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2. An Algebraic Approach to Sliding Mode Control

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2.1 Introduction

Recent developments in nonlinear systems theory propose the use of differential algebra for the conceptual formulation, clear understanding and definitive solution of long standing problems in the discipline of automatic control. Fundamental contributions in this area are due to Fliess (1986, 1987, 1988a, 1988b, 1989a, 1989b) while some other work has been independently presented by Pommaret (1983, 1986). Similar developments have resulted in a complete restatement of linear systems theory using the theory of Modules (see Fliess (1990c)).

In this chapter implications of the differential algebraic approach for the sliding mode control of *nonlinear* single-input single-output systems are reviewed. We also explore the implications of using module theory in the treatment of sliding modes for the case of (multivariable) *linear* systems.

Formalization of sliding mode control theory, within the framework of differential algebra and module theory, represents a theoretical need. All the basic elements of the theory are recovered from this viewpoint, and some fundamental limitations of the traditional approach are therefore removed.

For instance, input-dependent sliding surfaces are seen to arise naturally from this new approach. These manifolds are shown to lead to continuous, rather than bang-bang, inputs and chatter-free sliding regimes. Independence of the dimension of the desired ideal sliding dynamics with respect to that of the underlying plant, is also an immediate consequence of the proposed approach. A relationship linking controllability of a nonlinear system and the possibility of creating *higher order* sliding regimes is also established using differential algebra. The implications of the module theoretic approach to sliding regimes in linear systems seem to be multiple. Clear connections with decouplability, nonminimum phase problems, and the irrelevance of matching conditions from an input-output viewpoint, are but a few of the theoretical advantages with far reaching practical implications.

The first contribution using differential algebraic results in sliding mode control was given by Fliess and Messager (1990). These results were later extended and applied in several case studies by Sira-Ramírez et al (1992), Sira-Ramírez and Lischinsky-Arenas (1991) and Sira-Ramírez (1992a, 1992b, 1992c, 1993). Recent papers dealing with the multivariable linear systems case are those of Fliess and Messager (1991) and Fliess and Sira-Ramírez (1993). Extensions to pulse-width-modulation, and pulse frequency modulation control strategies may also be found in Sira-Ramírez (1992d, 1992e). Some of these results, obtained for sliding mode control, can be related to ideas presented by Emelyanov (1987, 1990) in his *binary systems* formulation of control problems. In Emelyanov's work, however, the basic developments are not drawn from differential algebra. The algebraic approach to sliding regimes in perturbed linear systems was studied by Fliess and Sira-Ramírez (1993a, 1993b). The theory is presented here in a tutorial fashion with a number of illustrative examples.

Section 2.2 is devoted to general background definitions used in the differential algebraic approach to nonlinear systems theory. Section 2.3 presents some of the fundamental implications of this new trend to sliding mode control analysis and synthesis. As a self-contained counterpart of the results for nonlinear systems, Sect. 2.4 is devoted to present the module theoretic approach to sliding mode control in linear systems. Sect. 2.5 contains some conclusions and suggestions for further work.

2.2 Basic Background to Differential Algebra

In this section we present in a tutorial fashion some of the basic background to differential algebra which is needed for the study of nonlinear dynamical systems. The results are gathered from Fliess's numerous contributions with little or no modification. Further details are found in Fliess (1988a, 1989a).

2.2.1 Basic Definitions

Definition 2.1 *An ordinary differential field K is a commutative field in which a single operation, denoted by "$\mathrm{d}/\mathrm{d}t$" or "$\cdot$", called derivation, is defined, which satisfies the usual rules:* $\mathrm{d}(ab+c)/\mathrm{d}t = (\mathrm{d}a/\mathrm{d}t)b + a(\mathrm{d}b/\mathrm{d}t) + \mathrm{d}c/\mathrm{d}t$ *for any a, b and c in K. If all elements c in K satisfy* $\mathrm{d}c/\mathrm{d}t = 0$, *then K is said to be a field of constants.*

Example 2.2 The field $\mathbb{R}$ of real numbers, with the operation of time differentiation $\mathrm{d}/\mathrm{d}t$, trivially constitutes a differential field, which is a field of constants. The field of rational functions in t with coefficients in $\mathbb{R}$, denoted by $\mathbb{R}(t)$, is a differential field with respect to time derivation. $\mathbb{R}(x)$ is also a differential field for any differentiable indeterminate x.

Definition 2.3 *Given a differential field L which contains K, we say L is a differential field extension of K, and denote it by L/K, if the derivation in K is a restriction of that defined in L.*

Example 2.4 $\mathbb{R}(t)/\mathbb{R}$ is a differential field extension over the set of real numbers. The differential field $\mathbb{R}(t)/Q(t)$ is also a differential field extension over the field $Q(t)$ of all rational functions in t with coefficients in the set of rational numbers Q. Similarly, the field $C(t)$ of rational functions in t with complex coefficients, is a differential field extension of, both $\mathbb{R}(t)$ and of $Q(t)$. Evidently, $C(t)/Q$ and $C(t)/C$ are also differential field extensions.

In the following developments u is considered to be a scalar differential indeterminate and k stands for an ordinary differential field with derivation denoted by $\mathrm{d}/\mathrm{d}t$.

Definition 2.5 *By $k\langle u\rangle$, we denote the differential field generated by u over the ground field k. i.e., the smallest differential field containing both k and u. This field is clearly the intersection of all differential fields which contain the union of k and u.*

Example 2.6 Consider the field of all possible rational expressions in u and its time derivatives, with coefficients in $\mathbb{R}$. This differential field is $\mathbb{R}\langle u\rangle$. A typical element in $\mathbb{R}\langle u\rangle$ may be

$$\frac{u^{(3)}}{u^2} + \frac{3u^2\dot{u} + \pi(\ddot{u})^{-1}u^4 - 1.02(\dot{u})^3}{\sqrt{7}u^{(5)} + u} - 5^{0.2}u \tag{2.1}$$

Example 2.7 Let $x_1, \ldots, x_n$ be differential indeterminates. Consider the differential field $k\langle u\rangle$. One may then extend $k\langle u\rangle$ to a differential field K containing all possible rational expressions in the variables $x_1, \ldots, x_n$, and their time derivatives, with coefficients in $k\langle u\rangle$. For instance, a typical element in $K/\mathbb{R}\langle u\rangle$ may be

$$\frac{u^2\left(x_1{}^3\right)^2 - \frac{1}{\log(\pi)}\left(\frac{2\ddot{u}+u^3u^{(7)}}{1+\pi u}\right)(\dot{x}_5)^2 x_6 + x_2}{\sqrt[5]{5}x_3x_4(\ddot{x}_1)^3 + u^{(3)} - e^2\dot{u}x_2} \tag{2.2}$$

A differential field K, like the one just described, is addressed as a *finitely generated field extension* over $\mathbb{R}\langle u\rangle$. In general, K does not coincide with $\mathbb{R}\langle u, x\rangle$ and it is somewhat larger since we find in K some other variables, like e.g. outputs, which may not be in $\mathbb{R}\langle u, x\rangle/\mathbb{R}\langle u\rangle$.

Definition 2.8 *Any element of a differential field extension, say L/K, has only two possible characterizations. Either it satisfies an algebraic differential equation with coefficients in K, or it does not. In the first case, the element is said to be differentially algebraic over K, otherwise it is said to be differentially transcendental over K. If the property of being differentially algebraic is shared by all elements in L, then L is said to be a differentially algebraic extension of K. If, on the contrary, there is at least one element in L which is differentially transcendent over K, then L is said to be a differentially transcendent extension of K.*

Example 2.9 Consider $k\langle u\rangle$, with k being a constant field. If x is an element which satisfies, $\dot{x} - ax - u = 0$, then x is differentially algebraic over $k\langle u\rangle$. However, since no further qualifications have been given, u is differentially transcendent over k.

Definition 2.10 *A differential transcendence basis of L/K is the largest set of elements in L which do not satisfy any algebraic differential equation with coefficients in K, i.e. they are not differentially K-algebraically dependent. A non-differential transcendence basis of L/K is constituted by the largest set of elements in L which do not satisfy any algebraic differential equation with coefficients in K. The number of elements constituting a differential transcendence basis is called the differential transcendence degree, and denoted by diff tr d °. The (non-differential) transcendence degree (tr d °) refers to the cardinality of a non-differential transcendence basis.*

Example 2.11 In the previous example the differential field extension $k\langle x,u\rangle/k\langle u\rangle$ is algebraic over $k\langle u\rangle$, but, on the other hand, $k\langle u\rangle/k$ is differentially transcendent over k, with u being the differential transcendence basis. Note that x is transcendent over $k\langle u\rangle$ as it does not satisfy any algebraic equation, but does satisfy a differential one. Hence, x is a non-differential transcendence basis of $k\langle x,u\rangle/k\langle u\rangle$. Evidently, *diff tr d* $^\circ k\langle x,u\rangle/k\langle u\rangle = 0$, and *tr d* $^\circ k\langle x,u\rangle/k\langle u\rangle = 1$

Theorem 2.12 *A finitely generated differential extension L/K is differentially algebraic if, and only if its (non-differential) transcendence degree is finite.*

Proof. See Kolchin (1973).

