{z}_n = z_{n+1}$$

$$\dot{z}_{n+1} = \frac{\partial c(\Phi(x),u)}{\partial x} F(x,u) \bigg|_{\substack{x=\Phi^{-1}(z) \\ u=g(z,z_{n+1})}} + \frac{\partial c(\Phi(x),u)}{\partial u} \bigg|_{\substack{x=\Phi^{-1}(z) \\ u=g(z,z_{n+1})}} v$$

$$:= f(z,z_{n+1}) + g(z,z_{n+1})v$$

Hence, if the original system is exactly linearizable by means of state coordinates transformations, and static state feedback, so is the extended system with respect to the new auxiliary input v. An equilibrium point of (20) is evidently given by $v = 0, u = U, x = X(U)$. We denote this equilibrium point by $((X(U),U),0)$.

Notice that the state coordinate transformation taking (19) into the linearizable form (20) is given by

$$\hat{z} := \begin{bmatrix} z \\ z_{n+1} \end{bmatrix} = \begin{bmatrix} \Phi(x) \\ c(\Phi(x),u) \end{bmatrix} := \hat{\Phi}(x,u) \tag{21}$$

which is evidently invertible whenever $\Phi(x)$ is invertible and $\partial c/\partial u$ is non zero. It is easy to see that $z = 0$ is an equilibrium point of (20). Let $p_n(\lambda)$ be an $n-th$ order Hurwitz polynomial with constant coefficients:

$$p_n(\lambda) = \lambda^n + a_n\lambda^{n-1} + \cdots + a_2\lambda + a_1 \tag{22}$$

If one considers now the following auxiliary output function of the system (20)

$$\sigma(z,z_{n+1}) = z_{n+1} + a_n z_n + \cdots + a_2 z_2 + a_1 z_1 \tag{23}$$

then the condition $\sigma = 0$ implies that the restricted motions of the system (20) satisfy the following asymptotically stable linear time-invariant dynamics

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_n &= -a_n z_n - \cdots - a_2 z_2 - a_1 z_1 \end{aligned} \tag{24}$$

The following proposition is a direct consequence of the preceding considerations and of Theorem 1.

Proposition 2: A system of the form (10) is locally asymptotically stabilizable to the equilibrium point $((X(U),U),0)$ if the control action u is specified as a dynamical PWM control policy given by the solution of the following explicit first order nonlinear differential equation with discontinuous right hand side:

$$\begin{aligned} f(\hat{\Phi}(x,u)) + g(\hat{\Phi}(x,u))\dot{u} &= -\sum_{i=1}^{n}(a_{i-1} + a a_i)h^{(i-1)}(x) - (a_n + a)h^{(n)}(x,u) \\ &\quad -W\, PWM_\tau \left[\sum_{i=1}^{n} a_i h^{(i-1)}(x) + h^{(n)}(x,u)\right] \end{aligned} \tag{25}$$

where $a_0 = 0$.

Proof: As in Proposition 1, above, imposing on the auxiliary output function $\sigma(z, z_{n+1})$, given in (23), the asymptotically stable discontinuous PWM controlled PWM controlled dynamics, defined by (1), one immediately obtains, in terms of the transformed coordinates (z, z_{n+1}), a nonlinear algebraic equation for the required control input v. Rewriting the obtained static controller expression in terms of the original state and input coordinates (x, u), the controller adopts the dynamical form of equation (25).

Notice that since $g = \partial c/\partial u$ is assumed to be nonzero, the controller (2.25) is locally well defined and no *impasse points* needs to be considered (see Fliess and Hassler [18]).

APPLICATIONS TO SINGLE-LINK RIGID AND FLEXIBLE JOINT MANIPULATORS

PWM Control of a Single Link Rigid Robotic Manipulator

Consider the following nonlinear dynamical model of a single link robotic manipulator (Khalil [19]):

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g}{L}\sin x_1 - \frac{k}{M}x_2 + \frac{1}{ML^2}u \end{aligned} \qquad (26)$$

where x_1 is the link angular position, x_2 is the angular velocity and u represents the applied torque. The mass M is assumed to be concentrated at the tip of the manipulator. The constant k is the viscous damping coefficient.

It is desired to synthesize, both, a static and a dynamical PWM feedback control policy which drives the angular position of the system to a constant desired angular position x_{1d}.

