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8. Robust Observer-Controller Design for Linear Systems

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8.1 Introduction

Sliding mode observation and control schemes for both linear and nonlinear systems have been of considerable interest in recent times. Discontinuous nonlinear control and observation schemes, based on sliding modes, exhibit fundamental robustness and insensitivity properties of great practical value (see Utkin (1992), and also Canudas de Wit and Slotine (1991)). A fundamental limitation found in the sliding mode control of linear perturbed systems and in sliding mode feedforward regulation of observers for linear perturbed systems, is the necessity to satisfy some structural conditions of the "matching" type. These conditions have been recognized in the work of Utkin (1992), Walcott and Żak (1988) and Dorling and Zinober (1983). Such structural constraints on the system and the observer have also been linked to *strictly positive real* conditions in Walcott and Żak (1988) and in the work of Watanabe et al (1992). More recently a complete Lyapunov stability approach for the design of sliding observers, where the above-mentioned limitations are also apparent, was presented by Edwards and Spurgeon (1993).

Here a different approach to the problem of output feedback control for any controllable and observable, perturbed linear system is taken. For the sake of simplicity, single-input single-output perturbed plants are considered, but the results can be easily generalized to multivarable linear systems.

Using a *Matched Generalized Observer Canonical Form* (MGOCF), similar to those developed by Fliess (1990a), it is found that for the sliding mode state observation problem in observable systems, the structural conditions of the matching type are largely irrelevant. This statement is justified by the fact that a perturbation input "rechannelling" procedure *always* allows one to obtain a matched realization for the given system. Such rechannelling is never carried out in practice and its only purpose is to obtain a reasonable estimate (bound) of the influence of the perturbation inputs on the state equations of the proposed canonical form. It is shown that the chosen matched output reconstruction error feedforward map, which is a design quantity, uniquely determines the stability features of the reduced order sliding state estimation error dynamics. The state vector of the proposed realization is, hence, robustly asymptotically estimated, independently of whether or not the matching conditions are satisfied by the original system.

The sliding mode output regulation problem for controllable and observable minimum phase systems is then addressed, using a combination of a sliding mode observer and a sliding mode controller. For this, a suitable modification of the MGOCF is proposed. The resulting matched canonical form turns out quite surprisingly to be in a traditional Kalman state space representation form. The obtained Matched Output Regulator Canonical Form (MORCF) is constructed in such a way that it is always matched with respect to the "re-channelled" perturbation inputs. The output signal of the system, expressed now in canonical form, is shown to be controlled by a suitable dynamical "pre-compensator" input, which is physically realizable. For the class of systems treated, the combined state estimation and control problem (i.e. output regulation problem) is therefore always robustly solvable by means of a sliding mode scheme, independently of any matching conditions.

In Sect. 8.2 the role of the matching conditions in sliding mode controller, sliding mode observer and sliding mode output regulation designs, is examined from a classical state space representation viewpoint. This section addresses the rather restrictive nature of the structural conditions that guarantee the robust reconstruction and robust regulation of the system state vector components. In essence, these conditions imply that the feedforward output error injection map of the observer must be in the range space of the perturbation input distribution map of the system. For guaranteeing robustness in a sliding mode control problem, the matching conditions demand that the perturbation input channel map must be in the range space of the control input channel map. For the observer design in particular, these matching conditions imply that the freedom in choosing the stability features of the reduced order ideal sliding reconstruction error dynamics, is severely curtailed and the structure of the system must, by itself, guarantee asymptotic stability of the reduced order observation error dynamics. If the matching condition is not satisfied, then the observation error is dependent upon the external perturbations, and accurate state reconstruction is not feasible.

In Sect. 8.3 the MGOCF, based on the input-output description of the given system, is proposed and it is shown that the matching conditions can always be satisfied while placing no restrictions on the stabilizability of the feedforward regulated error dynamics. This result constitutes the "dual", in a certain sense, to that recently published by Fliess and Messager (1991), involving sliding mode controllers for linear time-invariant controllable systems. Sect. 8.4 presents the MORCF for minimum phase controllable and observable systems. The proposed canonical form is shown to be suitable for the simultaneous design of a robust sliding mode observer/sliding mode controller scheme, independently of any matching conditions. A tutorial design example which considers the design of a sliding mode controller for a power converter demonstrates the theoretical results of this chapter in Sect. 8.5. In Sect. 8.6 conclusions are drawn and further research is suggested.

8.2 Matching Conditions in Sliding Mode State Reconstruction and Control of Linear Systems

Here the classical approaches to sliding mode controller and observer design using the traditional Kalman state variable representation of linear time-invariant systems are presented. Within this constrained formulation, robust observation and control schemes are feasible if, and only if, certain structural conditions are satisfied. The structural conditions for the sliding mode controller design restrict the system's input disturbance distribution map to the range of the control input distribution map. Similar conditions for the sliding mode observer design demand that the observer's feedforward output error injection map be in the range of the system's input disturbance distribution map.

Consider a controllable and observable n-dimensional linear system of the form

$$\begin{aligned} \dot{x} &= Ax + bu + \gamma\xi \\ y &= cx \end{aligned} \tag{8.1}$$

where u and ξ are, respectively, the scalar control input signal and the (bounded) scalar external perturbation input signal. The output y is also assumed to be a scalar quantity. All matrices have the appropriate dimensions. The column vector γ is referred to as the *perturbation input distribution map*, while b is called the *control input distribution map*. The system (8.1) is assumed to be *relative degree* one, i.e. the scalar product $cb \neq 0$. It is assumed, without loss of generality, that $cb > 0$. Furthermore, it is assumed that the underlying input-output system is *minimum phase*.