Definition 2.13 *A dynamics is defined as a finitely generated differentially algebraic extension $K/k\langle u\rangle$ of the differential field $k\langle u\rangle$.*

The input u is regarded as an *independent* indeterminate. This means that u is a *differentially transcendent* element of K/k, i.e. u does not satisfy any algebraic differential equation with coefficients in k. It is easy to see, that if u is a differential transcendent element of $k\langle u\rangle$, then it is also a differential transcendence element of $K/k\langle u\rangle$.

The following result is quite basic:

Proposition 2.14 *Suppose $x = (x_1, x_2, ..., x_n)$ is a non-differential transcendence basis of $K/k\langle u\rangle$, then, the derivatives $\mathrm{d}x_i/\mathrm{d}t; (i = 1,\ldots,n)$ are $k\langle u\rangle$ -algebraically dependent on the components of x.*

Proof. This is immediate.

One of the consequences of all these results, discussed by Fliess (1990a) is that a more general and natural representation of nonlinear systems requires *implicit* algebraic differential equations. Indeed, from the preceeding proposition, it follows that there exist exactly n polynomial differential equations with coefficients in k, of the form

$$P_i\left(\dot{x}_i, x, u, \dot{u}, \ldots, u^{(\alpha)}\right) = 0 \;; \qquad i = 1,\ldots,n \tag{2.3}$$

implicitly describing the controlled dynamics with the inclusion of input time derivatives up to order α.

It has been shown by Fliess and Hassler (1990) that such implicit representations are not entirely unusual in physical examples. The more traditional form of the state equations, known as *normal form*, is recovered in a local fashion, under the assumption that such polynomials locally satisfy the following rank condition

$$\text{rank}\begin{bmatrix} \frac{\partial P_1}{\partial \dot{x}_1} & 0 & \cdots & 0 \\ 0 & \frac{\partial P_2}{\partial \dot{x}_2} & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\partial P_n}{\partial \dot{x}_n} \end{bmatrix} = n \tag{2.4}$$

The time derivatives of the x_i's may then be solved locally

$$\dot{x}_i = p_i(x, u, \dot{u}, \ldots, u^{(\alpha)}) = 0 \quad i = 1, \ldots, n \tag{2.5}$$

It should be pointed out that even if (2.3) is in polynomial form, it may happen that (2.5) is not. The representation (2.5) is known as the *Generalized State Representation* of a nonlinear dynamics.

2.2.2 Fliess's Generalized Controller Canonical Forms

The following theorem constitutes a direct application of the theorem of the *differential primitive element* which may be found in Kolchin (1973). This theorem plays a fundamental role in the study of systems dynamics from the differential algebraic approach (Fliess 1990a).

Theorem 2.15 *Let $K/k\langle u\rangle$ be a dynamics. Then, there exists an element $\xi \in K$ such that $K = k\langle u, \xi\rangle$ i.e., such that K is the smallest field generated by the indeterminates u and ξ.*

Proof. See Fliess (1990a).

The (non-differential) transcendence degree n of $K/k\langle u\rangle$ is the smallest integer n such that $\xi^{(n)}$ is $k\langle u\rangle$-algebraically dependent on ξ, $\mathrm{d}\xi/\mathrm{d}t, \ldots, \mathrm{d}^{(n-1)}\xi/\mathrm{d}t^{(n-1)}$. We let $q_1 = \xi$, $q_2 = \mathrm{d}\xi/\mathrm{d}t, \ldots$, $q_n = \mathrm{d}^{(n-1)}\xi/\mathrm{d}t^{(n-1)}$. It follows that $q = (q_1, \ldots, q_n)$ also qualifies as a (non-differential) transcendence basis of $K/k\langle u\rangle$.Hence, one obtains a nonlinear generalization of the controller canonical form, known as the *Global Generalized Controller Canonical Form* (GGCCF)

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= q_3 \end{aligned}$$

$$
\begin{aligned}
&\vdots \\
C(\dot{q}_n, q, u, \dot{u}, \ldots, u^{(\alpha)}) &= 0
\end{aligned}
\qquad (2.6)
$$

where C is a polynomial with coefficients in k. If one can solve locally for the time derivative of q_n in the last equation of 2.6, one obtains locally an explicit system of first order differential equations, known as the *Local Generalized Controller Canonical Form* (LGCCF)

$$
\begin{aligned}
\dot{q}_1 &= q_2 \\
\dot{q}_2 &= q_3 \\
&\vdots \\
\dot{q}_n &= c(q, u, \dot{u}, \ldots, u^{(\alpha)})
\end{aligned}
\qquad (2.7)
$$

Remark. We assume throughout that $\alpha \geq 1$, i.e. the input u explicitly appears before the n-th derivative of the differential primitive element. The case $\alpha = 0$ corresponds to that of exactly linearizable systems under state coordinate transformations and static state feedback. One may still obtain the same smoothing effect of dynamical sliding mode controllers which we shall derive in this article, by considering arbitrary *prolongations* of the input space (i.e. addition of integrations before the input signal). This is accomplished by successively considering the *extended system* (Nijmeijer and Van der Schaft 1990), and proceeding to use the same differential primitive element yielding the LGCCF of the original system.

Example 2.16 Consider the second order system
$\dot{x}_1 = x_2 + u$, $\dot{x}_2 = u$. Then one may consider $\xi = x_1$ as a differential primitive element. In this case the GCCF of the system is simply $\dot{\xi}_1 = \xi_2$, $\dot{\xi}_2 = u + \dot{u}$

2.2.3 Input-Output Systems

Definition 2.17 *(Fliess 1988) Let k be a differential ground field and let u be a differential transcendent element over k. A single input-single output system consists of*

(i) a given input u

(ii) an output y, belonging to a universal differential field extension U, such that y is differentially algebraic over the differential field $k\langle u\rangle$, which denotes the smallest differential field containing, both k and u.

Remark. An input-output system may be viewed as a finitely generated differential field extension $k\langle y, u\rangle / k\langle u\rangle$. The differential field $k\langle y, u\rangle$ is, hence, differentially algebraic over $k\langle u\rangle$, i.e. y satisfies an algebraic differential equation with coefficients in $k\langle u\rangle$.

Definition 2.18 *Let $k\{y,u\}$ denote the differential ring generated by y and u and let U be a universal differential field. A differential homomorphism $\psi : k\{y,u\} \mapsto U$ is defined as a homomorphism which commutes with the derivation defined on $k\{y,u\}$, i.e.*

$$\forall \ \theta \in k\{u,y\}, \qquad \psi\left(\dot{\theta}\right) = \frac{d}{dt}\psi(\theta) \tag{2.8}$$

Definition 2.19 *A differential k-specialization of the differential ring $k\{y,u\}$ is a differential homomorphism $\psi : k\{y,u\} \mapsto U$, taking $k\{y,u\}$ into the universal differential field U, which leaves the elements of the ground field k invariant, i.e.*

$$\forall \ a \in k, \ \ \psi(a) = a \tag{2.9}$$

The differential transcendence degree of the extension over k, of the differential quotient field $Q(\psi(k\{y,u\}))$, is nonnegative and it is never higher than the differential transcendence degree of $k\langle u\rangle/k$ (i.e. $\textit{diff tr d}\,^\circ Q(\psi(k\{y,u\})/k \leq \textit{diff tr d}\,^\circ k\langle u\rangle/k = 1$). One frequently takes ψ as the identity mapping.

Remark. Differential specializations have been found to have a crucial relevance in the definition of the zero dynamics (Fliess 1990b). Indeed, consider the input-output system $k\langle y,u\rangle/k\langle u\rangle$. Let J be the largest differential subfield of $k < u,y >$ which contains $k\langle y\rangle$ and such that $J/k\langle y\rangle$ is differentially algebraic. Notice that J is not, in general, equal to $k\langle y,u\rangle$, unless the system is left invertible. Consider now the differential homomorphism $\psi : k\{y,u\} \mapsto U$, such that $\psi(y) = 0$. Hence, $\psi(y^{(\beta)}) = 0$, for all $\beta \geq 1$. It follows that $\psi(k\{y,u\}) = k\{u\}$ and the quotient field $Q(\psi(k\{y,u\}))/k$ coincides with the differential field extension $k\langle u\rangle/k$. Extend now the corresponding differential specialization ψ to the differential field J, in a trivial manner, and obtain a *smaller* differential field J^*. The specialized extension J^*/k , which is evidently differentially algebraic, is called the *zero dynamics*.

In the language of differential algebra, feedback is also accounted for, in all generality, by means of differential specializations (Fliess 1989a). This most appealing way of treating the fundamental concept of control theory is stated as follows:

Definition 2.20 *A closed-loop control is a differential k-specialization $\psi : k\{y,u\} \mapsto U$ such that $\textit{diff tr d}\,^\circ Q(\psi(k\{y,u\}))/k = 0$. We refer to such feedback loops as pure feedback loops. In such a case, the specialized elements $\psi(u)$, $\psi(y)$ satisfy an ordinary algebraic differential equation. Whenever $\textit{diff tr d}\,^\circ k\langle\psi(y)\rangle/k$ is zero, the closed-loop is said to be degenerate .*

We are mainly interested in those cases for which $\textit{diff tr d}\,^\circ Q(\psi(k\{y,u\}))/k = 0$. However, let v be a scalar differential transcendent element of $k\langle v\rangle/k$, such that $\psi(u)$, $\psi(y)$ are differentially algebraic over $k\langle v\rangle$. Then, if *diff tr d* $^{\circ}Q(\psi(k\{y,v\}))/k = 1$, the underlying differential specialization ψ leads to a regular feedback loop with an (independent) external input v (Fliess 1987).