Static PWM Controller Design. Let $z_1 = x_1 - x_{1d}$. It is easy to see that z_1 qualifies as a differential primitive element for (26) and that to obtain the FGCCF of (26) it simply requires the use of the following (trivial) state coordinate transformation $z_1 = x_1 - x_{1d}, z_2 = x_2$:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\frac{g}{L}\sin(z_1 + x_{1d}) - \frac{k}{M}z_2 + \frac{1}{ML^2}u \end{aligned} \qquad (27)$$

Let $s(z) = z_2 + a_1 z_1$ with $a_1 > 0$, be an auxiliary output function for system (27). Notice that if $s(x)$ is stabilized to zero, the constrained dynamics evolves according to $dz_1/dt = -a_1 z_1$ (or $dx_1/dt = -a_1(x_1 - x_{1d})$) thus achieving the desired regulation task. Imposing on the auxiliary output function $s(z)$ the asymptotically stable dynamics of the PWM controlled system (1) one obtained the following static PWM controller

$$\begin{aligned} u &= ML^2[-aa_1 z_1 + \left(\frac{k}{M} - a - a_1\right) z_2 \\ &\quad + \frac{g}{L}\sin(z_1 + x_{1d}) - W\ PWM_\tau(z_2 + a_1 z_1)] \end{aligned} \qquad (28)$$

which, in original coordinates is rewritten as

$$u = ML^2 \left\{ -aa_1(x_1 - x_{1d}) + \left(\frac{k}{M} - a - a_1\right) x_2 \right. \qquad (29)$$

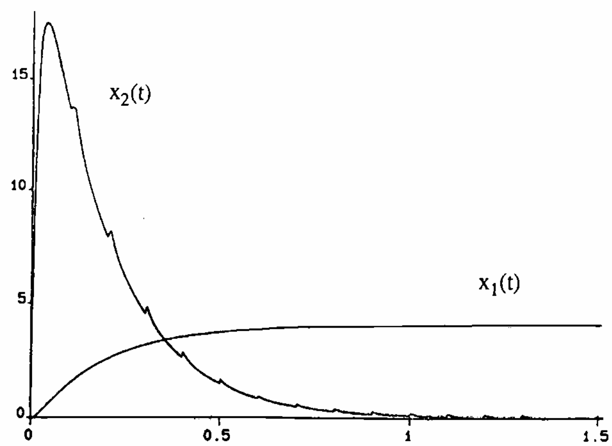

Figure 3. Angular position and angular velocity of static PWM controlled rigid robotic manipulator.

$$+\frac{g}{L}\sin(x_1) - WPWM_\tau[x_2 + a_1(x_1 - x_{1d})]\Big\}$$

Simulations were run for the above PWM controlled manipulator (26), (29), with the following parameters: $M = 0.01068$ $[Kg], L = 0.5$ $[m], k = 0, x_{1d} = 4$ $[rad], g = 9.8$ $[m/s^2]$. The auxiliary output function was synthesized with $a_1 = 60$ $[s^{-1}]$. The static PWM controller parameters were chosen as: $a = 5.4$ $[s^{-1}], W = 100$ $[rad/s], T = 0.1$ $[s], r = 0.01$ $[s/rad]$. In this case the sufficient condition of Theorem 1 is verified as $rW/a = 0.185 < \tanh(0.27) = 0.2636$. Figure 3 depicts the state trajectories of the controlled system clearly showing convergence to the desired angular position.

The angular velocity is also shown to converge to zero with significant chattering. Figure 4 shows a magnified view of a portion of the *discontinuous* applied torque input signal u, as generated by (29), and of the PWM signal.

In spite of the fact that the discontinuities associated to the input torque variable u do not noticeably propagate towards the angular position variable (due to the two natural integration steps existing between the angular acceleration and the angular position) the bang-bang input behavior is deemed as highly undesirable. These discontinuities not only cause wear and tear, but, also, they represent an opportunity for unnecessary excitation of high-frequency unmodelled dynamics of the mechanical system.

Dynamical PWM Controller Design. Consider now the extended system of (26):

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g}{L}\sin x_1 - \frac{k}{M}x_2 + \frac{1}{ML^2}u \\ \dot{u} &= v \end{aligned} \quad (30)$$

Taking again the angular position error $x_1 = x_{1d}$ as a differential primitive element

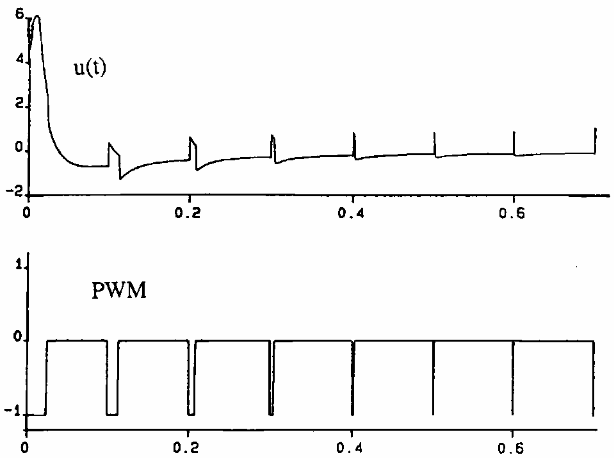

Figure 4. Discontinuous control input torque signal (magnified view of only a portion) and PWM control signal for static PWM controlled robotic manipulator.

