VARIABLE STRUCTURE CONTROL FOR ROBOTICS AND AEROSPACE APPLICATIONS

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Editor

STUDIES IN AUTOMATION AND CONTROL

10

ELSEVIER

Variable Structure Control for Robotics
and Aerospace Applications / K.K.D. Young (Editor)

Chapter 3

NONLINEAR PULSE WIDTH MODULATION CONTROLLER DESIGN

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3.1 INTRODUCTION

In this chapter a design method, based on an infinite frequency average model, is proposed for the specification of stabilizing Pulse-Width-Modulation (PWM) feedback controllers for Nonlinear Dynamical Systems. It is first shown that the infinite frequency average model of the PWM controlled system coincides with Filippov's geometrical average dynamics of the original discontinuous PWM system. The average model captures the essential qualitative stability properties of the feedback controlled system and thus considerably simplifies the design task. Some satellite control examples are provided.

Three classes of nonlinear controlled systems stand out as genuine representatives of discontinuous feedback control strategies. These are: Variable Structure Systems (VSS) undergoing sliding regimes, plants regulated by Pulse Width Modulation (PWM) techniques and, finally, systems regulated by Pulse Frequency Modulation (PFM) controllers. For a complete account on the basic results and main applications of VSS in Sliding Mode, the reader is referred to the work of Utkin (refs. 1-3), PFM Controlled Systems will not be treated in this chapter.

PWM controlled systems constitute a class of nonlinear periodically sampled-data control systems. The sampled output error being the difference between the desired and the actual plant output signals, is translated into a pulse control signal whose pulse width is proportional to the error signal. PWM controlled systems, as VSS in sliding mode, are typically robust with respect to plant parameter variations and external perturbation signals.

Early contributions to the study of PWM controllers are those of Nelson (ref. 4), Kadota and Bourne (ref. 5), Jury and Nishimura (ref. 6), Tsypkin (ref. 7), Further developments were contributed, later on, by Skoog (ref. 8), Skoog and Blankenship (ref. 9), Friedland (ref. 10), Min *et al* (ref. 11), Marino *et al* (ref. 12), and La Cava *et al.* (ref. 13). In all these works, emphasis was primarily placed on the discrete-time aspects of such controllers. (See also Czaki (ref. 14) pp. 591). The proposed discrete-time analysis technique fitted the problem quite naturally due to the inherent sampling process associated with every PWM control scheme. The method, besides unnecessarily limiting the considerations to the case of linear plants, also, generally, led to quite tortuous calculations which usually resorted to a not always easily justifiable approximation scheme.

In recent papers, however, Sira-Ramirez (ref. 15-17) has explored a different design approach by using the **geometric** properties of **average** PWM controlled responses (obtained by an infinite **duty cycle** frequency assumption). The results, aside from allowing a simpler analysis of nonlinear PWM controlled systems through their average responses, also found that actual PWM controlled responses exhibit sliding mode trajectories about integral manifolds of the average PWM controlled system model.

Roughly speaking, two classes of PWM controlled systems can be studied; those corresponding to an ON-OFF controlled switch (the control variable can be made to take values in the discrete set {0,1}) and those including a control variable taking values in the discrete set {-1,0,+1} (herein addressed as an ON-OFF-ON controlled switch. See, for instance, (ref. 9) and (ref. 13)). Typically, DC to DC Power Converters, switch controlled networks, such as Switched Capacitor Circuits, spacecraft vehicles equipped with a main thruster controlling a smooth vertical descend on a planet, and local quantizers of the type used in Delta Modulation Circuits for analog signal encoding, correspond to the ON-OFF class of switch-controlled systems. On the other hand, symmetric gas reaction jets controlling reorientation or detumbling maneuvers in satellites, some torque actuators, used for control of joint positions in robotic manipulators, and input-relief arrangements of fluid control valves, are expressible as switched controlled systems of the ON-OFF-ON class.