Definition 2.21 *An input-output system $k\langle y,u\rangle/k\langle u\rangle$ is invertible if u is differentially algebraic over $k\langle y\rangle$, i.e. if diff tr d $^{\circ}k\langle y,u\rangle/k\langle y\rangle = 0$. It is easy to see that every nontrivial single-intput single-output system is always invertible.*

2.3 A Differential Algebraic Approach to Sliding Mode Control of Nonlinear Systems

In this section we present some applications of the results of the differential algebraic approach, proposed by Fliess for the study of control systems, to characterize in full generality, sliding mode control of nonlinear systems.

2.3.1 Differential Algebra and Sliding Mode Control of Nonlinear Dynamical Systems

Consider a (nonlinear) dynamics $K/k\langle u\rangle$. Furthermore, let, $\zeta = (\zeta_1, ..., \zeta_n)$ be a non-differential transcendence baisis for K, i.e. the transcendence degree of $K/k\langle u\rangle$ is then assumed to be n.

Definition 2.22 *A sliding surface candidate is any non k-algebraic element σ of $K/k\langle u\rangle$ such that its time derivative $d\sigma/dt$ is $k\langle u\rangle$-algebraically dependent on ζ , i.e. there exists a polynomial S over k such that*

$$S(\dot{\sigma}, \zeta, u, \dot{u}, \ldots, u^{(\alpha)}) = 0 \tag{2.10}$$

Remark. In the traditional definition of the sliding mode for systems in *Kalman form* with state ζ, the time derivative of the sliding surface was required to be only algebraically dependent on ζ and u. Hence, all the resulting sliding mode controllers were necessarily static. One can generalize this definition using differential algebra. The differential algebraic approach naturally points to the possibilities of *dynamical sliding mode controllers* specially in the case of nonlinear systems, where elimination of input derivatives from the system model may not be possible at all (see Fliess and Hasler (1990) for a physical example).

Proposition 2.23 *The element σ in $K/k\langle u\rangle$ is a sliding surface candidate if it is k-algebraically dependent on all the elements of a transcendence basis ζ.*

Proof. The time derivative of σ is k-algebraically dependent on the derivatives of every element in the transcendence basis ζ. Therefore, $d\zeta/dt$ is $k\langle u\rangle$-algebraically dependent on ζ

The condition in the above proposition is clearly not necessary as σ may well be k-algebraically dependent only on some elements of the transcendence basis ζ, and still have $d\sigma/dt$ being $k\langle u\rangle$-algebraically dependent on ζ. Imposing on σ a ***discontinuous sliding dynamics*** of the form

$$\dot{\sigma} = -W \text{sign}\ \sigma \tag{2.11}$$

one obtains from (2.10) an *implicit dynamical sliding mode controller* given by

$$S(-W\text{sign}(\sigma), \zeta, u, \dot{u}, \ldots, u^{(\alpha)}) = 0 \tag{2.12}$$

which is an implicit time-varying discontinuous ordinary differential equation for the control input u. The two *structures* associated with the underlying variable structure control system are represented by the following pair of implicit dynamical controllers

$$\begin{aligned} S(-W, \zeta, u, \dot{u}, \ldots, u^{(\alpha)}) &= 0 \qquad \text{for } \sigma > 0 \\ S(W, \zeta, u, \dot{u}, \ldots, u^{(\alpha)}) &= 0 \qquad \text{for } \sigma < 0 \end{aligned} \tag{2.13}$$

each one valid, respectively, on one of the *regions* $\sigma > 0$ and $\sigma < 0$. Precisely when $\sigma = 0$ neither of the control structures is valid. One then ideally characterizes the motions by formally assuming $\sigma = 0$ and $d\sigma/dt = 0$ in (2.10). We formally define the ***equivalent control dynamics*** as the dynamical state feedback control law obtained by letting $d\sigma/dt$ become zero in (2.12), and consider the resulting implicit differential equation for the equivlent control, here denoted by u_{eq}

$$S(0, \zeta, u_{eq}, \dot{u}_{eq}, \ldots, u_{eq}^{(\alpha)}) = 0 \tag{2.14}$$

According to the initial conditions of the state ζ and the control input and its derivatives, one obtains in general, $\sigma =$ *constant*. Hence, the sliding motion ideally taking place on $\sigma = 0$ may be viewed as a particular case of the motions of the system obtained by means of the equivalent control.

Note that whenever $\partial S/\partial\dot{\sigma} \neq 0$, one locally obtains from the implicit equation (2.10)

$$\dot{\sigma} = s(\zeta, u, \dot{u}, \ldots, u^{(\alpha)}) \tag{2.15}$$

The corresponding dynamical sliding mode feedback controller, satisfying (2.11), is given by

$$s(\zeta, u, \dot{u}, \ldots, u^{(\alpha)}) = -W \text{sign}\ \sigma \tag{2.16}$$

Furthermore, if $\partial\sigma/\partial u^{(\alpha)} \neq 0$, one obtains locally a time-varying state space representation for the dynamical sliding mode controller (2.16) in *normal* form

$$\begin{aligned} \dot{u}_1 &= u_2 \\ \dot{u}_2 &= u_3 \\ &\vdots \\ \dot{u}_\alpha &= \theta(u_1, \dots, u_\alpha, \zeta, W\text{sign}\ \sigma) \end{aligned} \tag{2.17}$$

All discontinuities arising from the bang-bang control policy (2.11) are seen to be confined to the highest derivative of the control input through the nonlinear function θ. The output u of the dynamical controller is clearly the outcome of α integrations performed on such a discontinuous function θ and for this reason u is, generically speaking, sufficiently continuous.

2.3.2 Dynamical Sliding Regimes Based on Fliess's GCCF

The general results on canonical forms for nonlinear systems, presented in Sect. 2.2, have an immediate consequence in the definition of sliding surfaces for stabilization and tracking problems. We explore the stabilization problem below.

Consider a system of the form (2.7) and the following sliding surface coordinate function, expressed in terms of the generalized phase coordinates q

$$\sigma = c_1 q_1 + c_2 q_2 + \cdots + c_{n-1} q_{n-1} + q_n \tag{2.18}$$

where the scalar coefficients c_i ($i = 1, \cdots, n-1$) are chosen in such a manner that the polynomial

$$p(s) = c_1 + c_2 s + \cdots + c_{n-1} s^{n-2} + s^{n-1} \tag{2.19}$$

in the complex variable s, is Hurwitz. Imposing on the sliding surface coordinate function σ the discontinuous dynamics (2.11), then the trajectories of σ are seen to exhibit, within finite time T given by $T = W^{-1}|s(0)|$, a sliding regime on $\sigma = 0$. Substituting in (2.11) the expression (2.18) for σ, and using (2.7), one obtains after some straightforward algebraic manipulations, the implicit dynamical sliding mode controller

$$\begin{aligned} c(q, u, \dot{u}, \dots, u^{(\alpha)}) &= c_{n-1}\dot{\sigma} + c_1 c_{n-1} q_1 + (c_2 c_{n-1} - c_1) q_2 + \cdots \\ &\quad + (c_{n-2} c_{n-1} - c_{n-3}) q_{n-2} + (c_{n-1}^2 - c_{n-2}) q_{n-1} \\ &\quad - W\text{sign}\ \sigma \\ &= -c_1 q_2 - c_2 q_3 - \cdots - c_{n-2} q_{n-1} - c_{n-1} q_n \\ &\quad - W\text{sign}\ \sigma \end{aligned} \tag{2.20}$$

Evidently, under ideal sliding conditions $\sigma = 0$, the variable q_n no longer qualifies as a state variable for the system since it is expressible as a linear combination of the remanining states and, hence, q_n is no longer a non-differentially transcendent element of the field extension K. The ideal (autonomous) closed-loop dynamics may then be expressed in terms of a *reduced* non-differential transcendence basis of K/k which only includes the remaining $n-1$ phase coordinates associated with the original differential primitive element. This leads to the ideal sliding dynamics

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= q_3 \\ &\vdots \\ \dot{q}_{n-1} &= -c_1 q_1 - c_2 q_2 - \cdots - c_{n-2} q_{n-2} - c_{n-1} q_{n-1} \end{aligned} \tag{2.21}$$

The characteristic polynomial of (2.21) is evidently given by (2.19) and hence the (reduced) autonomous closed-loop dynamics is asymptotically stable to zero. Note that, by virtue of (2.18), the condition $\sigma = 0$ holds, and due to the asymptotic stability of (2.21), the variable q_n also tends to zero in an asymptotically stable fashion. The equivalent control, denoted by u_{eq}, is a ***virtual*** feedback control action achieving ideally smooth evolution of the system on the constraining sliding surface $\sigma = 0$, provided initial conditions are precisely set on such a switching surface. The equivalent control is formally obtained from the condition $d\sigma/dt = 0$, i.e.

$$\begin{aligned} c(q, u, \dot{u}_{eq}, \ldots, u_{eq}^{(\alpha)}) &= c_1 c_{n-1} q_1 + (c_2 c_{n-1} - c_1) q_2 + \cdots \\ &\quad + (c_{n-2} c_{n-1} - c_{n-3}) q_{n-2} + (c_{n-1}^2 - c_{n-2}) q_{n-1} \end{aligned} \tag{2.22}$$

Since q asymptotically converges to zero, the solutions of the above time-varying implicit differential equation, describing the evolution of the equivalent control, asymptotically approach the solutions of the following autonomous implicit differential equation

$$c(0, u, \dot{u}, \ldots, u^{(\alpha)}) = 0 \tag{2.23}$$

Equation (2.23) constitutes the *zero dynamics* (Fliess 1990b) associated with the problem of zeroing the differential primitive element, considered now as an (auxiliary) output of the system. Note that (2.23) may also be regarded as the zero dynamics associated with the zeroing of the sliding surface coordinate function σ. If (2.23) locally asymptotically approaches a constant equilibrium point $u = U$, then the system is said to be ***locally minimum phase*** around such an equilibrium point, otherwise the system is said to be ***non-minimum phase***. The equivalent control is, thus, locally asymptotically stable to U, whenever the underlying input-output system is minimum phase.