x_1. The resulting FGCCF of the extended system is now obtained as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= -\frac{g}{L} z_2 \cos(z_1 + x_{1d}) - \frac{k}{M} z_3 + \frac{1}{ML^2} v \end{aligned} \tag{31}$$

with $z_1 = x_1 - x_{1d}, z_2 = x_2, z_3 = -(g/L)\sin x_1 - (k/M)x_2 + [(1/ML^2)]u$. Let the auxiliary output function $\sigma(z)$ be defined as: $z_3 + a_2 z_2 + a_1 z_1$, with a_2 and a_1 positive constants, chosen in the standard second order system form, with damping factor ζ and natural frequency ω_n : $a_2 = 2\zeta\omega_n$ and $a_1 = \omega_n^2$. Notice that if $\sigma(z)$ is stabilized to zero, the constrained dynamics evolves in accordance to the asymptotically stable second order dynamics: $dz_1/dt = z_2$; $dz_2/dt = -2\zeta\omega_n z_2 - \omega_n^2 z_1$ thus achieving the desired control task: $z_2 = x_2 \to 0$ and $z_1 = x_1 - x_{1d} \to 0$.

Imposing on $\sigma(z)$ the same asymptotically stable dynamics of the PWM controlled system (1) one obtains the following static PWM controller for the extended system

$$\begin{aligned} v &= ML^2 \Big[-\left(a + \frac{k}{M} + 2\zeta\omega_n\right) z_3 - (2\zeta\omega_n a + \omega_n^2) z_2 - a\omega_n^2 z_1 \\ &\quad + \frac{g}{L} z_2 \cos(z_1 + x_{1d}) - W\ PWM_\tau(z_3 + 2\zeta\omega_n z_2 + \omega_n^2 z_1) \Big] \end{aligned} \tag{32}$$

which, in original coordinates, is rewritten as a dynamical PWM controller given by the solution of the following time-varying ordinary differential equation for the control input u with discontinuous (PWM) right hand side

$$\begin{aligned} \dot{u} &= ML^2 \Big\{ -\left(a + \frac{k}{M} + 2\zeta\omega_n\right) \left(-\frac{g}{L}\sin x_1 - \frac{k}{M} x_2 + \frac{1}{ML^2} u\right) \\ &\quad - \left(2\zeta\omega_n a + \omega_n^2\right) x_2 - a\omega_n^2 (x_1 - x_{1d}) + \frac{g}{L} x_2 \cos(x_1) \\ &\quad - W\ PWM_\tau \left[-\frac{g}{L}\sin x_1 - \frac{k}{M} x_2 + \frac{1}{ML^2} u + 2\zeta\omega_n x_2 + \omega_n^2 (x_1 - x_{1d})\right] \Big\} \end{aligned} \tag{33}$$

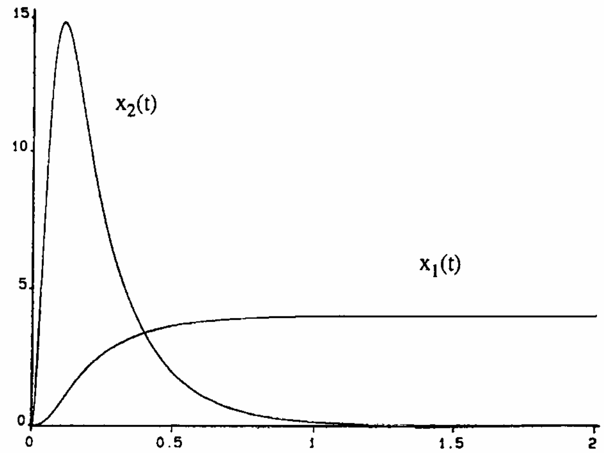

Figure 5. Angular Position and angular velocity of dynamical PWM controlled rigid robotic manipulator.

Simulations were run for the dynamically PWM controlled manipulator (26), (33) with the same physical parameter values as before. The auxiliary output function σ was synthesized such that the corresponding characteristic polynomial of the linearized system is $p_2(\lambda) = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2$, with $\zeta = 0.8$ and $\omega_n = 28.0$ $[rad/s]$. The static PWM controller parameters were chosen in the same manner as in the previous example. Figure 5 depicts the state trajectories of the controlled system clearly showing convergence to the desired angular position and to zero angular velocity with no chattering being exhibited now.

Figure 6 shows the PWM signal and the substantially smoothed out (chattering-free) applied torque input signal u, as generated by the dynamical PWM controller (33). The effect of adding an integrator to the original input u of the system results in a low pass filtering effect on the generated auxiliary input v of the extended system.