8.2.1 Matching Conditions in Sliding Mode Controller Design

Suppose it is desired by means of state feedback to zero the output y of the given system. It is well known that if the system (8.1) is unperturbed (i.e. $\xi = 0$), then a variable structure feedback control law of the form

$$u = -\frac{1}{cb}(cAx + K \text{ sign } y) \tag{8.2}$$

where $K > 0$ is a constant design gain, accomplishes the desired control objective in finite time. The output signal y satisfies then the following dynamics

$$\dot{y} = -K \text{ sign } y \tag{8.3}$$

It can be shown under rather mild assumptions that the regulated output variable y of the perturbed system (8.1) still converges to zero in finite time, when the controller (8.2) is used. Indeed the resulting controlled behaviour of the output signal when the controller (8.2) is used in the system (8.1) is given by

$$\dot{y} = c\gamma\xi - K \text{ sign } y \tag{8.4}$$

Let the absolute value of the perturbation input ξ be bounded by a constant $M > 0$. Then, for $K > M|c\gamma|$, the feedback control policy (8.2) is seen to create in finite time a ***sliding regime*** on the hyperplane represented by $y = 0$, irrespective of the particular values adopted by ξ.

The ***ideal sliding dynamics*** satsified by the controlled state vector x are obtained from the following invariance conditions (Utkin 1992)

$$y = 0 \ , \qquad \dot{y} = 0 \tag{8.5}$$

These conditions imply the existence of a "virtual" perturbation-dependent value of the regulating input u, known as the ***equivalent control***, and denoted by u_{eq} (see Utkin (1992)), which replaces the discontinuous feedback control action on the sliding hyperplane $y = 0$ and helps in describing, in an average sense, the dynamical behaviour of the constrained system. From (8.1) and $\dot{y} = 0$ in (8.5) one obtains

$$u_{eq} = -\frac{cAx}{cb} - \frac{c\gamma}{cb}\xi \tag{8.6}$$

Substituting (8.6) into (8.1) yields

$$\dot{x} = (I - \frac{bc}{cb})Ax + (I - \frac{bc}{cb})\gamma\xi \tag{8.7}$$

which represents a ***redundant*** dynamics taking place on any of the linear varieties $y =$ constant. In particular, when the initial conditions are such that $y = cx = 0$, then (8.7) in combination with $y = 0$ is called the ***reduced order ideal sliding dynamics***.

Note that the matrix $P = [I - (bc)/(cb)]$ is a ***projection operator*** along the range space of b onto the null space of c (El-Ghezawi et al 1983), i.e.

$$Pb = 0 \ , \qquad Px = x \qquad \forall\, x \quad \text{s.t.} \quad cx = 0$$

Thus, in general, the reduced order ideal sliding dynamics will be dependent upon the perturbation signal ξ. However, under structural constraints on the distribution maps b and γ, known as the ***matching conditions***, it is possible to obtain a reduced order ideal sliding dynamics (8.7) which is free of the influence of the perturbation signal ξ. One may establish that the ideal sliding dynamics (8.7) are independent of ξ if, and only if,

$$\gamma = \rho b \tag{8.8}$$

for some constant scalar ρ. In other words, the ideal sliding dynamics are independent of ξ if, and only if, the range spaces of the maps γ and b coincide. The proof is as follows. If the matrix feeding the perturbations ξ into the (average) sliding dynamics equation (8.7) is identically zero, then no perturbations are present in the average system behaviour. This would require the following identity to hold

$$(I - \frac{bc}{cb})\gamma = 0 \tag{8.9}$$

which simply means that γ may be expressed as $\gamma = \rho b$ where $\rho = (c\gamma)/(cb)$. On the other hand if γ is a column vector of the form $\gamma = \rho b$, then

$$(I - \frac{bc}{cb})\gamma = (b - \frac{bc}{cb}b)\rho = (b - b)\rho = 0$$

If the matching condition (8.8) is satisfied, the ideal sliding dynamics is specified by the following constrained dynamics

$$\begin{aligned} \dot{x} &= (I - \frac{bc}{cb})Ax \\ y &= cx = 0 \end{aligned} \tag{8.10}$$

The robust sliding mode controller design problem, for systems satisfying the matching condition (8.8), consists of specifying an output vector c (i.e. a ***sliding surface*** $y = cx = 0$) and a discontinuous state feedback control policy u of the form (8.2), such that the reduced order ideal sliding dynamics (8.10) is guaranteed to exhibit asymptotically stable behaviour to zero. As may easily be seen, such a stability property is a ***structural property*** associated with the particular form of the maps A, c and γ. It can be shown that the asymptotic stability of (8.10) can be guaranteed if a ***strictly positive real condition***, associated with the constrained system, is satisfied (see Utkin (1992)).

8.2.2 Matching Conditions in Sliding Mode Observer Design

An asymptotic observer for the system (8.1), including an external feedforward compensation signal v, may be proposed as follows

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + bu + h(y - \hat{y}) + \lambda v \\ \hat{y} &= c\hat{x} \end{aligned} \tag{8.11}$$

The vector h is called the ***vector of observer gains*** and the column vector λ is the ***feedforward injection map***.

The ***state reconstruction error***, defined as $e = x - \hat{x}$, obeys the following dynamical behaviour, from (8.1) and (8.11)

$$\begin{aligned} \dot{e} &= (A - hc)e + \gamma\xi - \lambda v \\ e_y &= ce \end{aligned} \tag{8.12}$$

The signal $e_y = y - \hat{y}$ is called the ***output reconstruction error***.

Because of the observability assumption on the system (8.1), there always exists a vector of observer gains h which assigns any arbitrarily prespecified set of n eigenvalues (with complex conjugate pairs) to the matrix $(A - hc)$.

The robust sliding mode observer design problem consists of specifying a vector of observer gains h, a feedforward injection map λ and a discontinuous feedforward injection policy v, based solely on output reconstruction error measurements e_y, such that the reconstruction error dynamics (8.12) is guaranteed to exhibit asymptotically stable behaviour to zero, in spite of all possible bounded values of the external perturbation input signal ξ.