In this chapter, it will be first shown, in full generality, that in a VSS, undergoing structural changes according to an ON-OFF PWM scheme an **infinite sampling frequency** assumption reduces the feedback system model precisely to a **Filippov's Geometric Average model** of the discontinuous PWM system. Filippov's scalar function defining the average vector field, as a convex combination of the intervening structures, is shown to be coincident with the prescribed **duty ratio** function of the original PWM scheme. It immediately follows that a corresponding **sliding regime** is exhibited by the actual PWM controlled system trajectories about integral manifolds of the Infinite Duty Cycle Frequency Average PWM model (hereinafter referred to as the **Average PWM model**, or **Filippov's average model**). The **equivalent control** (Utkin, (ref. 18)) associated with the corresponding **Ideal Sliding Mode** is none other than the duty ratio itself. Hence, the Average PWM model has the primordial characteristic of entirely capturing all the relevant qualitative features of the actual discontinuous PWM controlled system. This fact being much in accordance with the manner in which the Ideal Sliding Dynamics captures the essential qualitative features of the actual (chattering) sliding motions about a switching surface.

The above results are extended to a well known class of output error feedback ON-OFF-ON PWM switched-controlled plants (ref. 9) and a design method is proposed for controlled systems of this nature. The specification of the PWM controller is made on the basis of the **Average PWM model**. The average model is obtained by formally replacing the discontinuous PWM regulator by a nonlinear, memoryless, piecewise smooth controller of the saturation type. It is claimed that the stabilizing design tasks for the actual PWM controller are much easier to handle on the basis of such average model.

Section 3.2 contains a general theory of VSS of the ON-OFF PWM type and its connections with Filippov's geometrical averaging technique. It also considers an extension of the obtained results to a general class of ON-OFF-ON PWM systems. The results are then specialized to a rather typical representative of output error ON-OFF-ON PWM controlled systems (ref. 9). A high-gain design approach, entirely based on the average model, is proposed for that particular class of PWM systems. Examples related to satellite control are presented in Section 3.3. Section 3.4 contains the conclusions of the chapter. while general results, relevant to VSS in sliding mode, are collected in the Appendix.

3.2 DEFINITIONS AND BASIC RESULTS

3.2.1 Generalities about Nonlinear ON-OFF PWM Controlled Systems

Consider the nonlinear discontinuously controlled system, described by:

$$\frac{dx}{dt} = f(x) = \begin{cases} f_1(x) & \text{for} \quad t_k < t \le t_k + \tau[x(t_k)]\,T \\ f_2(x) & \text{for} \quad t_k < t \le t_k + \tau[x(t_k)]\,T \end{cases} \tag{3.1}$$

where $f_1(x)$ and $f_2(x)$ are smooth vector fields defined on R^n. The t_k's represent regularly spaced instants of time where an ideal sampling process takes place and by means of which the value of the **duty ratio function**, $\tau(x)$, is determined in correspondence with the value of the sampled state vector, $x(t_k)$. This duty ratio function is assumed to take values in the bounded interval $[0,1]$ of the real line. The regions where $\tau(x)$ is fixed at either 0 or 1, constitute the **saturation regions** of the PWM controller. The sampling period T, also known as the **duty cycle** , is assumed to be constant, and sufficiently small as compared with the time constants associated with the dynamics of the controlled system. Unless otherwise stated, it will be assumed that our considerations are restricted to a region of the state space where the duty ratio function is not saturated i.e., $\tau(x)$ takes values in the open interval (0,1).

In terms of an ideal switching function u, taking values in the discrete set {0,1}, the above system can be equivalently represented as :

$$\frac{dx}{dt} = u\, f_1(x) + (1-u) f_2(x) \tag{3.2}$$

with a switching control policy of the form:

$$u = \begin{cases} 1 & \text{for} \quad t_k < t \le t_k + \tau[x(t_k)]\, T \\ 0 & \text{for} \quad t_k < t \le t_k + \tau[x(t_k)]\, T \end{cases} \tag{3.3}$$

The following lemma is a straightforward consequence of the Fundamental Theorem of Calculus.

Lemma 1 Let f be a smooth vector field and let $I_f(t) := \int_0^t f(x(s))ds$. Then for any smooth, strictly positive, function $\mu(x)$:

$$\lim_{T\to 0,\, t_k \to t} \frac{\{I_f\,(t_k + \tau[x(t_k)]\,T - I_f\,(t_k)\}}{T} = \tau[x(t)]\, f(x(t)) \tag{3.4}$$

The next theorem determines the smooth character, and the nature, of the infinte-frequency average dynamics of (3.2),(3.3) under nonsaturating conditions.