PWM Control of a Single Link Flexible Joint Robotic Manipulator

Consider the following damping-free nonlinear dynamical model of a single link robotic manipulator with a flexible joint (Spong and Vidyasagar [20], Khalil [19]):

$$\begin{aligned} I\ddot{q}_1 + MgL\sin q_1 + \kappa(q_1 - q_2) &= 0 \\ J\ddot{q}_2 - \kappa(q_1 - q_2) &= u \end{aligned} \tag{34}$$

where q_1 is the angular position of the link, q_2 is the motor shaft's angular position and u represents the generated torque applied to the shaft (sec [20] for details). The constants I and M represent, respectively, the moment of inertia and the mass of the link. J is the moment of inertia of the link and the motor shaft. The constant κ is the elasticity coefficient associated to the flexible joint.

A state space model of the above system is readily obtained by defining $x_1 = q_1, x_2 = dx_1/dt, x_3 = q_2, x_4 = dx_3/dt$ as shown below. (See also Korasani [21], for a different state space model).

$$\dot{x}_1 = x_2 \tag{35}$$

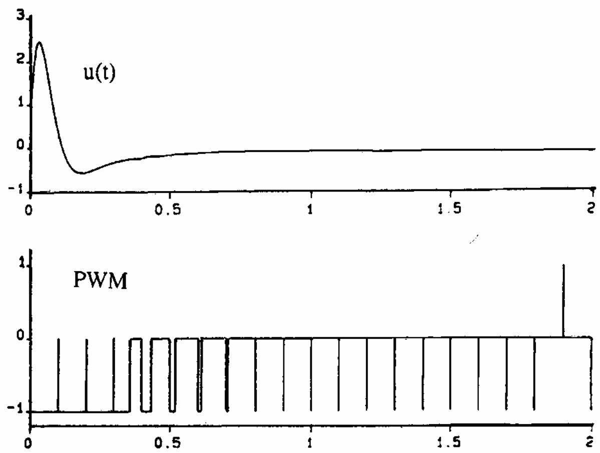

Figure 6. Chattering free control input torque signal and PWM signal for dynamical PWM controlled rigid robotic manipulator.

$$\begin{aligned}\dot{x}_2 &= -\frac{MgL}{I}\sin(x_1) - \frac{\kappa}{L}(x_1 - x_3)\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= \frac{\kappa}{J}(x_1 - x_3) + \frac{1}{J}u\end{aligned}$$

It is desired to synthesize, both, a static and a dynamical PWM feedback control policy which drives the angular position of the system to a desired constant angular position x_{1d}. We summarize below the steps leading to both controllers designs.

Static PWM Controller Design

State coordinate transformation and its inverse to obtain FGCCF of the flexible link manipulator model:

$$\begin{aligned}z_1 &= x_1 - x_{1d}\\ z_2 &= x_2\\ z_3 &= -\frac{MgL}{I}\sin(x_1) - \frac{\kappa}{I}(x_1 - x_3)\\ z_4 &= -\frac{MgL}{I}x_2\cos(x_1) - \frac{\kappa}{I}(x_2 - x_4)\end{aligned} \tag{36}$$

$$\begin{aligned}x_1 &= z_1 + x_{1d}\\ x_2 &= z_2\\ x_3 &= \frac{1}{\kappa}[Iz_3 + MgL\sin(z_1 + x_{1d})] + z_1 + z_{1d}\\ x_4 &= \frac{1}{\kappa}[Iz_4 + MgLz_2\cos(z_1 + x_{1d})] + z_2\end{aligned} \tag{37}$$

Fliess generalized controller canonical form:

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= \frac{MgL}{I}\left[z_2^2 + \frac{MgL}{I}\cos(z_1 + z_{1d}) + \frac{\kappa}{I}\right]\sin(z_1 + x_{1d}) \\
&\quad - \left[\frac{\kappa}{I} + \frac{\kappa}{J} + \frac{MgL}{I}\cos(z_1 + x_{1d})\right]\left[z_3 + \frac{MgL}{I}\sin(z_1 + x_{1d})\right] + \frac{\kappa}{IJ}u
\end{aligned} \tag{38}
$$

Auxiliary output function:

$$
s(z) = z_4 + a_3 z_3 + a_2 z_2 + a_1 z_1 \tag{39}
$$

with the a's chosen so that $p_3(\lambda) = \lambda^3 + a_3\lambda^2 + a_2\lambda + a_1$ is a Hurwitz polynomial (for instance $p(\lambda) = (\lambda + a)(s^2 + 2\zeta\omega_n\lambda + \omega_n^2)$, with $a > 0$ and $1 > \zeta > 0, \omega_n > 0$). In original coordinates:

$$
\begin{aligned}
s(x) &= -\frac{MgL}{I}x_2\cos(x_1) - \frac{\kappa}{I}(x_2 - x_4) - \frac{MgLa_3}{I}\sin(x_1) \\
&\quad -\frac{\kappa a_3}{I}(x_1 - x_3) + a_2 x_2 + a_1(x_1 - x_{1d})
\end{aligned}
$$