One may be tempted to postulate, for the sake of physical realizability of the sliding mode controller, that a sliding mode control strategy is properly defined whenever the zero dynamics associated with the system is constituted by an asymptotically stable motion towards equilibrium. In other words, the input-output system should be minimum phase. It must be pointed out, however, that non-minimum phase systems might make perfect physical sense and that, in some instances, instability of a certain state variable or input does not necessarily imply disastrous effects on the controlled system (for an example of this frequently overlooked fact, see Sira-Ramírez (1991, 1993)).

2.3.3 Some Formalizations of Sliding Mode Control for Input-Output Nonlinear Systems

Definition 2.24 *Consider a differential k-specialization ϕ, mapping $k\{y\} \mapsto U$, such that diff tr d $^{\circ}Q(\phi(k\{y\}))/k = 0$. The elements $\sigma \in Q(\phi(k\{y\}))/k$ are referred to as ideal sliding dynamics , or sliding surfaces. Note that $Q(\phi(k\{y\}))/k = k\langle\phi(y)\rangle/k$. We will be using the identity map for the mapping ϕ from now on. A sliding surface σ is, therefore, directly taken from the specialized extension $k\langle y\rangle/k$, as $\sigma = 0$.*

Definition 2.25 *Let σ be an element of $k\langle y\rangle/k$ such that $\sigma = 0$ represents a desirable ideal sliding dynamics. An equivalent control, corresponding to σ, is said to exist for the system $k\langle y,u\rangle/k\langle u\rangle$, if there exists a differential k-specialization $\psi : k\{y,u\} \mapsto U$, which represents a pure feedback loop, such that $d\sigma/dt$ is identically zero. A sliding regime is said to exist on $\sigma = 0$ if $\sigma \in k\langle\psi(y)\rangle/k$ and diff tr $d^{\circ}k\langle\psi(y)\rangle/k = 0$. The differential k-specialization $\psi : k\{y,u\} \mapsto U$, may be computed, in principle, from the condition $d\sigma/dt = 0$.*

Sliding mode control thus leads to a very special class of degenerate feedback in which the resulting closed-loop system ideally satisfies a preselected autonomous algebraic differential equation. Note that, in this setting and at least for single-input single-output systems, the order of the highest derivative of the output y in the differential equation representing the ideal sliding dynamics, is not necessarily restricted to be smaller than the highest order of the derivative of y in the differential equation defining the input-output system. The following helps to formalize this issue.

Definition 2.26 *An element r in the differential field $k\langle y\rangle/k$ is said to be a prolongation of an element $\rho \in k\langle y\rangle/k$, if r is obtained by a finite number of time differentiations performed on ρ, i.e. if there exist a nonnegative integer, L such that $r = \rho^{(L)}$. The integer L, of required differentiations, is called the length of the prolongation. Similarly, given an input-output system $k\langle y,u\rangle/k\langle u\rangle$ a prolonged system is obtained by straightforward differentiation of the input-output relation (Nijmeijer and Van der Schaft 1990). All prolongations of an input-output system rest in the differential field extension: $k\langle y,u\rangle/k\langle u\rangle$.*

Proposition 2.27 *Let $k\langle y,u\rangle/k\langle u\rangle$ be an invertible system, then any prolongation of the system, of finite length, is also invertible.*

Proof. It is easy to see that *diff tr d* $^{\circ}k\langle y\rangle/k$ is invariant with respect to prolongations.

Theorem 2.28 *Modulo singularities in the actual computation of the required control input, and the need for suitable prolongations, the equivalent control always exists for a given element $\sigma \in k\langle y\rangle$.*

Proof. The result is obviously true from the fact that the single input-single output system $k\langle y,u\rangle/k\langle u\rangle$ is trivially invertible, modulo the possible local singularities.

Example 2.29 Consider the first order input-output system $\dot{y} = u$ and the asymptotically stable second order ideal sliding dynamics $\sigma = \ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = 0, \xi > 0,\ \omega_n > 0$. The dynamical feedback (equivalent) controller $\ddot{u} = -2\xi\omega_n\dot{u} - \omega_n^2 u$, obtained from $\dot{\sigma} = \ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = 0$, defines the equivalent control for arbitrary initial conditions in u.

Remark. We have defined sliding motions in a quite general and relaxed sense. Essentially, we have required only that the ideal (autonomous) sliding dynamics be synthesizable, in principle, by pure feedback. The process of actually achieving a sliding regime on such a desirable autonomous dynamics may then be carried out through discontinous or continuous (e.g. high gain) feedback control of a static or dynamic nature. Owing to the generally local nature of the invertibility of a given system, as well as the possible presence of singularities, it may actually happen that finding well-defined discontinuous or continuous feedback policies, which eventually result in closed-loop compliance with the ideal sliding dynamics, may not be possible at all due to singularities.

Consider now a regular feedback loop with an external input v, obtained from the differential k-specializations ϕ^+, and ϕ^- mapping $k\{y\} \mapsto U$, such that

$$\textit{diff tr } d^\circ\ Q(\phi^+(k\{y\}))/k = \textit{diff tr } d^\circ\ Q(\phi^-(k\{y\}))/k = 1 \qquad (2.24)$$

In particular, let the external input v be obtained as $v = -W\text{sign}(\sigma)$. The controlled elements $\sigma \in Q(\phi^+(k\{y\}))/k$ and $\sigma \in Q(\phi^-(k\{y\}))/k$ are referred to as *controlled motions towards sliding*, and the differential specializations ϕ^+ and ϕ^- constitute the sliding mode control strategy .

Example 2.30 Consider again the single integrator system with a higher order sliding surface. A sliding regime is achieved on $\sigma = 0$ in finite time by imposing on σ the discontinuous dynamics $d\sigma/dt = -W\text{sign}\ \sigma$, i.e. $\dot{\sigma} = y^{(3)} + 2\xi\omega_n\ddot{y} + \omega_n^2\dot{y} = -W\text{sign}\ \left(\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y\right)$. Using suitably prolonged system equations, one obtains the dynamical sliding mode controller $\ddot{u} = -2\xi\omega_n\dot{u} - \omega_n^2 u - W\text{sign}\left(\dot{u} + 2\xi\omega_n u + \omega_n^2 y\right)$.

2.3.4 An Alternative Definition of the Equivalent Control Dynamics

One may generate a differential algebraic extension of $k\langle u\rangle$ by adjoining the sliding surface element σ to u, and considering $k\langle u,\sigma\rangle$ as an input-output system. The differential field extension $k\langle u,\sigma\rangle/k\langle u\rangle$ is indeed an input-output system, or, more precisely, an input-sliding surface system. The element σ is then a non-differential transcendence element of the field extension $k\langle u,\sigma\rangle/k\langle u\rangle$. It therefore satisfies an algebraic differential equation with coefficients in $k\langle u\rangle$. This means that there exists a polynomial with coefficients in k such that

$$P(\sigma,\dot{\sigma},\ldots,\sigma^{(\rho)},u,\dot{u},\ldots,u^{(\gamma)})=0 \tag{2.25}$$

where we have implicitly assumed that ρ is the smallest integer such that $d^{\rho}\sigma/dt^{\rho}$ is algebraically dependent upon $\sigma,\dot{\sigma},\ldots,\sigma^{(\rho)},u,\dot{u},\ldots,u^{(\gamma)}$. This general characterization of sliding surface coordinate functions has not been clearly established in the sliding mode control literature. Obtaining a differential equation for the sliding surface coordinate σ, which is independent of the system state, has direct implications for the area of *higher order* sliding motions (see Chang (1991)), for a second order sliding motion example) and some recent developments in *binary control systems*. We will explore only the first issue in Section 2.3.5. A state-independent implicit definition of the *equivalent control dynamics* can then be immediately obtained from (2.25) by setting σ and its time derivatives to zero

$$P(0,0,\ldots,0,u,\dot{u},\ldots,u^{(\gamma)})=0 \tag{2.26}$$

2.3.5 Higher Order Sliding Regimes

Recently some effort has been devoted to the smoothing of system responses to sliding mode control policies through so called *higher order* sliding regimes. *Binary control systems*, as applied to variable structure control, are also geared towards obtaining asymptotic convergence towards the sliding surface, in a manner that avoids control input chattering through integration. These two develpments are also closely related to the differential algebraic approach. In the following paragraphs we explain in complete generality how the same ideas may be formally derived from differential algebra.