Consider the time derivative of the output reconstruction error signal

$$\begin{aligned} \dot{e}_y &= c(A - hc)e + c\gamma\xi - c\lambda v \\ &= cAe - che_y + c\gamma\xi - c\lambda v \end{aligned} \tag{8.13}$$

We assume, without loss of generality, that the quantity $c\lambda$ is nonzero and positive (i.e. $c\lambda > 0$). As before, let the absolute value of the the perturbation input ξ be bounded by a constant $M > 0$. Also let W be a sufficiently large positive scalar constant. Then, a discontinuous feedforward input v of the form

$$v = W \text{sign } e_y \tag{8.14}$$

is seen to create a *sliding regime* on a bounded region of the reconstruction error space. Such a region would necessarily be contained in the hyperplane $e_y = 0$.

As may be easily verified, from (8.13) and (8.14), in the region characterized by $e_y = 0$ and $|cAe| + |c\gamma\xi| \leq Wc\lambda$, the above choice of the feedforward signal v results in the *sliding condition* $e_y\dot{e}_y < 0$ (see Utkin (1992)) being satisfied. Using the known bound M on the signal ξ, such a region can be expressed as

$$|cAe| \leq Wc\lambda - M|c\gamma|$$

Thus, the discontinuous feedforward policy (8.14) drives the output observation error e_y to zero in finite time, irrespective of both the initial conditions of e and the values of the perturbation input ξ, provided $c\lambda W > |c\gamma|M$.

The ***ideal reduced order sliding behaviour*** of the state reconstruction error signal e is obtained from the following version of the ***invariance conditions***

$$e_y = 0 \ , \qquad \dot{e}_y = 0 \tag{8.15}$$

The conditions (8.15) imply a "virtual" perturbation-dependent value of the output error feedforward injection signal v, which constitutes the ***equivalent feedforward*** signal, denoted by v_{eq}. This "virtual" feedforward signal is useful in describing the average behaviour of the error system (8.12) when regulated by the feedforward signal v. Using (8.13) and (8.15) one readily obtains

$$v_{eq} = \frac{cAe}{c\lambda} + \frac{c\gamma}{c\lambda}\xi \tag{8.16}$$

Substitution of the equivalent feedforward signal expression (8.16) in the state observation error equation (8.12), leads to the following (redundant) ***ideal sliding error dynamics***, taking place on a bounded region of $e_y = 0$

$$\dot{e} = (I - \frac{\lambda c}{c\lambda})Ae + (I - \frac{\lambda c}{c\lambda})\gamma\xi \tag{8.17}$$

Note that the matrix $S = [I - (\lambda c)/(c\lambda)]$ is a *projection operator* along the range space of λ onto the null space of c, i.e.

$$S\lambda = 0 \ , \qquad Sx = x \qquad \forall x \qquad \text{s.t. } cx = 0$$

The reduced order ideal sliding error dynamics will, in general, be dependent upon the perturbation signal ξ. However, under a structural constraint on the distributions maps γ and λ, known as the *matching condition*, it is possible to obtain an ideal sliding error dynamics (8.17) which is free of the influence of the perturbation signal ξ. One may establish that the ideal sliding error dynamics (8.17) is independent of ξ if, and only if,

$$\gamma = \mu\lambda \tag{8.18}$$

for some constant scalar μ. In other words, the sliding error dynamics is independent of ξ if, and only if, the range spaces of the maps γ and λ coincide. The proof of this result is similar to the one carried out for the sliding mode controller case in Sect. 8.2.2 and is omitted.

If the matching condition (8.18) is satisfied, then the reconstruction error dynamics is specified by the following constrained dynamics

$$\begin{aligned} \dot{e} &= (I - \frac{\gamma c}{c\gamma})Ae \\ e_y &= ce = 0 \end{aligned} \tag{8.19}$$

The resulting reduced order unforced error dynamics obtained from (8.19), must be asymptotically stable. As can be seen, such a stability property is a structural property linked to the particular form of the maps A, c and γ. It can be shown that the asymptotic stability of (8.19) can be guaranteed if a ***strictly positive real condition***, associated with the constrained system, is satisfied (see also Walcott and Żak (1988)).

8.2.3 The Matching Conditions for Robust Output Regulation

If the state variables x of the system are not available for measurement, then the variable structure feedback control law (8.2) must be modified to include the dynamical observer states, instead of those of the given system. The *estimated* variable structure feedback control law is now

$$\hat{u} = -\frac{1}{cb}(cA\hat{x} + K \text{ sign } y) \tag{8.20}$$

The regulated state variables x now obey the following variable structure controlled dynamics

$$\begin{aligned} \dot{x} &= Ax - \frac{b}{cb}(cA\hat{x} + K \text{ sign } y) \\ &= (I - \frac{bc}{cb})Ax + \frac{b}{cb}cAe - \frac{bK}{cb} \text{ sign } y \end{aligned} \tag{8.21}$$

where e is the state reconstruction error dynamics.

The output signal evolution is therefore governed by the dynamical system

$$\dot{y} = cAe - K \text{ sign } y \tag{8.22}$$

Since the observation error e is guaranteed to converge asymptotically to zero, the output signal y is clearly seen to converge to zero in finite time, provided a sufficiently large value of K is chosen.

It is clear that the ideal sliding dynamics simultaneously taking place on $y = 0$ and $e_y = 0$, will be independent of the perturbation input ξ if, and only if, the matching conditions (8.8) and (8.18) are satisfied, i.e. if the maps γ and λ are both in the range space of the control input distribution channel map b.