Static PWM controller in transformed coordinates:

$$
\begin{aligned}
u &= -\frac{JMgL}{\kappa}\left[z_2^2 + \frac{MgL}{I}\cos(z_1 + x_{1d}) + \frac{\kappa}{I}\right]\sin(z_1 + x_{1d}) \\
&\quad + \left[1 + \frac{I}{J} + \frac{MgL}{\kappa}\cos(z_1 + x_{1d})\right]\left[z_3 + \frac{MgL}{I}\sin(z_1 + x_{1d})\right] \\
&\quad - \frac{IJ}{\kappa}\left[(a + a_3)z_4 + (a_2 + aa_3)z_3 + (a_1 + aa_2)z_2 + aa_1 z_1\right] \\
&\quad - W\ PWM_\tau[z_4 + a_3 z_3 + a_2 z_2 + a_1 z_1]
\end{aligned} \tag{40}
$$

Static PWM controller in original coordinates:

$$
\begin{aligned}
u &= -\frac{JMgL}{\kappa}\left[x_2^2 + \frac{MgL}{I}\cos(x_1) + \frac{\kappa}{I}\right]\sin(x_1) + \left[1 + \frac{I}{J} + \frac{MgL}{\kappa}\cos(x_1)\right] \times \\
&\quad \left[-\frac{MgL}{I}\sin(x_1) - \frac{\kappa}{I}(x_1 - x_3) + \frac{MgL}{I}\sin(x_1)\right] \\
&\quad - \frac{IJ}{\kappa}\left\{(a + a_3)\left[\frac{MgL}{I}x_2\cos(x_1) + \frac{\kappa}{I}(x_2 - x_4)\right]\right. \\
&\quad \left. + (a_2 + aa_3)\left[\frac{MgL}{I}\sin(x_1) + \frac{\kappa}{I}(x_1 - x_3)\right] + (a_1 + aa_2)x_2 + aa_1[x_1 - x_{1d}]\right\} \\
&\quad - W\ PWM_\tau\left[-\frac{MgL}{I}x_2\cos(x_1) - \frac{\kappa}{I}(x_2 - x_4)\right.
\end{aligned} \tag{41}
$$

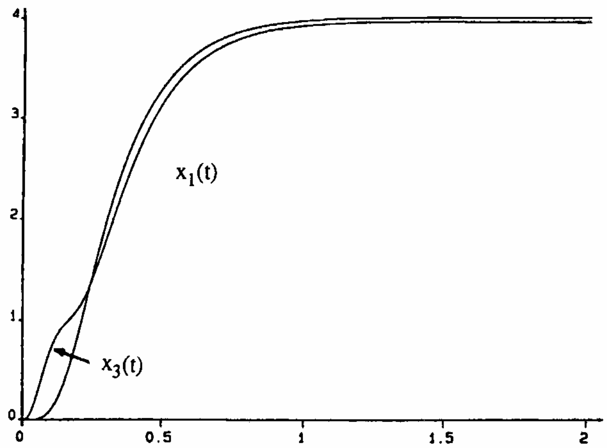

Figure 7. Angular position of link and motor shaft for static PWM controlled flexible robotic manipulator.

$$-a_3\frac{MgL}{I}\sin(x_1) - a_3\frac{\kappa}{I}(x_1 - x_3) + a_2 x_2 + a_1(x_1 - x_{1d})\Big]$$

Simulations were run for the static PWM controlled manipulator (35)-(40) with the following parameters: $M = 0.01068$ $[Kg], L = 0.5$ $[m], \kappa = 1.0$ $Nm/rad, x_{1d} = 4$ $[rad], g = 9.8$ $[m/s^2], J = 0.07$ $[Nms^2/rad]$. The auxiliary output function was synthesized such that the corresponding characteristic polynomial is given by $p_2(\lambda) = (\lambda + b)(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2)$ with $b = 30, \zeta = 0.8, \omega_n = 28.0$. The static PWM controller parameters chosen again as: $a = 5.4$ $[s^{-1}], W = 10^4$ $[rad/s], T = 0.1$ $[s], r = 10^{-4}$ $[s/rad]$.

Figure 7 depicts the controlled trajectories of the angular positions x_1 and x_3 clearly showing convergence of x_1 to the desired angular position. Due to the assumed flexibility, the shaft's angular position x_3 exhibits a small steady state error of 0.052 $[rad]$ with respect to x_1. Figure 8 shows the link and shaft's angular velocities x_2 and x_4. The effects of the PWM discontinuities are clearly portrayed in the shaft's angular velocity response.