Theorem 1 In the regions where the PWM controller is not saturated, as the sampling frequency 1/T tends to infinity in system (3.2),(3.3), the discontinuous system is substituted by **Filippov's average model** (See the Appendix):

$$dx/dt = \tau(x)\, f_1(x) + [1-\tau(x)] f_2(x) = f_{av}(x) \tag{3.5}$$

with a corresponding convex combination function $\mu(x)$ represented by the duty ratio function $\tau(x)$. Moreover, in such a region, a sliding regime is exhibited by the actual PWM controlled system (3.2),(3.3) about an integral manifold $S := \{ x \in R^n : s(x) = 0 \}$ of (3.5).

Proof From (3.2), (3.3), the state x at time $t_k + T$ is exactly computed as:

$$x(t_k+T) = x(t_k) + \int_{t_k}^{t_k+\tau[x(t_k)]\,T} f_1(x(\sigma))d\sigma + \int_{t_k+\tau[x(t_k)]}^{t_k+T} f_2(x(\sigma))d\sigma$$

$$= x(t_k) + \int_{t_k}^{t_k+\tau[x(t_k)]\,T} f_1(x(\sigma))d\sigma + \int_{t_k}^{t_k+T} f_2(x(\sigma))d\sigma \;-\int_{t_k+\tau[x(t_k)]}^{t_k+T} f_2(x(\sigma))d\sigma$$

assuming that $\tau(x)$ is neither 0 or 1 in the region of interest, and using the result of lemma 1, one has :

$$\lim_{T\to 0,\, t_k\to t} \frac{[x(t_k+T) - x(t_k)]}{T} =$$

$$= \lim_{T\to 0,\, t_k\to t} \frac{\left[\int_{t_k}^{t_k+\tau[x(t_k)]\,T} f_1(x(\sigma))d\sigma + \int_{t_k}^{t_k+T} f_2(x(\sigma))d\sigma \;-\int_{t_k+\tau[x(t_k)]}^{t_k+T} f_2(x(\sigma))d\sigma\right]}{T}$$

$$= \tau(x(t))f_1(x(t)) + [1-\tau(x(t)]f_2(x(t))$$

or:

$$\frac{dx}{dt} = \tau(x)\, f_1(x) + [1-\tau(x)]f_2(x) =: f_{av}(x) \qquad (3.5)$$

i.e., the infinite frequency model of (3.2)-(3.3) coincides with Filippov's Average model (See Appendix) in which the convex combination function $\mu(x)$, defining the average vector field $f_{av}(x)$, is precisely taken as the duty ratio function $\tau(x)$.

From the results of theorem A.2, and the assumption that the duty ratio function is locally bounded in the open interval (0,1), it follows that a sliding regime exists locally on the manifold S for the VSS (3.2),(3.3). The equivalent control $u^{EQ}(x)$, associated with such a sliding regime, is simply obtained from the invariance conditions (3.A.6) of the ideal sliding mode taking place on S :

$$ds/dt = \langle ds,\, u^{EQ}(x)\, f_1(x) + [1-u^{EQ}(x)\,]\, f_2(x) \rangle$$
$$= u^{EQ}(x) \langle ds,\, f_1(x) \rangle + [1-u^{EQ}(x)\,] \;\langle ds,\, f_2(x) \rangle = 0$$

The corresponding equivalent control is then obtained as :

$$u^{EQ}(x) = - \langle ds,\, f_2(x) \rangle / [\,\langle ds,\, f_1(x)-f_2(x) \rangle\,]$$

It follows, from the uniqueness of the equivalent control and (3.A.7), that :

$$u^{EQ}(x) = \tau(x) \tag{3.6}$$

i.e., the **equivalent control** of the sliding motion associated with (3.2),(3.3) is then, precisely, constituted by the **duty ratio** associated to the PWM control scheme. The corresponding ideal sliding dynamics is then represented by :

$$dx/dt = u^{EQ}(x)\, f_1(x) + [1-u^{EQ}(x)\,]\, f_2(x)$$

$$= \tau(x)\, f_1(x) + [1-\tau(x)]\, f_2(x)$$

which is just the Average PWM model (3.5).