Consider (2.25) with σ as an output and rewrite in the following *Global Generalized Observability Canonical Form* (GGOCF) (Fliess 1990a)

$$\begin{aligned}\dot{\sigma}_1 &= \sigma_2\\ \dot{\sigma}_2 &= \sigma_3\\ &\vdots\\ P(\sigma_1, \;\ldots\; &,\sigma_\rho,\dot{\sigma}_\rho,u,\dot{u},\ldots,u^{(\gamma)})=0\end{aligned} \tag{2.27}$$

As before, an explicit LGOCF can be obtained for the element σ whenever $\partial P/\partial\dot{\sigma}_\rho\neq 0$

$$\begin{aligned}\dot{\sigma}_1 &= \sigma_2\\ \dot{\sigma}_2 &= \sigma_3\\ &\vdots\\ \dot{\sigma}_\rho &= p\left(\sigma_1,\ldots,\sigma_\rho,u,\dot{u},\ldots,u^{(\gamma)}\right)\end{aligned} \tag{2.28}$$

Definition 2.31 *An element σ of the dynamics $K/k\langle u\rangle$ admits a ρ-th order sliding regime if the GOCF (2.29) associated with σ is ρ-th order.*

One defines a ρ-th order *sliding surface candidate* as any arbitrary (algebraic) function of σ and its time derivatives up to $(r-1)$-st order. For obvious reasons the most convenient type of function is represented by a suitable linear combination of σ and its time derivatives, which achieves stabilization

$$s = m_1\sigma_1 + m_2\sigma_2 + \cdots + m_{n-1}\sigma_{\rho-1} + \sigma_\rho \tag{2.29}$$

First-order sliding motion is then imposed on this linear combination of generalized phase variables by means of the discontinuous sliding mode dynamics

$$\dot{s} = -M\text{sign } s \tag{2.30}$$

This policy results in the implicit dynamical higher order sliding mode controller

$$\begin{aligned} p(\sigma_1, \ldots, \sigma_\rho, u, \dot{u}, \ldots, u^{(\gamma)}) &= -m_1\sigma_2 - m_2\sigma_3 - \cdots - m_{\rho-2}\sigma_{\rho-1} - m_{\rho-1}\sigma_\rho \\ &\quad -M\text{sign }(s) \end{aligned} \tag{2.31}$$

As previously discussed, s goes to zero in finite time and, provided the coefficients in (2.29) are properly chosen, an ideal asymptotically stable motion can be then obtained for s, which is governed by the autonomous linear dynamics

$$\begin{aligned} \dot{\sigma}_1 &= \sigma_2 \\ \dot{\sigma}_2 &= \sigma_3 \\ &\vdots \\ \dot{\sigma}_{\rho-1} &= -m_1\sigma_1 - \cdots - m_{\rho-1}\sigma_{\rho-1} \end{aligned} \tag{2.32}$$

2.3.6 Sliding Regimes in Controllable Nonlinear Systems

The differentially algebraic closure of the ground field k in the dynamics K is defined as the differential field κ, where $K \supseteq \kappa \supseteq k$, consisting of the elements of K which are differentially algebraic over k. The field k is differentially algebraically closed if and only if $k = \kappa$.

The following definition is taken from Fliess (1991) (see also Pommaret (1991)).

Definition 2.32 *The dynamics $K/k\langle u\rangle$ is said to be algebraically controllable if and only if the ground field k is differentially algebraically closed in K.*

Algebraic controllability implies that all elements of K are necessarily influenced by the input u, since they satisfy a differential equation which is not independent of u and possibly some of its time derivatives.

Proposition 2.33 *A higher order sliding regime can be created for any element s of the dynamics $K/k\langle u\rangle$ if and only if $K/k\langle u\rangle$ is controllable.*

Proof. Sufficiency is obvious from the fact that s satisfies a differential equation with coefficients in $k\langle u\rangle$. For the necessity of the condition, suppose, contrary to what is asserted, that $K/k\langle u\rangle$ is not controllable, but that a higher order sliding regime can be created on any element of the differential field extension $K/k\langle u\rangle$. Since k is not differentially algebraically closed, there are elements in K, which belong to a differential field κ containing k, which satisfy differential equations with coefficients in k. Clearly these elements are not related to the control input u through differential equations. It follows that a higher order sliding regime cannot be imposed on such elements. A contradiction is established.

In this more relaxed notion of sliding regime, one may say that sliding mode behaviour can be imposed on any element of the dynamics of the system, if and only if the system is controllable. The characterization of sliding mode existence through controllability, is a direct consequence of the differential algebraic approach.

2.4 A Module Theoretic Approach to Sliding Modes in Linear Systems

The particularization of the differential algebraic approach to the case of linear systems applies the notion of *Modules of Khäler differentials.* This theory establishes far reaching properties of the linearized version of the system to those of the underlying nonlinear system (see Fliess (1991) for details). It turns out that, in its own right, the theory of linear systems can be handled in a self contained manner, from the theory of modules over rings of finite linear differential operators. This approach discards the need to relate the linear system to some linearizability properties of an underlying nonlinear system generating it, which operates in the vicinity of an equilibrium point. Due to the wide spread knowledge about linear systems, this latter approach is preferred in the presentation that follows.

In this section we address the algebraic approach to sliding mode control of linear systems. We first provide some background definitions of the relevant topics in algebra. The reader is referred to the book by Adkins and Weintraub (1992) for a fundamental background. We shall be closely following the work of Fliess (1990c) for the portion containing background material on the applications of module thoery to linear systems. The algebraic approach to sliding mode control is taken from Fliess and Sira-Ramírez (1993a, 1993b).

Definition 2.34 *A ring $(R,+,\cdot)$ is a set R with two binary operations*

$$+ \quad : R \rightarrow R(addition)$$

$$\cdot \quad : R \rightarrow R(multiplication)$$

such that $(R, +)$ *is an abelian group with a* zero. *Multiplication and addition satisfy the usual properties of associativity and distributivity.*

Here we shall be dealing only with commutative rings with *identity*.

Example 2.35

The set $2Z$ of *even* integers is a ring without an identity. The set of all square $n \times n$ matrices defined over the field of real numbers. The set of all polynomials in an indeterminate x

Definition 2.36 *Let* R *be an arbitrary ring with identity. A left* R*-module is an abelian group* M *together with a scalar multiplication map*

$$\cdot \ : R \times M \rightarrow M$$

which satisfies the following axioms $\forall a, b \in R, \ m, n \in M$

$$\begin{aligned} a(m+n) &= am + an \\ (a+b)m &= am + bm \\ (ab)m &= a(bm) \\ 1m &= m. \end{aligned}$$

Example 2.37

Let F be a *field*, then an F-module V is called a *vector space* over F. Let R be an arbitrary ring. The set of matrices $M_{m,n}(R)$ is a left R-module via left scalar multiplication of matrices. Any subgroup $N \subset M$ which is closed under scalar multiplication by elements in R is itself a module, called a *submodule* of M.

If $S \subset M$, then $[S]$ denotes the intersection of all submodules of M containing S. We may say that $[S]$ is the "smallest" submodule, with respect to inclusion, containing the set S. The submodule $[S]$ is also called the *submodule of* M *generated by* S.

Definition 2.38 M *is* fintely generated *if* $M = [S]$ *for some finite subset* S *of* M*. The elements of* S *are called the "generators" of* M*. The rank of a module* M *is the cardinality of the minimal set of generators of* M *in* S.

We denote by $k\left[\frac{d}{dt}\right]$ the *ring of finite linear differential operators.* These are operators of the following form

$$\sum_{finite} a_\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}, \quad a_\alpha \in k$$

The ring $k\left[\frac{\mathrm{d}}{\mathrm{d}t}\right]$ is commutative if, and only if, k is a field of constants. We will be primarily concerned with rings of linear differential operators with real coefficients. This necessarily restricts the class of problems treated to linear, time-invariant, systems. The results, however, can be extended to time-varying systems by using rings defined over *principal ideal domains* (see Fliess (1990c)).

Definition 2.39 *Let M be a left $k\left[\frac{\mathrm{d}}{\mathrm{d}t}\right]$-module. An element $m \in M$ is said to be* torsion *if and only if there exists $\pi \in k\left[\frac{\mathrm{d}}{\mathrm{d}t}\right], \pi \neq 0$, such that $\pi m = 0$ i.e. m satisfies a linear differential equation with coefficients in k.*

Definition 2.40 *A module T such that all its elements are torsion is said to be a torsion module.*

Definition 2.41 *A finite set of elements in a $k[\frac{\mathrm{d}}{\mathrm{d}t}]$-module M constitutes a* basis *if every element in the module may be uniquely expressed as a $k[\frac{\mathrm{d}}{\mathrm{d}t}]$-linear combination of such elements. A module M is said to be* free *if it has a basis.*

Proposition 2.42 *Let M be a finitely generated left $k\left[\frac{\mathrm{d}}{\mathrm{d}t}\right]$-module. M is torsion if and only if the dimension of M as a k-vector space is finite .*

Definition 2.43 *The set of all torsion elements of a module M is a submodule T called the* torsion submodule *of M.*

Definition 2.44 *A module M is said to be free if and only if its torsion submodule is trivial.*

Theorem 2.45 *Any finitely generated left $k\left[\frac{\mathrm{d}}{\mathrm{d}t}\right]$-module M can be decomposed into a direct sum*

$$M = T \oplus \Phi$$

where T is the torsion submodule and Φ is a free submodule.