Dynamical PWM Controller Design

The extended system:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{MgL}{I}\sin(x_1) - \frac{\kappa}{L}(x_1 - x_3) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{\kappa}{J}(x_1 - x_3) + \frac{1}{J}u \\
\dot{u} &= v
\end{aligned} \tag{42}$$

The (PWM) signal and the discontinuous applied torque input u, as generated by (41), are shown in Figure 9.

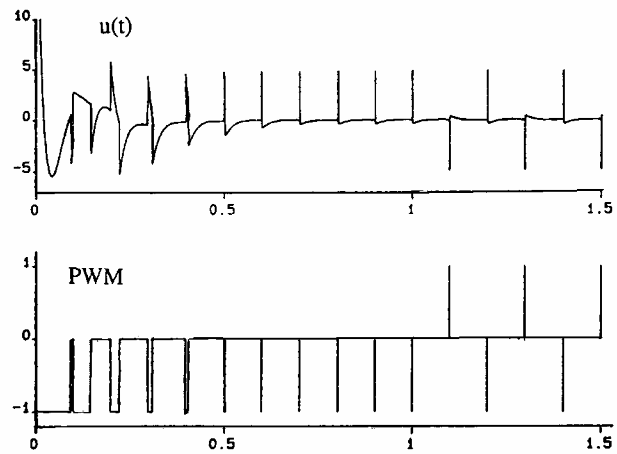

Figure 8. Angular velocities of link and motor shaft for static PWM controlled flexible robotic manipulator.

State coordinate transformation and its inverse to obtain FGCCF of the extended manipulator system:

$$\begin{aligned}
z_1 &= x_1 - x_{1d} \\
z_2 &= x_2 \\
z_3 &= -\frac{MgL}{I}\sin(x_1) - \frac{\kappa}{I}(x_1 - x_3) \\
z_4 &= -\frac{MgL}{I}x_2\cos(x_1) - \frac{\kappa}{I}(x_2 - x_4) \\
z_5 &= \left(\frac{MgL}{I}\right)^2 \sin(x_1)\cos(x_1) + \frac{MgL\kappa}{I^2}(x_1 - x_3)\cos(x_1) \\
&\quad + \frac{MgL}{I}x_2^2\sin(x_1) + \frac{MgL\kappa}{I^2}\sin(x_1) + \left(\frac{\kappa}{I}\right)^2(x_1 - x_3) \\
&\quad + \frac{\kappa^2}{IJ}(x_1 - x_3) + \frac{\kappa}{IJ}u
\end{aligned} \tag{43}$$

$$\begin{aligned}
x_1 &= z_1 + x_{1d} \\
x_2 &= z_2 \\
x_3 &= \frac{1}{\kappa}[Iz_3 + MgL\sin(z_1 + x_{1d})] + z_1 + x_{1d}
\end{aligned} \tag{44}$$

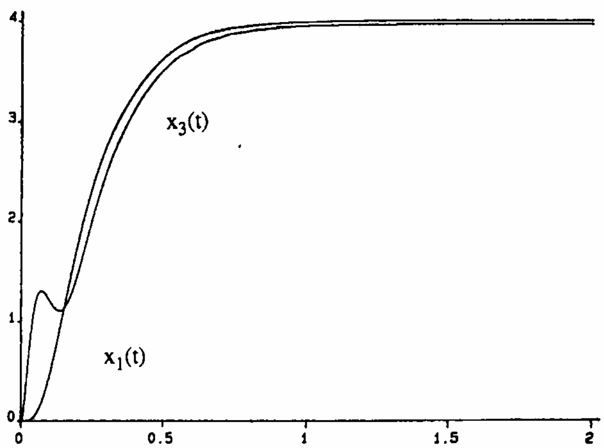

Figure 9. Discontinuous control input torque signal (magnified view of only a portion) and PWM control signal for static PWM controlled flexible robotic manipulator.

$$x_4 = \frac{1}{\kappa}[Iz_4 + MgLz_2 \cos(z_1 + x_{1d})] + z_2$$

$$\begin{aligned} u &= z_5 - \left(\frac{MgL}{I}\right)^2 \sin(z_1 + x_{1d}) \cos(z_1 + x_{1d}) \\ &+ \frac{1}{\kappa}[Iz_3 + MgL \sin(z_1 + z_{1d})] \left[\frac{MgL\kappa}{I^2} + \left(\frac{k}{I}\right)^2 + \frac{k^2}{IJ}\right] \cos(z_1 + x_{1d}) \\ &- \frac{MgL}{I} z_2^2 \sin(z_1 + x_{1d}) - \frac{MgL\kappa}{I^2} \sin(z_1 + x_{1d}) \end{aligned}$$

Fliess generalized controller canonical form of the extended system:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= z_5 \\ \dot{z}_5 &= c(z, v) \end{aligned} \tag{45}$$

The function $c(z, v)$ is easily obtained from (42), (43) and it only requires long but straightforward manipulations. It is not presented here in the interest of brevity.