8.3 A Generalized Matched Observer Canonical Form for State Estimation in Perturbed Linear Systems

Suppose a linear system of the form (8.1) is given such that the matching condition discussed in Sect. 8.2.3 does not yield an asymptotically stable reduced observation error system (8.19). By resorting to an input-output description of the perturbed system, one can find a canonical state space realization, in generalized state coordinates, which *always* satisfies the matching condition of the form (8.18) while producing a prespecified asymptotically stable constrained error dynamics. The state of the matched canonical realization can therefore always be estimated robustly.

By means of straightforward state vector elimination, the input-output representation of the linear time-invariant perturbed system (8.1) is assumed to be in the form

$$\begin{aligned} y^{(n)} + k_n y^{(n-1)} + \cdots + k_2\dot{y} + k_1 y &= \beta_0 u + \beta_1 \dot{u} + \cdots + \beta_{n-1}u^{(n-1)} \\ &\quad + \gamma_0\xi + \gamma_1\dot{\xi} + \cdots + \gamma_q\xi^{(q)} \end{aligned} \tag{8.23}$$

where ξ represents the bounded external perturbation signal and the integer q satisfies, without loss of generality, $q \leq n-1$.

The *Generalized Matched Observer Canonical Form* (GMOCF) of the above system is given by the following generalized state representation model (see Fliess (1990a) for a similar canonical form)

$$\begin{aligned} \dot{\chi}_1 &= -k_1\chi_n + \beta_0 u + \beta_1\dot{u} + \cdots + \beta_{n-1}u^{(n-1)} + \lambda_1\eta \\ \dot{\chi}_2 &= \chi_1 - k_2\chi_n + \lambda_2\eta \end{aligned}$$

$$
\begin{aligned}
&\vdots \\
\dot{\chi}_{n-1} &= \chi_{n-2} - k_{n-1}\chi_n + \lambda_{n-1}\eta \\
\dot{\chi}_n &= \chi_{n-1} - k_n\chi_n + \eta \\
y &= \chi_n
\end{aligned}
\tag{8.24}
$$

where η is an "auxiliary" perturbation signal, modelling the influence of the external signal ξ on every equation of the proposed system realization.

The relation existing between the signal η and its generating signal ξ, is obtained by computing the input-output description of system (8.24) in terms of the perturbation input η. The input-output description of the hypothesized model (8.24) is then compared with that of the original system (8.23). This procedure results in a scalar linear time-invariant differential equation for η which accepts the signal ξ as an input.

The models presented below constitute realizations of such an input-output description, according to the order q of the differential polynomial for ξ in (8.23).

For $q < n-1$, the perturbation input η is obtained as the output of the following dynamical system

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_{n-1} &= -\lambda_1 z_1 - \lambda_2 z_2 - \cdots - \lambda_{n-1} z_{n-1} + \xi \\
\eta &= \gamma_0 z_1 + \gamma_1 z_2 + \cdots + \gamma_{q-1} z_q
\end{aligned}
\tag{8.25}
$$

For $q = n-1$ the state space realization corresponding to (8.25) is simply

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_{n-1} &= -\lambda_1 z_1 - \lambda_2 z_2 - \cdots - \lambda_{n-1} z_{n-1} + \xi \\
\eta &= (\gamma_0 - \gamma_{n-1}\lambda_1) z_1 + (\gamma_1 - \gamma_{n-1}\lambda_2) z_2 + \cdots \\
&\quad + (\gamma_{n-2} - \gamma_{n-1}\lambda_{n-1}) z_{n-1} + \gamma_{n-1}\xi
\end{aligned}
\tag{8.26}
$$

Assumption 8.1 *Suppose the components of the auxiliary perturbation distribution channel map $\lambda_1, \ldots, \lambda_{n-1}$ in (8.24) are such that the following polynomial, in the complex variable s, is Hurwitz*

$$
p_r(s) = s^n + \lambda_{n-1} s^{n-2} + \cdots + \lambda_2 s + \lambda_1 \tag{8.27}
$$

Equivalently, Assumption 8.1 implies that the output η of the system (8.25) (or that of system (8.26)), generating the auxiliary perturbation η, is a *bounded* *signal* for every bounded external perturbation signal ξ. If, for instance, ξ satisfies $|\xi| < N$, then, given N, the signal η satisfies $|\eta| \leq M$ for some positive constant M. An easily computable, although conservative, estimate for M is given by $M = \sup_{\omega \in [0,\infty)} | N\,G(j\omega)|$ where $G(s)$ is the Laplace transfer function relating η to ξ in the complex frequency domain.

Remark. It should be stressed that the purpose of having a state space model for the auxiliary perturbation signal η, accepting as a forcing input the signal ξ, is to be able to estimate a bound for the influence of ξ on the proposed state realization (8.24) of the original system (8.1).

An observer for the system realization (8.24) is proposed as follows

$$\begin{aligned}
\dot{\hat{\chi}}_1 &= -k_1\hat{\chi}_n + \beta_0 u + \beta_1 \dot{u} + \cdots + \beta_{n-1}u^{(n-1)} \\
&\qquad\qquad + h_1(y-\hat{y}) + \lambda_1 v \\
\dot{\hat{\chi}}_2 &= -k_2\hat{\chi}_n + \hat{\chi}_1 + h_2(y-\hat{y}) + \lambda_2 v \\
&\ \vdots \\
\dot{\hat{\chi}}_{n-1} &= -k_{n-1}\hat{\chi}_n + \hat{\chi}_{n-2} + h_{n-1}(y-\hat{y}) + \lambda_{n-1} v \\
\dot{\hat{\chi}}_n &= -k_n\hat{\chi}_n + \hat{\chi}_{n-1} + h_n(y-\hat{y}) + v \\
\hat{y} &= \hat{\chi}_n
\end{aligned} \tag{8.28}$$

Note that exactly the same output error feedforward distribution map for the signal v has been chosen as the one corresponding to the auxiliary perturbation input signal η in (8.24). Consequently, the proposed canonical form (8.24) for the system always satisfies the matching condition (8.8). The crucial point is that the matched error feedforward distribution map can always be conveniently chosen to guarantee asymptotic stability of the ideal sliding error dynamics.