2.4.1 Quotient Modules

Let M be an R-module and let $N \subset M$ be a submodule of M, then N is a subgroup of the abelian group M and we can form the *quotient group M/N* as the set of all *cosets*

$$M/N = \{m + N \ ; \ \text{for} \ m \in M\} \tag{2.33}$$

They evidently accept the operation of addition as a well defined (commutative) operation

$$(m + N) + (p + N) = (m + p) + N$$

The elements $m+N$ of M/N can now be endowed with an R-module structure by defining scalar products in a manner inherited from M, namely,

$$a(m+N) = am + N \ ; \ \forall a \in R \ \text{ and } \ m \in M$$

The elements $m' = m(\text{mod}N)$ are called the *residues* of M in M/N. The map $M \to M/N$ taking $m \mapsto m' = m + N$ is called the *canonical projection*.

2.4.2 Linear Systems and Modules

Linear systems enjoy a particularly appealing characterization from the algebraic viewpoint. This has been long recognized since the work of Kalman (1970). More recently Fliess (1990c) has provided a rather different approach to such characterization, which still uses modules but in a different context. In this section we follow the work of Fliess (1990c) with little or no modifications.

Definition 2.46 *A linear system is a finitely generated left $k\left[\frac{\mathrm{d}}{\mathrm{dt}}\right]$-module Λ.*

Example 2.47 (Fliess, 1990c) Consider a system Σ as a finite set of quantities $w = (w_1, \ldots, w_q)$ which are related by a set of homogeneous linear differential equations over k.

Let

$$E_\alpha\left(w_i^{(\nu_j)}\right) = \sum_{finite} a_{\alpha,i,j} w_i^{(\nu_j)} = 0, \ (a_{\alpha,i,j} \in k)$$

Consider the left $k\left[\frac{\mathrm{d}}{\mathrm{dt}}\right]$-module $\mathcal{F}$ spanned by $\overline{w} = (\overline{w}_1, \ldots, \overline{w}_q)$ and let $\Xi \subset \mathcal{F}$ be the submodule spanned by

$$e_\alpha = E_\alpha\left(\overline{w}_i^{(\nu_j)}\right) = \sum_{finite} a_{\alpha,i,j} \overline{w}_i^{(\nu_j)}, \ (a_{\alpha,i,j} \in k)$$

The quotient module $\Lambda = \mathcal{F}/\Xi$ is the module corresponding to the system. It is easy to see that the canonical image (residue) of w in $\mathcal{F}/\Xi$ satisfies the system equations.

2.4.3 Unperturbed Linear Dynamics

Definition 2.48 *A linear dynamics $\mathcal{D}$ is a linear system $\mathcal{D}$ where we distinguish a finite set of quantities, called the inputs $u = (u_1, \ldots, u_m)$, such that the module $\mathcal{D}/[u]$ is torsion.*

The set of inputs u are said to be *independent* if and only if $[u]$ is a *free* module. An output vector $y = (y_1, \ldots, y_p)$ is a finite set of elements in $\mathcal{D}$.

Example 2.49 (Fliess 1990c) Consider the single input single output system

$$a(\frac{\mathrm{d}}{\mathrm{dt}})y = b(\frac{\mathrm{d}}{\mathrm{dt}})u \qquad a, b \in k[\frac{\mathrm{d}}{\mathrm{dt}}] \ , \ a \neq 0$$

Take as the free left $k[\frac{\mathrm{d}}{\mathrm{dt}}]$-module $\mathcal{F} = [\overline{u}, \overline{y}]$ spanned by $\overline{u}$, $\overline{y}$. Let $\Xi \subset \mathcal{F}$ be the submodule spanned by $a(\frac{\mathrm{d}}{\mathrm{dt}})\overline{y} - b(\frac{\mathrm{d}}{\mathrm{dt}})\overline{u}$. The quotient module $\mathcal{D} = \mathcal{F}/\Xi$ is the system module. Let u, y be the *residues* of $\overline{u}, \overline{y}$ in $\mathcal{D}$. Then u, y satisfy the system equations. If we let $\underline{y}$ be the residue of y in $\mathcal{D}/[u]$, then $\underline{y}$ satisfies $a(\frac{\mathrm{d}}{\mathrm{dt}})\underline{y} = 0$, which is torsion.

2.4.4 Controllability

Definition 2.50 *A linear system is said to be controllable if and only if its associated module Λ is* free.

Example 2.51 The system given by $\dot{w}_1 = w_2$ is controllable since its associated module is not torsion.

Definition 2.52 *A linear dynamics $\mathcal{D}$, with input u, is said to be* controllable *if and only if the associated linear system is controllable.*

Example 2.53 The linear dynamics $\dot{x}_1 = u$ is controllable, since its associated linear system is described by a free module. The module decomposition $\mathcal{D} = \Phi \oplus T$ shows that a system is controllable if and only if T is trivial.

Example 2.54 The linear system $\dot{w}_1 = w_2 \ ; \ \dot{w}_2 = -w_2$ is uncontrollable since its associated module can be decomposed as $[w_1] \oplus [w_2]$ with $[w_2]$ being evidently torsion.

2.4.5 Observability

Definition 2.55 *A linear dynamics $\mathcal{D}$ with input u and output y, is said to be observable if and only if $\mathcal{D} = [u, y]$. The quotient module $\mathcal{D}/[u, y]$ is trivial.*

Example 2.56 The linear dynamics $\dot{x}_1 = x_2 \ ; \ \dot{x}_2 = u \ ; \ y = x_1$ is observable since $x_1 = y \ ; \ x_2 = \dot{y}$.

If the system is unobservable then $[u, y] \subset \mathcal{D}$ and the quotient module $\mathcal{D}/[u, y]$ is torsion.

Example 2.57 The linear dynamics $\dot{x}_1 = x_1 \ ; \ \dot{x}_2 = u \ ; \ y = x_2$ is unobservable since $x_1 \notin [u, y]$ and the residues $\overline{x}_1\overline{x}_2$ in the quotient module $\mathcal{D}/[u, y]$ satisfy $\dot{\overline{x}}_1 - \overline{x}_1 = 0$ and $\dot{\overline{x}}_2 = 0$ which is torsion but nontrivial.

2.4.6 Linear Perturbed Dynamics

Here we will introduce the basic elements that allow us to treat sliding mode control of perturbed linear systems from an algebraic viewpoint. The basic developments and details may also be found in Fliess and Sira-Ramírez (1993b)

Definition 2.58 *A linear perturbed dynamics $\overline{\mathcal{D}}$ is a module where we distinguish a control input vector $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_m)$ and perturbation inputs $\overline{\xi} = (\overline{\xi}_1, \ldots, \overline{\xi}_m)$ such that*

$$\overline{\mathcal{D}}/[\overline{u}, \overline{\xi}] = \textit{torsion}.$$

Consider the canonical epimorphism

$$\phi : \overline{\mathcal{D}} \rightarrow \overline{\mathcal{D}}/[\overline{\xi}] = \mathcal{D}$$

Since $[\overline{u}] \cap [\overline{\xi}] = 0$, then $\phi\mid_{[\overline{u}]}$ and $\phi\mid_{[\overline{\xi}]}$ are isomorphisms, i.e.

$$[\overline{u}] \simeq [u] \;\; ; \;\; [\overline{\xi}] \simeq [\xi]$$

This means that we should not distinguish between "perturbed" and "unperturbed" versions of the control input (i.e. between $\overline{u}$ and u), nor between similar versions of the perturbation input ($\overline{\xi}$ and ξ). Since $\mathcal{D}/[u]$ is torsion, we call $\mathcal{D}$ the unperturbed linear dynamics with u being the unperturbed control.

Control and perturbation inputs are not assumed to interact, thus the condition

$$[\overline{\xi}] \cap [\overline{u}] = \{0\}$$

appears to be quite natural. It will be assumed furthermore assumed that $[\overline{u}]$ is free. This means that we are essentially considering linear systems with unrestricted control inputs. Note, however, that perturbations are not necessarily independent in the sense that they might indeed satisfy some (linear unknown) set of differential equations. For this reason we assume here that $[\overline{\xi}]$ is not necessarily free, i.e. it may be torsion. It is reasonable to assume that the unperturbed version of the system, $\mathcal{D}$ is controllable, i.e. $\mathcal{D}$ is free. Regulation of uncontrollable systems is only possible in quite limited and unrealistic cases.

2.4.7 A Module-Theoretic Characterization of Sliding Regimes

The work presented here is taken from Fliess and Sira-Ramírez (1993a), where an algebraic characterization of sliding regimes is presented in terms of module theory.

Definition 2.59 *Let $\overline{\mathcal{D}}$ be a linear perturbed dynamics, such that $\mathcal{D}$ is controllable. We define a submodule $\overline{S}$ of $\overline{\mathcal{D}}$ as a sliding submodule if the following conditions holds* *(i) The sliding module does not contain elements which are driven exclusively by the perturbations. This condition is synthesized by* $[\overline{S}] \cap [\overline{\xi}] = 0$

(ii) The canonical image S *of* $\overline{S}$ *in* $\mathcal{D} = \overline{\mathcal{D}}/[\overline{\xi}]$ *is a rank* m *free submodule, i.e. the quotient module*

$$\mathcal{D}/S \text{ is torsion.}$$

This condition means that all the control effort is spent in making the system behave as elements that are found in S.