Auxiliary output function for the extended system:

$$\sigma(z) = z_5 + a_4 z_4 + a_3 z_3 + a_2 z_2 + a_1 z_1 \tag{46}$$

with the a's chosen so that $p_4(\lambda) = \lambda^4 + a_4\lambda^3 + a_3\lambda^2 + a_2\lambda + a_1$ is a Hurwitz polynomial (for instance $p(\lambda) = (\lambda^2 + 2\zeta_1\omega_{1n}\lambda + \omega_{1n}^2)(\lambda^2 + 2\zeta_2\omega_{2n}\lambda + \omega_{2n}^2)$, with $1 > \zeta_1, \zeta_2 > 0, \omega_{1n}, \omega_{2n} > 0$). In original coordinates:

$$\begin{aligned}
\sigma(x) \;=\;& \left(\frac{MgL^2}{I}\right)\sin(x_1)\cos(x_1) + \frac{MgL\kappa}{I^2}(x_1 - x_3)\cos(x_1) \\
&+\frac{MgL}{I}x_2^2\sin(x_1) + \frac{MgL}{I}\left(\frac{\kappa}{I} - a_3\right)\sin(x_1) + \left[\left(\frac{\kappa}{I}\right)^2 - \frac{\kappa}{I}a_3\right](x_1 - x_3) \\
&+\frac{\kappa^2}{IJ}(x_1 - x_3) + \frac{\kappa}{IJ}u - \frac{MgLa_4}{I}x_2\cos(x_1)\frac{\kappa a_4}{I}(x_2 - x_4) \\
&+a_2x_2 + a_1(x_1 - x_{1d})
\end{aligned} \tag{47}$$

Dynamical PWM controller in original coordinates:

$$\begin{aligned}
\dot{u} \;=\;& \frac{IJ}{\kappa}\left\{-W\,PWM[\sigma(x)] - a_4\frac{MgL}{I}\left[\frac{MgL}{I}\sin(x_1) + \frac{\kappa}{I}(x_1 - x_3)\cos(x_1)\right]\right. \\
&-a_4x_2^2\sin(x_1) + \frac{\kappa}{I}\left[\frac{MgL}{I}\sin(x_1) + (x_1 - x_3)\left(\frac{\kappa}{I} + \frac{\kappa}{J}\right) + \frac{u}{J}\right] \\
&+a_3\left[\frac{MgL}{I}x_2\cos(x_1)\frac{\kappa}{I}(x_2 - x_4)\right] + a_2\left[\frac{MgL}{I}\sin(x_1) + \frac{\kappa}{I}(x_1 - x_3)\right] - a_1x_2 \\
&-\frac{MgL}{I}\left\{x_2^3\cos(x_1) - 2x_2\left[\frac{MgL}{I}\sin(x_1) + \frac{\kappa}{I}(x_1 - x_3)\right]\sin(x_1)\right. \\
&-\left[\frac{MgL}{I}x_2\cos(x_1) + \frac{\kappa}{I}(x_2 - x_4)\right]\cos(x_1) \\
&\qquad\qquad\qquad\left. + \left[\frac{MgL}{I}\sin(x_1) + \frac{\kappa}{I}(x_1 - x_3)\right]x_2\sin(x_1)\right\} \\
&-\frac{\kappa}{I}\left[\frac{MgL}{I}x_2\cos(x_1) - \left(\frac{\kappa}{J} + \frac{\kappa}{I}\right)(x_2 - x_4)\right] + \frac{MgL}{I}x_2^2\sin(x_1) \\
&\left.+\frac{MgL\kappa}{I^2}\sin(x_1) + \frac{MgL\kappa}{I^2}\sin(x_1) + \left(\frac{\kappa}{I}\right)^2(x_1 - x_3) + \frac{\kappa^2}{IJ}(x_1 - x_3) + \frac{\kappa}{IJ}u\right\}
\end{aligned} \tag{48}$$

Simulations were run for the dynamically PWM controlled manipulator (42), (48) with the same parameter values as before. The auxiliary output function was synthesized such that the corresponding characteristic polynomial is given by $p_4(\lambda) = (\lambda^2+2\zeta_1\omega_{1n}\lambda+\omega_{1n}^2)(\lambda^2+2\zeta_2\omega_{2n}\lambda+\omega_{2n}^2)$ with $\zeta_1 = 0.8, \zeta_2 = 0.9, \omega_{1n} = 28.0\ [rad/s], \omega_{2n} = 15.25\ [rad/s]$. The PWM controller parameters were chosen as before. Figure 10 depicts the controlled trajectories of the angular positions x_1 and x_3 clearly showing convergence of x_1 to the desired reference value.