Use of (8.28) results in the following feedforward regulated reconstruction error dynamics

$$\begin{aligned}
\dot{\epsilon}_1 &= -(k_1+h_1)\epsilon_n + \lambda_1(\eta - v) \\
\dot{\epsilon}_2 &= \epsilon_1 - (k_2+h_2)\epsilon_n + \lambda_2(\eta - v) \\
&\ \vdots \\
\dot{\epsilon}_{n-1} &= \epsilon_{n-2} - (k_{n-1}+h_{n-1})\epsilon_n + \lambda_{n-1}(\eta - v) \\
\dot{\epsilon}_n &= \epsilon_{n-1} - (k_n+h_n)\epsilon_n + (\eta - v) \\
\epsilon_y &= \epsilon_n
\end{aligned} \tag{8.29}$$

where ϵ_i represents the state estimation error components $\chi_i - \hat{\chi}_i$, for $i = 1, \ldots, n$.

In order to have a reconstruction error transient response associated with a preselected nth order characteristic polynomial, such as

$$p(s) = s^n + \alpha_n s^{n-1} + \cdots + \alpha_2 s + \alpha_1 \ , \tag{8.30}$$

the gains h_i $(i = 1, \ldots, n)$ should be appropriately chosen as $h_i = \alpha_i - k_i$ $(i = 1, \ldots, n)$.

The feedforward output error injection signal v is chosen to be the discontinuous regulation policy

$$v = W\text{sign } \epsilon_y = W\text{sign } \epsilon_n \tag{8.31}$$

where W is a positive constant. From the final equation in (8.29) it is seen that, for a sufficiently large gain W, the proposed choice of the feedforward signal v results in a sliding regime on a region properly contained in the set expressed by

$$\epsilon_n = 0 \ , \qquad |\epsilon_{n-1}| \leq W - M \tag{8.32}$$

The equivalent feedforward signal, v_{eq}, is obtained from the ***invariance conditions*** (see also Canudas de Wit and Slotine (1991))

$$\epsilon_n = 0 \ , \qquad \dot{\epsilon}_n = 0 \tag{8.33}$$

One obtains from (8.33) and the last of (8.29)

$$v_{eq} = \eta + \epsilon_{n-1} \tag{8.34}$$

The equivalent feedforward signal is, generally speaking, dependent upon the perturbation signal η. It should be remembered that the equivalent feedforward signal v_{eq} is a *virtual* feedforward action that needs not be synthesized in practice, but one which helps to establish the salient features of the *average* behaviour of the sliding mode regulated observer. The resulting dynamics governing the evolution of the error system in the sliding region are then ideally described by

$$\begin{aligned}
\dot{\epsilon}_1 &= -\lambda_1 \epsilon_{n-1} \\
\dot{\epsilon}_2 &= \epsilon_1 - \lambda_2 \epsilon_{n-1} \\
&\vdots \\
\dot{\epsilon}_{n-1} &= \epsilon_{n-2} - \lambda_{n-1} \epsilon_{n-1} \\
\epsilon_y &= \epsilon_n = 0
\end{aligned} \tag{8.35}$$

and exhibits, in a natural manner, a feedforward error injection structure of the "auxiliary output error" signal ϵ_{n-1}, through the design gains $\lambda_1, \ldots, \lambda_{n-1}$. As a result, the roots of the characteristic polynomial in (8.27) determining the behaviour of the homogeneous reduced order system (8.35), are completely determined by a suitable choice of the components of the feedforward vector, $\lambda_1, \ldots, \lambda_{n-1}$.

An asymptotically stable behaviour to zero of the estimation error components $\epsilon_1, \ldots, \epsilon_{n-1}$ is therefore achievable since the output observation error ϵ_n undergoes a sliding regime on the relevant portion of the "sliding surface" $\epsilon_n = 0$. The states of the estimator (8.28) are then seen to converge asymptotically towards the corresponding components of the state vector of the system realization (8.24).

The characteristic polynomial (8.27) of the reduced order observation error dynamics (8.35) coincides entirely with that of the transfer function relating the auxiliary perturbation model signal η to the actual perturbation input ξ. Hence, appropriate choice of the design parameters $\lambda_1, \ldots, \lambda_{n-1}$ not only guarantees asymptotic stability of the sliding error dynamics, but also ensures boundedness of the auxiliary perturbation input signal η, for any given bounded external perturbation ξ.

Remark. In general, the observed states of the matched generalized state space realization are different from the states of the particular realization (8.1). The state χ in (8.24) may even be devoid of any physical meaning. A linear relationship can always be established between the originally given state vector x of system (8.1) and the state χ, reconstructed from the canonical form (8.24). However, generally speaking, such a relationship allows a *perturbation dependent* state coordinate transformation and cannot be used in practice. Nevertheless, it will be shown that a suitable modification of the proposed matched canonical form is effective in implementing a combined observer-controller output feedback sliding mode regulator.

8.4 A Matched Canonical Realization for Sliding Mode Output Feedback Regulation of Perturbed Linear Systems

Consider a linear system of the form (8.1). It will be shown that by resorting to an input-output description of the perturbed system, one can find a canonical state space realization which *always* satisfies the matching conditions of the form (8.8) and (8.18), while producing a prespecified asymptotically stable reduced order state and observation error sliding dynamics. The state of the matched canonical realization can therefore always be robustly estimated and controlled.