It is convenient to assume that the unperturbed version of the system is observable; $\mathcal{D} = [u, y]$. *This guarantees that elements in the sliding module* S *may be obtained, if necessary, from asymptotic estimation procedures.*

$\mathcal{D}/S$ is the unperturbed (residual) *sliding dynamics* while $\overline{\mathcal{D}}/\overline{S}$ is the *perturbed sliding dynamics*. The canonical image of $\overline{u}$ in $\overline{\mathcal{D}}/\overline{S}$ is the *perturbed equivalent control*, denoted by $\overline{u}_{eq}$. The canonical image of u on $\mathcal{D}/S$ is addressed simply as the *equivalent control*, u_{eq}. Note that $\overline{u}_{eq}$ generally depends on the perturbation inputs $\overline{\xi}$, while u_{eq}, is perturbation independent.

Example 2.60 Consider the linear perturbed dynamics $\dot{\overline{y}} = \overline{u} + \overline{\xi}$. In this case $\overline{\mathcal{D}} = [\tilde{u}, \tilde{y}, \tilde{\xi}]/[\tilde{e}]$, with $\tilde{e} = \dot{\tilde{y}} - \tilde{u} - \tilde{\xi}$. The module $\overline{\mathcal{D}}/[\overline{u}, \overline{\xi}] = \textit{torsion}$ and $\mathcal{D}$ is rank 1, with u acting as a basis. $\mathcal{D}$ is also controllable. The condition $\dot{\overline{y}} = -\overline{y}$ may be regarded as a *desirable* asymptotically stable dynamics. Consider $\overline{S} = [\overline{s}] = [\overline{y} + \overline{u}]$. It is easy to see that $\overline{S} \subset \overline{\mathcal{D}}$ with rank $S = 1$, while $\overline{S} \cap [\overline{\xi}] = 0$. Finally, the residue $\underline{y}$ of y in $\mathcal{D}/[y + u]$ satisfies : $\dot{\underline{y}} = -\underline{y}$, which is torsion. Note that the unperturbed equivalent control satisfies $\dot{u}_{eq} + u_{eq} = 0$, while the perturbed equivalent control satisfies $\dot{\overline{u}}_{eq} + \overline{u}_{eq} = -\overline{\xi}$.

2.4.8 The Switching Strategy

Let $z = (z_1, \ldots, z_m)$ be a *basis* of S and $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_m)$ be a *basis* of $\overline{S}$. The basis z is the image of $\overline{z}$ under $\phi|_{\overline{S}}$. The input-output system relating u to z is *right* and *left invertible*, and hence *decouplable*. Therefore the multivariable case reduces to the single-input single-output case. The basis z (resp. $\overline{z}$) is unique up to a constant factor.

Example 2.61 Consider the previous example, $\dot{\overline{y}} = \overline{u} + \overline{\xi}$, with sliding module S generated by $s = u + y$. The element $z = u + y$ is a basis for S, while $\overline{z} = \overline{u} + \overline{y}$ is a basis for $\overline{S}$. The relation between z and u is trivially invertible. A switching strategy is obtained by condsidering $\dot{z} = -W\text{sign}z$, with $W > 0$ a sufficiently large constant. This choice results in the discontinuous controller, $\dot{u} + u = -W$ sign $(u+y)$. The response of the perturbed basis to the synthesized controller is governed by $\dot{\overline{z}} = \overline{\xi} - W$ sign $\overline{z}$.

2.4.9 Relations with Minimum Phase Systems and Dynamical Feedback

Definition 2.62 *Let $[u, S]$ stand for the module generated by u and S. The sliding module S is said to be minimum phase if and only if one of the following conditions are satisfied*

(i) $[u] = S$

(ii) If $[u] \not\subset S$ *then the endomorphism* τ*, defined as*
$\tau : [u, S]/S \rightarrow [u, S]/S,$ *has eigenvalues with negative real parts.*

The first condition means that the elements of the vector u can be expressed as a (decoupled) $k[\frac{d}{dt}]$-linear combination of the basis elements in S. The second condition means that some Hurwitz differential polynomial associated with u can be expressed as a decoupled $k[\frac{d}{dt}]$ linear combination of the basis elements in S.

Example 2.63 In the previous example the basis z for S was taken to be $z = u + y$ and evidently $[u] \not\subset S$, since u is not expressible as a $k[\frac{d}{dt}]$ linear combination of z. Definitely $[u] \subset [u, S] = [u, z]$ since $\dot{z} = \dot{u} + u$. The residue $\underline{u}$ of u in $[u, z]/[z]$ satisfies the linear system equation $\dot{\underline{u}} + \underline{u} = 0$ and therefore the sliding module is minimum phase.

2.4.10 Non-Minimum Phase Case

Let S be non-minimum phase. One may replace z by some other output $\sigma \in \mathcal{D}$, which is for instance a basis of $[u, z]$ and such that the transfer function relating u and σ is minimum phase.

It is easy to see, due to linearity, that the convergence of σ ensures that of z. Thus the minimum phase case is recovered. If the resulting numerator of the transfer function, relating σ and u, is not constant, then switchings will be taken by the highest order derivative of the control signal. This gives naturally the possibility of smoothed sliding mode controllers (see Sira-Ramirez (1992a, 1992c, 1993)).

2.4.11 Some Illustrations

Example 2.64 Consider the perturbed linear dynamics, $\dot{\overline{y}} = \overline{u} + \xi$, and the (desired) unperturbed *second* order dynamics given by $\ddot{\overline{y}} + 2\zeta\omega_n\dot{\overline{y}} + \omega_n^2\overline{y} = 0$. Consider the sliding module $S \subset \mathcal{D}$, generated by $z = \dot{u} + 2\zeta\omega_n u + \omega_n^2 y$. The element z is a basis for S and $\overline{z} = \dot{\overline{u}} + 2\zeta\omega_n\overline{u} + \omega_n^2\overline{y}$ is a basis for $\overline{S}$. The residue $\underline{y}$ of y in $\mathcal{D}/S$ satisfies the relation $\ddot{\underline{y}} + 2\zeta\omega_n\dot{\underline{y}} + \omega_n^2\underline{y} = 0$, which is certainly torsion and asymptotically stable to zero.

Evidently $[u] \not\subset [z]$. In order to obtain the necessary inclusion, consider the module $[u, z]$. Here one finds that the relationship between u and the basis element z for S, is given by $\dot{z} = \ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u$ Taking the quotient $[u, z]/[z]$, one is left with the torsion system $\ddot{\underline{u}} + 2\zeta\omega_n\dot{\underline{u}} + \omega_n^2\underline{u} = 0$.

The linear map associated to $\frac{\mathrm{d}}{\mathrm{dt}}$ is represented by the matrix

$$\tau = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}$$

which has eigenvalues with negative real parts. The sliding module S is therefore minimum phase.

Let W be a positive constant parameter. A dynamical sliding mode controller, which is robust with respect to ξ, is given by

$$\dddot{\overline{u}} + 2\zeta\omega_n\ddot{\overline{u}} + \omega_n^2\dot{\overline{u}} = -W\mathrm{sign}(\ddot{\overline{u}} + 2\zeta\omega_n\dot{\overline{u}} + \omega_n^2\dot{\overline{y}}).$$

Use of the proposed dynamical switching strategy on the system leads to the following regulated dynamics for $\overline{z}$,

$$\dot{\overline{z}} = \ddot{\xi} + 2\zeta\omega_n\dot{\xi} + \omega_n^2\xi - W\mathrm{sign}\ \overline{z}.$$

For sufficiently high values of the gain parameter W, the element $\overline{z}$ goes to zero in finite time, and the desired (torsion) dynamics is achieved.

Example 2.65 Consider the nonminimum phase system $\ddot{\overline{y}} + 2\zeta\omega_n\dot{\overline{y}} + \omega_n^2\overline{y} = \dot{\overline{u}} - \beta\overline{u} + \xi$, (with $\beta > 0$), and the desired dynamics $\dot{\overline{y}} + \alpha\overline{y} = 0$; $\alpha > 0$. Evidently, $z = \dot{y} + \alpha y$ is a basis for the sliding submodule S, and $z = 0$ is deemed to be desirable.

However, as before, $[u] \not\subset S$. The relationship between z and u is readily obtained as $\ddot{u} + (\alpha - \beta)\dot{u} - \alpha\beta u = \ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z$. The canonical image $\underline{u}$ of u in $[u, z]/[z]$ leads to the following unstable (torsion) dynamics $\ddot{\underline{u}} + (\alpha - \beta)\dot{\underline{u}} - \alpha\beta\underline{u} = (\frac{\mathrm{d}}{\mathrm{dt}} + \alpha)(\frac{\mathrm{d}}{\mathrm{dt}} - \beta)\underline{u} = 0$. The sliding module is therefore nonminimum phase.