The region of existence of a sliding motion is determined by the region on S where conditions (3.A.3) are satisfied. From the results of Theorem A.2 the portion of S on which $\tau(x)$ satisfies :

$$0 < \tau(x) = u^{EQ}(x) < 1$$

determines such an existence region. The duty ratio evidently satisfies the above condition, along the integral manifold S, in all regions of the state space where the PWM controller is not saturated. æ

Corollary 1 Provided $0 < \tau(x) < 1$, Filippov's average model corresponding to a PWM controlled system of the form:

$$dx/dt = f(x) + g(x)\, u, \tag{3.7}$$

with u given as in (3.3), is obtained by formally substituting the discontinuous control variable u by the duty ratio function $\tau(x)$. i.e.,

$$dx/dt = f(x) + g(x)\, \tau(x) = f_{av}(x) \tag{3.8}$$

Moreover, in such a non-saturation region, the actual controlled system (3.7),(3.3) exhibits a sliding motion about an integral manifold $S = \{ x : s(x) = 0 \}$ satisfying the condition $\langle ds, f_{av}(x) \rangle = 0$.

Proof Immediate upon letting $f_1(x) = f(x) + g(x)$ and $f_2(x) = f(x)$ and using the results of Theorem 1.

<u>3.2.2 Generalities about ON-OFF-ON PWM Controlled Systems.</u>

Consider the nonlinear PWM controlled system:

$$\frac{dx}{dt} = f(x) = \begin{cases} f_1(x) + f_2(x)\mathrm{sign}[e(t_k)] & \text{for} \quad t_k < t \le t_k + \tau[x(t_k)]\,T \\ f_2(x) & \text{for} \quad t_k < t \le t_k + \tau[x(t_k)]\,T \end{cases} \tag{3.9}$$

where $f_1(x)$ and $f_2(x)$ are smooth vector fields and $e(x)$ is a known smooth scalar function of x.

The above system can be expressed, in terms of a switch position function u taking values in the discrete set { -1,0, +1 } , as :

$$dx/dt = f_1(x) + u\, f_2(x) \tag{3.10}$$

with :

$$u = \mathrm{PWM}_\tau\, [e(t_k)] = \begin{cases} \mathrm{sign}[e(t_k)] & \text{for} \quad t_k < t \le t_k + \tau[x(t_k)]\,T \\ 0 & \text{for} \quad t_k < t \le t_k + \tau[x(t_k)]\,T \end{cases} \tag{3.11}$$

Notice that for $e(x) > 0$ and $e(x) < 0$ the PWM controlled system is, in each case, an ON-OFF PWM controlled system of the form (3.7). It is easy to see, from the results of Corollary 1, that the Average PWM model of (3.10) (3.11), in the region where $0 < \tau(x) < 1$, is simply described by :

$$dx/dt = f_1(x) + f_2(x)\, \tau(x)\, \mathrm{sign}[e(x)] \tag{3.12}$$

<u>Corollary 2</u> In those regions of the state space where $0 < \tau(x) < 1$, the state trajectories of the ON-OFF-ON PWM controlled system (3.10)-(3.11) exhibit a sliding mode behavior about integral manifolds of the Average PWM model (3.12).

3.2.3 An output error feedback PWM Controlled System.

Consider a nonlinear PWM feedback controlled system defined in R^n, and described by (figure 3.1):

$$dx/dt = f(x) + g(x)u$$
$$y = h(x)$$
$$e = y_d(t) - y$$
$$u = M\ PWM_\tau\ [e(t_k)] \qquad (3.13)$$

with f and g being smooth vector fields, while h is a smooth scalar output function. The control input u is a discontinuous scalar control function obtained as the output of a Pulse-Width-Modulator excited by the output error e. The error signal e is obtained, at each instant, as the difference between the desired output value $y_d(t)$ and the actual output value y(t). The sampling process associated to the PWM process is assumed to take place at regularly spaced time intervals of fixed duration T, i.e., $t_{k+1} = t_k + T$. M is a positive constant gain representing the maximum allowable input magnitude. The cases in which M is a static nonlinear operator, representing an amplitude modulation block, or a linear dynamic "shaping filter", or a compensating network, can be treated with little additional effort by the techniques used in this chapter.