By means of straightforward state vector elimination, the input-output representation of the linear time-invariant perturbed system (8.1) is assumed to be of the form given by (8.23). The *Matched Output Regulator Canonical Form* (MORCF) of the above system is given by the following state representation model

$$\begin{aligned}
\dot{\chi}_1 &= -k_1\chi_n + \lambda_1(\eta + \vartheta) \\
\dot{\chi}_2 &= \chi_1 - k_2\chi_n + \lambda_2(\eta + \vartheta) \\
&\vdots \\
\dot{\chi}_{n-1} &= \chi_{n-2} - k_{n-1}\chi_n + \lambda_{n-1}(\eta + \vartheta) \\
\dot{\chi}_n &= \chi_{n-1} - k_n\chi_n + (\eta + \vartheta) \\
y &= \chi_n
\end{aligned} \tag{8.36}$$

where ϑ is an "auxiliary" input interpreted as a *precompensator* input. Note that the auxiliary input distribution map of the proposed canonical form is chosen to match precisely that of the auxiliary (rechannelled) perturbation input η. This guarantees that the realization is matched and that the sliding mode controller will be robust with respect to such perturbations. It is easy to see by computing the input-output representation of the matched realization (8.36), that the auxiliary input ϑ is related to the original control input u by means of the following proper transfer function

$$\frac{\hat{u}(s)}{\hat{\vartheta}(s)} = \frac{s^{n-1} + \lambda_{n-1}s^{n-2} + \cdots + \lambda_1}{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} \tag{8.37}$$

We refer to (8.37) as the *precompensator* transfer function. Alternatively, a state space realization of the dynamical precompensator is given by

$$\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= \zeta_3 \\
&\vdots \\
\dot{\zeta}_{n-2} &= \zeta_{n-1} \\
\dot{\zeta}_{n-1} &= -\frac{b_0}{b_{n-1}}\zeta_1 - \frac{b_1}{b_{n-1}}\zeta_2 - \cdots - \frac{b_{n-3}}{b_{n-1}}\zeta_{n-2} - \frac{b_{n-2}}{b_{n-1}}\zeta_{n-1} + \frac{1}{b_{n-1}}\vartheta \\
u &= (\lambda_1 - \frac{b_0}{b_{n-1}})\zeta_1 + (\lambda_2 - \frac{b_1}{b_{n-1}})\zeta_2 + \cdots \\
&\quad + (\lambda_{n-1} - \frac{b_{n-2}}{b_{n-1}})\zeta_{n-1} + \frac{1}{b_{n-1}}\vartheta
\end{aligned} \tag{8.38}$$

The perturbation input η in (8.36) is, as before, an "auxiliary" perturbation signal, modelling the influence of the external signal ξ on every equation of the proposed system realization. It is straightforward to verify that the signal η in (8.36) is obtained from the signal ξ in the same manner as it was obtained in (8.25) (or (8.26)).

The components of the auxiliary perturbation distribution channel map $\lambda_1, ..., \lambda_{n-1}$ in (8.36), are such that the characteristic polynomial in the complex variable s is Hurwitz. This, in turn, guarantees a truly minimum phase dynamical precompensator (8.37) (or (8.38)). The minimum phase condition on the zeroes of the precompensator transfer function also guarantees simultaneously that the the output η of system (8.25) or (8.26), generating the auxiliary perturbation η, is a ***bounded signal*** for every bounded external perturbation signal ξ.

8.4.1 Observer Design

An observer for the system realization (8.36) is proposed as follows

$$
\begin{aligned}
\dot{\hat{\chi}}_1 &= -k_1\hat{\chi}_n + h_1(y-\hat{y}) + \lambda_1(v+\vartheta) \\
\dot{\hat{\chi}}_2 &= -k_2\hat{\chi}_n + \hat{\chi}_1 + h_2(y-\hat{y}) + \lambda_2(v+\vartheta) \\
&\vdots \\
\dot{\hat{\chi}}_{n-1} &= -k_{n-1}\hat{\chi}_n + \hat{\chi}_{n-2} + h_{n-1}(y-\hat{y}) + \lambda_{n-1}(v+\vartheta) \\
\dot{\hat{\chi}}_n &= -k_n\hat{\chi}_n + \hat{\chi}_{n-1} + h_n(y-\hat{y}) + (v+\vartheta) \\
\hat{y} &= \hat{\chi}_n
\end{aligned} \tag{8.39}
$$

Note that exactly the same output error feedforward distribution map λ for the signal v has been chosen as that corresponding to the auxiliary perturbation input signal η and to the control input distribution map in (8.36). As a consequence, the matching conditions (8.8) and (8.18) are satisfied by the proposed matched canonical realization (8.36). Use of the observer (8.39) results exactly in the same sliding mode feedforward regulated reconstruction error dynamics already given in (8.29).

A reconstruction error transient response may be chosen which is associated with a preselected n th order characteristic polynomial, such as (8.30), by means of the appropriate choice of the observer gains h_i, $i = 1, \ldots, n$.

The feedforward output error injection signal v is chosen, as before, as a discontinuous regulation policy of the variable structure type

$$v = W \text{sign } \epsilon_y = W \text{sign } \epsilon_n \tag{8.40}$$

with W being a positive constant. For a sufficiently large gain W, the proposed choice of the feedforward signal v results in a sliding regime on a region properly contained in the set

$$\epsilon_n = 0 \ , \qquad |\epsilon_{n-1}| \leq W - M \tag{8.41}$$

The resulting reduced order dynamics governing the evolution of the sliding mode regulated error system in the computed sliding region of the error space, is then ideally described by the same asymptotically stable unforced differential equation as in (8.35).

8.4.2 Sliding Mode Controller Design

We first show that the proposed matched canonical form (8.24) also facilitates the design of a sliding mode controller when all states of the realization are directly measurable. Once the sliding mode controller based on full state feedback information has been obtained, a similar sliding mode controller in which all the required state variables are derived from the observer, will be developed.