Take a new basis σ of S such that $\dot{\sigma} = \beta\sigma + \alpha\dot{y} + \alpha y$. Note that $z = \dot{\sigma} - \beta\sigma$ and $\overline{z} = \dot{\overline{\sigma}} - \beta\overline{\sigma}$. One now has $\ddot{\sigma} + 2\zeta\omega_n\dot{\sigma} + \omega_n^2\sigma = \dot{u} + \alpha u$. The residue of u in $[u, \sigma]/[\sigma]$ satisfies $\dot{\underline{u}} + \alpha\underline{u} = 0$, and the sliding module is now minimum phase.

A robust dynamical sliding mode controller may now be synthesized which guarantees asymptotic convergence of $\overline{\sigma}$ to zero, and hence of $\overline{z}$ to zero. The desired unforced dynamics is, therefore, asymptotically attainable by means of dynamical sliding modes.

2.5 Conclusions

The differential algebraic approach to system dynamics provides both theoretical and practical grounds for the development of the sliding mode control of nonlinear dynamical systems. More general classes of sliding surfaces, which include inputs and possibly their time derivatives, have been shown naturally to allow for chatter-free sliding mode controllers of dynamical nature. Although equivalent smoothing effects can be similarly obtained by simply resorting to appropriate system extensions or prolongations of the input space, the theoretical simplicity and conceptual advantages stemming from the differential algebraic approach, bestow new possibilities for the broader area of discontinuous feedback control. For instance, the same smoothing effects and theoretical richness can be used for the appropriate formulation and study of many potential application areas based on pulse-width-modulated control strategies (Sira-Ramírez 1992d). The less explored pulse-frequency-modulated control techniques have also been shown to benefit from this new approach (Sira-Ramírez 1992e, Sira-Ramírez and Llanes-Santiago 1992). Possible extensions of the theory to nonlinear multivariable systems, and to infinite dimensional systems such as delay differential systems and systems described by partial differential equations, deserve attention.

Module Theory recovers and generalizes all known results of sliding mode control of linear multivariable systems. A more relaxed concept of sliding motions evolve in this context, as any desirable output dynamics is synthesizable by minimum phase sliding mode control. This statement is independent of the order of the desired dynamics. Generalizations demonstrate, for instance, that *matching conditions* are linked to particular state space realizations, but they have no further meaning from a general viewpoint. This fact has also been corroborated in recent developments in sliding observers (see Sira-Ramírez and Spurgeon (1993)). Multivariable sliding mode control problems have been shown to be always reducible to single-input single output problems in a natural manner.

Nonminimum phase problems have been shown to be handled by a suitable change of the output variable, whenever possible. The practical implications of this result seem to be multiple (see also Benvenuti et al (1992)). Extension of the results here presented to the case of time varying linear systems requires non-conmutative algebra.

An exciting area in which the algebraic approach may be used to full advantage is the area of sliding mode observers for linear systems. An interesting area rest on the extension of sliding mode theory from an algebraic viewpoint, to nonlinear multivariable sytems. The results so far seem to indicate that the class of systems to which the theory can be extended without unforseen complications is constrained to the class of *flat systems* (see Fliess et al (1991)).

References

Adkins, W. A., Weintraub, S.H. 1992, *Algebra : An approach via module theory*, Springer-Verlag, New York

Chang, L.W. 1991, A versatile sliding control with a second-order sliding condition. *Proc American Control Conference*, , Boston 54–55

Benvenuti, L., Di Benedetto, M. D., Grizzle, J. W. 1992, *Approximate output tracking for nonlinear non-minimum phase systems with applications to flight control*, Report CGR-92-20, Michigan Control Group Reports. University of Michigan, Ann Arbor, Michigan

Emelyanov, S.V. 1987, *Binary control systems*, MIR, Moscow

Emelyanov, S.V. 1990, The principle of duality, new types of feedback, variable structure and binary control, *Proc IEEE Int. Workshop on Variable Structure Systems and their Applications*, Sarajevo, 1–10

Fliess, M. 1986, A note on the invertibility of nonlinear input-output differential systems. *Systems and Control Letters* **8**, 147–151

Fliess, M. 1987, Nonlinear control theory and differential algebra: Some illustrative examples. *Proc IFAC, 10th Triennial World Congress*, Munich, 103–107

Fliess, M. 1988a, Nonlinear control theory and differential algebra, in *Modelling and adaptive control*, Byrnes, Ch. I. Kurzhanski, A., Lect. Notes in Contr. and Inform. Sci., **105**, Springer-Verlag, New York, 134–145

Fliess, M. 1988b, Généralisation non linéaire de la forme canonique de commande et linarisation par bouclage. *C.R. Acad. Sci. Paris* **I-308** , 377–379

Fliess, M. 1989a, Automatique et corps différentieles. *Forum Mathematicum* **1**, 227–238

Fliess, M. 1989b, Generalized linear systems with lumped or distributed parameters and differential vector spaces. *International Journal of Control* **49**, 1989–1999

Fliess, M. 1990a, Generalized controller canonical forms for linear and nonlinear dynamics. *IEEE Transactions on Automatic Control* **35**, 994–1001

Fliess, M. 1990b, What the Kalman state variable representaion is good for. *Proc IEEE Conference on Decision and Control,* **3**, Honolulu, 1282–1287

Fliess M. 1990c, Some basic structural properties of generalized linear systems *Systems and Control Letters* **15** 391–396.

Fliess, M. 1991, Controllability revisited, in *Mathematical System Theory : The Influence of R.E. Kalman*, ed. Antoulas, A.C., Springer-Verlag, New York, 463–474

Fliess, M., Hassler, M. 1990, Questioning the classical state-space description via circuit examples, in *Mathematical Theory of Networks and Systems*, eds. Kaashoek, M.A., Ram, A.C.M., van Schuppen, J.H., Progress in Systems and Control Theory, Birkhauser, Boston

Fliess, M., Lévine, J., Rouchon, P. 1991, A simplified approach of crane control via generalized state-space model. *Proc IEEE Conference on Decision and Control,* **1**, Brighton, England, 736–741

Fliess, M., Messager, F. 1990, Vers une stabilisation non lineaire discontinue, in *Analysis and Optimization of Systems*, eds. Bensoussan, A., Lions, J.L., Lect. Notes Contr. Inform. Sci., **144**, Springer-Verlag, New York, 778–787

Fliess, M., Messager, F. 1991, Sur la commande en régime glissant. *C. R. Acad. Sci. Paris* **I-313**, 951–956

Fliess, M., Sira-Ramírez, H. 1993a, Regimes glissants, structures variables linéaires et modules. *C.R. Acad. Sci. Paris*, submitted for publication

Fliess, M., Sira-Ramírez, H. 1993b. A Module Theoretic Approach to Sliding Mode Control in Linear Systems, *Proc IEEE Conference on Decision and Control,* , submitted for publication

Kalman, R., Falb, P., Arbib, M. 1970, *Topics in Mathematical Systems Theory,* McGraw-Hill, New York

Kolchin, E.R. 1973, *Differential algbebra and algebraic groups,* Academic Press, New York

Nijmeijer, H., Van der Schaft, A. 1990, *Nonlinear dynamical control systems,* Springer-Verlag, New York

Pommaret, J.F. 1983, *Differential galois theory,* Gordon and Breach, New York

Pommaret, J.F. 1986, Géométrie différentielle algébrique et théorie du contrôle. *C.R. Acad. Sci. Paris* **I-302**, 547–550

Sira-Ramírez, H. 1991, Dynamical feedback strategies in aerospace systems control: A differential algebraic approach. *Proc First European Control Conference,* Grenoble, 2238–2243

Sira-Ramírez, H. 1993, Dynamical variable structure control strategies in asymptotic output tracking problems. *IEEE Transactions on Automatic Control* , to appear

Sira-Ramírez, H. 1992a, Asymptotic output stabilization for nonlinear systems via dynamical variable structure control. *Dynamics and Control* **2**, 45–58

Sira-Ramírez, H. 1992b, The differential algebraic approach in nonlinear dynamical feedback controlled landing maneuvers. *IEEE Transactions on Automatic Control* **AC-37**, 1173–1180

Sira-Ramírez, H. 1992c, Dynamical sliding mode control strategies in the regulation of nonlinear chemical processes. *International Journal of Control* **56**, 1–21

Sira-Ramírez, H. 1992d, Dynamical pulse width modulation control of nonlinear systems. *Systems and Control Letters* **18**, 223–231.

Sira-Ramírez, H. 1992e, Dynamical discontinuous feedback control in nonlinear systems. *Proc IFAC Nonlinear Control Systems Conference,* Burdeaux, 471–476

Sira-Ramírez, H. 1993, A Differential Algebraic Approach to Sliding Mode Control of Nonlinear Systems. *International Journal of Control* **57**, 1039–1061

Sira-Ramírez, H., Ahmad, S., Zribi, M. 1992, Dynamical feedback control of robotic manipulators with joint flexibility. *IEEE Transactions on Systems Man and Cybernetics* **22**, 736–747

Sira-Ramírez, H., Lischinsky-Arenas, P. 1991, The differential algebraic approach in nonlinear dynamical compensator design for dc-to-dc power converters. *International Journal of Control* **54**, 111–134

Sira-Ramírez, H., Llanes-Santiago, O. 1992, An extended system approach to dynamical pulse-frequency-modulation control of nonlinear systems. *Proc IEEE Conference on Decision and Control,* **1**, Tucson, 2376–2380

Sira-Ramírez, H., Spurgeon, S.K. 1993, On the robust design of sliding observers for linear systems, submitted for publication