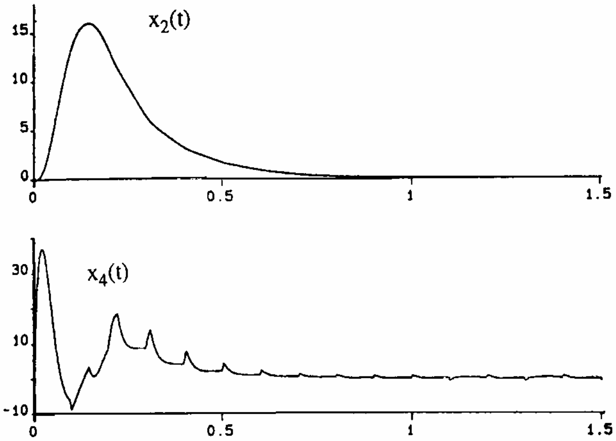

Figure 10. Angular position and angular velocity of dynamical PWM controlled flexible robotic manipulator.

Due to the assumed flexibility, the shaft's angular position x_3 exhibits a small steady state error of 0.052 $[rad]$ with respect to x_1. Figure 11 shows the link and shaft's angular velocities x_2 and x_4.

The (chattering-free) dynamically generated torque input signal u, as generated by (48), is shown in Figure 12.

CONCLUSIONS

A general stabilizing design procedure, based on dynamical PWM feedback control policies, has been presented for nonlinear systems describing some linearizable mechanical systems. A stabilizing discontinuous static controller of the PWM type is proposed for an elementary scalar system. Based on this simple result, a static PWM controller design can be obtained, for general high order nonlinear systems, by the zeroing of a suitably chosen auxiliary scalar output function. Zeroing of such an auxiliary output function induces an asymptotically stable motion for the constrained dynamics characterized by a linear time-invariant system with eigenvalues placeable at will. The results are easily implemented on a dynamical extension of the original nonlinear system which now results in a dynamical PWM feedback controller. In the dynamical controller case, the discontinuities, associated to the PWM regulator, take place on the one-dimensional state space of the dynamical controller, and not in the state space of the system. The resulting integrated control actions are, thus, *continuous* with substantially reduced (smoothed out) chattering.

In order to establish the salient stability features of the actual closed loop PWM controlled system, no need arises to resort to *average* controlled system considerations, nor high sampling frequency assumptions. As a matter of fact, the sampling frequency plays no crucial role in the stabilization features of the system, aside from the verification of a simple algebraic condition. New applications areas, such as nonlinear chemical process control, nonlinear electromechanical systems control etc., in which PWM con-

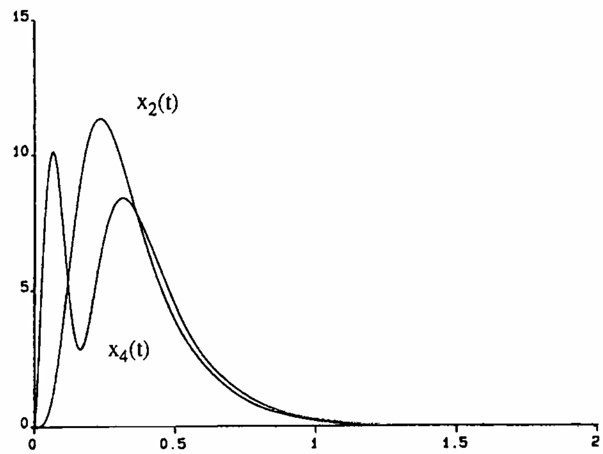

Figure 11. Angular velocities of link and motor shaft for dynamical PWM controlled flexible robotic manipulator.

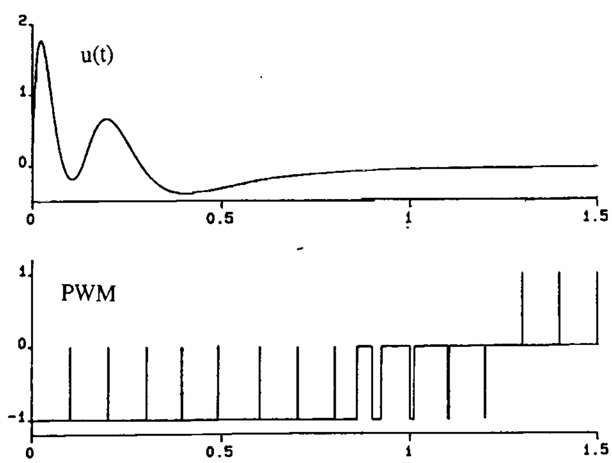

Figure 12. Chattering frec control input torque signal and PWM signal for dynamical PWM controlled flexible robotic manipulator.

trol was not traditionally feasible, can now benefit from the inherent robustness and high performance characteristics of this class of discontinuous control strategies.

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