8.4.2.1 Sliding Mode Controller Based on Full State Information A sliding mode controller may be obtained by considering the unperturbed version of the final equation in the canonical form (8.36), (i.e. from the differential equation governing the behaviour of the output $y = \chi_n$ with $\eta = 0$), and the discontinuous regulated policy proposed in (8.3). Such a sliding mode control policy is given by

$$\vartheta = k_n \chi_n - \chi_{n-1} - W \operatorname{sign} \chi_n \tag{8.42}$$

Using the above controller in the perturbed output equation, results in the following controlled output dynamics

$$\dot{\chi}_n = \eta - W \operatorname{sign} y \tag{8.43}$$

Therefore a sliding mode controller gain W, which is assumed to satisfy $W > M$, guarantees the convergence of y to zero in finite time, irrespectively of the bounded values of the computed perturbation effect η.

The invariance conditions $\chi_n = 0$, $\dot{\chi}_n = 0$ result in the following perturbation dependent equivalent auxiliary control input

$$\vartheta_{eq} = -\chi_{n-1} - \eta \tag{8.44}$$

The ideal sliding dynamics, obtained from substitution of (8.44) in the canonical realization (8.36), is

$$\begin{aligned} \dot{\chi}_1 &= -\lambda_1 \chi_{n-1} \\ \dot{\chi}_2 &= \chi_1 - \lambda_2 \chi_{n-1} \\ &\vdots \\ \dot{\chi}_{n-1} &= \chi_{n-2} - \lambda_{n-1} \chi_{n-1} \\ y &= \chi_n = 0 \end{aligned} \tag{8.45}$$

The characteristic polynomial of the constrained dynamics is given again by the Hurwitz polynomial (8.27), and the ideal sliding dynamics (8.45) is asymptotically stable to zero.

8.4.2.2 Sliding Mode Controller Based on Observer State Information If the state χ_{n-1} is not directly available for measurement, the feedback control (8.42) should be modified to employ the estimated state obtained from the sliding observer (8.39) as

$$\hat{\vartheta} = k_n y - \hat{\chi}_{n-1} - W \operatorname{sign} y \tag{8.46}$$

where the fact that the output y is clearly available for measurement, has been used. This control policy still results in finite time convergence of y to zero as can be seen from the closed-loop output dynamical equation

$$\begin{aligned} \dot{y} &= (\chi_{n-1} - \hat{\chi}_{n-1}) + \eta - W \operatorname{sign} y \\ &= \epsilon_{n-1} + \eta - W \operatorname{sign} y \end{aligned} \tag{8.47}$$

Since ϵ_{n-1} is decreasing asymptotically to zero, the output y is seen to go to zero in finite time for sufficiently large values of $W > M$.

The output observation error signal e_y, and the output signal y itself, are seen to converge to zero in finite time. The combined reduced order ideal sliding/ideal observer dynamics is obtained from the same invariance conditions $\chi_n = 0$, $\dot{\chi}_n = 0$ as before. This results in precisely the same equivalent control input and the same equivalent feedforward signals. The resulting reduced order ideal sliding/ideal observation error dynamics is still given by (8.35) and (8.45). The overall scheme is therefore asymptotically stable.

8.5 Design Example: The Boost Converter

Consider the average Boost converter model derived by Sira-Ramírez and Lischinsky-Arenas (1991)

$$\begin{aligned} \dot{z}_1 &= -\omega_0 z_2 + \mu\omega_0 z_2 + b \\ \dot{z}_2 &= \omega_0 z_1 - \omega_1 z_2 - \mu\omega_0 z_1 \end{aligned} \tag{8.48}$$

where $z_i, i = 1, 2$ denote the corresponding "averaged components" of the state vector x where $x_1 = I\sqrt{L}$, $x_2 = V\sqrt{C}$ represent the normalized input current and output voltage variables respectively. The quantity $b = E/\sqrt{L}$ is the normalised external input voltage. The LC (input) circuit natural oscillating frequency and the RC output circuit time constant are denoted by $\omega_0 = 1/\sqrt{LC}$ and $\omega_1 = 1/(RC)$ respectively. The variable μ is the control input. The equilibrium points of the average model (8.48) are obtained as

$$\mu = U \;\; ; \quad Z_1(U) = \frac{b\omega_1}{\omega_0{}^2(1-U)^2} \;\; ; \quad Z_2(U) = \frac{b}{\omega_0(1-U)} \tag{8.49}$$

where U denotes a particular constant value for the duty ratio function. The linearisation of the average PWM model (8.48) about the constant operating points (8.49) is given by

$$\begin{aligned} \dot{z}_{1\delta} &= -(1-U)\,\omega_0 z_{2\delta} + \frac{b}{1-U}\mu_\delta \\ \dot{z}_{2\delta} &= (1-U)\,\omega_0 z_{1\delta} - \omega_1 z_{2\delta} - \frac{b\omega_1}{(1-U)^2\omega_0}\mu_\delta \end{aligned} \tag{8.50}$$

with

$$\mu_\delta(t) = \mu(t) - U \;\; ; \quad z_{i\delta}(t) = z_i(t) - Z_i(U) \;\; , \quad i = 1, 2 \tag{8.51}$$

Taking the averaged normalised input inductor current z_1 as the system output in order to meet the relative degree 1 and minimum phase assumptions, the following input/output relationship is obtained

$$\frac{z_{1\delta}(s)}{\mu_\delta(s)} = \omega_0 Z_2(U)\frac{s + 2\omega_0}{s^2 + \omega_1 s + (1-U)^2\omega_0{}^2} \tag{8.52}$$

The controller/observer pair (8.46), (8.39) is now implemented on the average boost converter model. For simulation purposes nominal parameter values of $R = 30\Omega$, $C = 20\mu$F, $L = 20$mH and $E = 15$V are assumed. The desirable set point for the average normalized input inductor current is $z_1 = 0.4419$ which corresponds to a constant value $U = 0.6$. In order to demonstrate the robustness of the approach, the effects of noise on both the input current and output voltage dynamics will be considered. The system representation then becomes, from (8.52),

$$\begin{aligned} \dot{z}_{1\delta} &= -632.46 z_{2\delta} + 265.17\mu_\delta + \alpha\xi \\ \dot{z}_{2\delta} &= 632.46 z_{1\delta} - 1666.67 z_{2\delta} - 698.77\mu_\delta + \beta\xi \end{aligned} \tag{8.53}$$

Here α and β define the noise distribution channel which is not necessarily matched. The polynomial (8.27) which defines the auxiliary perturbation distribution map is chosen to be

$$p_r(s) = s + 3000 \tag{8.54}$$

The rate of decay of the reconstruction error dynamics (8.30) is determined by the roots of the following characteristic polynomial

$$p(s) = s^2 + 8500s + 18000000 \tag{8.55}$$

Using (8.54) and (8.55) an observer (8.39) for the system is given by

$$\begin{aligned} \dot{\hat{\chi}}_1 &= 400000\hat{\chi}_2 + 17600000(y - \hat{y}) + 3000(v + \vartheta) && (8.56) \\ \dot{\hat{\chi}}_2 &= -1666.67\hat{\chi}_2 + \hat{\chi}_1 + 6833.33(y - \hat{y}) + (v + \vartheta) \\ \hat{y} &= \hat{\chi}_2 \\ v &= W_{obs}\,\mathrm{sign}\,(y - \hat{y}) && (8.57) \end{aligned}$$

The following state-space realisation may be used to determine the plant input μ_δ

$$\begin{aligned} \dot{w} &= -3333.33w + 0.0038\vartheta && (8.58) \\ \mu_\delta &= -333.33z + 0.0038\vartheta \\ \vartheta &= -W_{con}\,\mathrm{sign}\,y - \hat{\chi}_1 + 1666.67y \end{aligned}$$

The magnitude of the discontinuous gain elements W_{con} and W_{obs} were chosen to be 120 and 220 respectively. These were tailored to provide the required speeds of response as well as appropriate disturbance rejection capabilities. Using a disturbance distribution map defined by $\alpha = 0.01$ and $\beta = -0.02$, which is clearly unmatched with respect to the input and output distributions of the system realisation (8.53), and a high frequency cosine representing the system noise, the following simulation results were obtained. Fig. 8.1 shows the convergence of the estimated inductor current to the actual inductor current. A sliding mode is reached whereby $z_1(t) - Z_i(t) = 0$. The required set point is thus attained and maintained despite the disturbance which is acting upon the system. Fig. 8.2 shows the control effort μ. The discontinuous nature of this signal supports the assertion that a sliding mode has been attained.

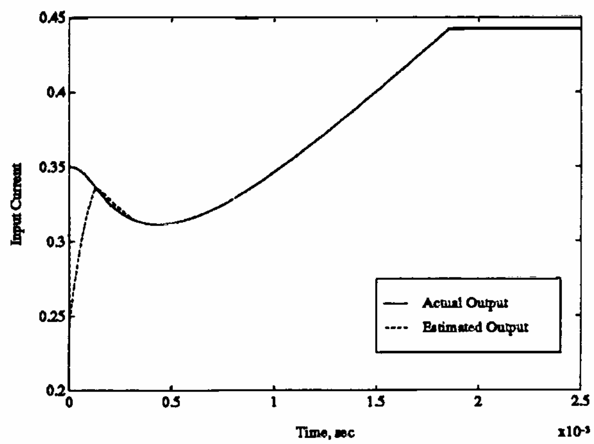

Fig. 8.1. Response of the actual and estimated average normalized inductor current

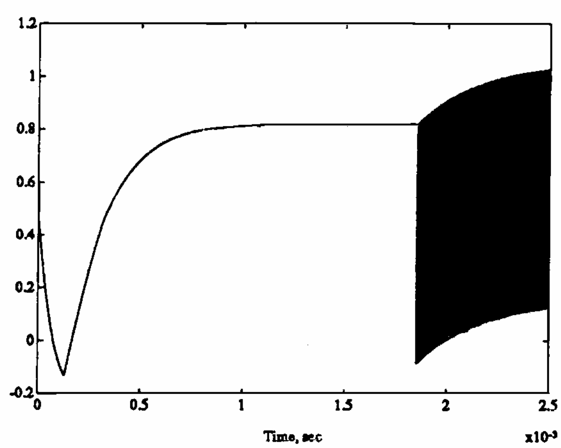

Fig. 8.2. Response of the control effort μ

8.6 Conclusions

It has been shown that, when using a sliding mode approach, structural conditions of the *matching type*, are largely irrelevant for robust state reconstruction and regulation of linear perturbed systems. The class of linear systems for which robust sliding mode output feedback regulation can be obtained, independently of any *matching conditions*, comprises the entire class of controllable (stabilizable) and observable (reconstructible) linear systems with the appropriate relative degree and minimum phase condition.

This result, first postulated by Sira-Ramírez and Spurgeon (1993b), is of particular practical interest when the designer has the freedom to propose a convenient state space representation for a given unmatched system. This is in total accord with the corresponding results found in Fliess and Messager (1991), and in Sira-Ramírez and Spurgeon (1993b) regarding, respectively, the robustness of the sliding mode control of perturbed controllable linear systems, expressed in the *Generalized Observability Canonical Form*, and the dual result for the sliding mode observation schemes based on the *Generalized Observer Canonical Form*.

Sliding mode output regulator theory (i.e. addressing an observer-controller combination) for linear systems may also be examined from an algebraic viewpoint using *Module Theory* (see Fliess (1990b)). The conceptual advantages of using a module theoretic approach to sliding mode control were recently addressed by Fliess and Sira-Ramírez (1993) and Sira-Ramírez in Chapter 2. The module theoretic approach can also provide further generalizations and insights related to the results presented.

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