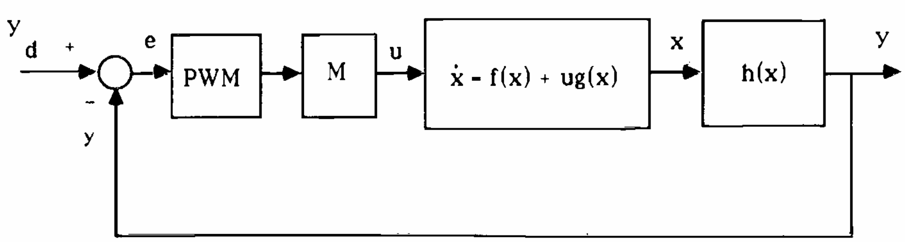

Figure 3.1. Nonlinear PWM controlled system

The PWM control operator, $PWM_\tau [e]$, characterizing an ON-OFF-ON switch is defined as in (3.11) (See refs. (9,12) and (ref. 13)), with $\tau[e(t_k)]$ as the error dependent **duty ratio** function defined by :

$$\tau|e(t_k)| = \begin{cases} \beta|e(t_k)| & \text{for } |e(t_k)| \le 1/\beta \\ 1 & \text{for } |e(t_k)| > 1/\beta \end{cases} \tag{3.14}$$

with β being a positive constant. Notice that :

$$\tau|e(t_k)|\text{sign}|e(t_k)| = \text{sat}|e(t_k),\beta| := \begin{cases} \beta e(t_k) & \text{for } |e(t_k)| \le 1/\beta \\ \text{sign}|e(t_k)| & \text{for } |e(t_k)| > 1/\beta \end{cases} \tag{3.15}$$

A variety of other interesting PWM models can be found in (refs. 12-13)

The basis of a design technique for the above class of PWM controlled systems, is given by the corollary below which immediately follows from the results of Section 3.2.2.

Corollary 3 As the sampling frequency 1/T tends to infinity, the description of the nonlinear controlled system (3.13) coincides with :

$$\begin{aligned} dx/dt &= f(x) + g(x)v \\ y &= h(x) \\ e &= y_d(t) - y \\ v &= M\,\text{sat}(e,\beta) \end{aligned} \tag{3.16}$$

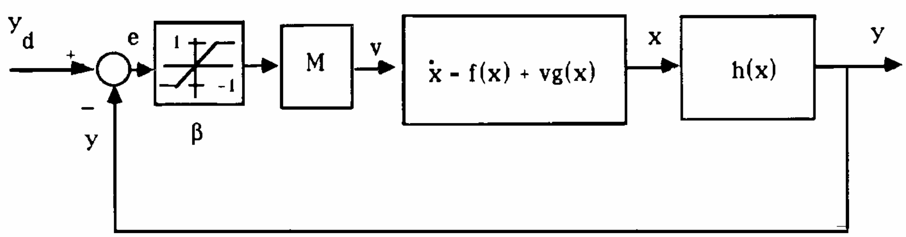

Figure 3.2. Average Model of ON-OFF-ON PWM Controlled System

Remark The behavior of the infinite frequency sampled system is described by a nonlinear system which includes a **continuous piece-wise smooth** control v, generated as the output of a memoryless nonlinear function of the saturation type.

The saturation function, in turn, is excited by the output error signal e. In other words, to evaluate the **average behavior** of the PWM controlled system, the PWM controller is simply substituted by a nonlinear memoryless saturating controller.

The fundamental qualitative stability characteristics of the actual nonlinear PWM controlled system (3.13) are entirely captured by the average model (3.16). This result follows immediately from the sliding mode considerations related to the actual PWM controlled system. In particular, a necessary condition for asymptotic stability of the actual PWM system (3.13), toward the state space manifold represented by $e = 0$, is given by the asymptotic stability of the average model towards the same manifold. The actual PWM controlled response can be made to follow, arbitrarily closely, in a sliding mode fashion (See (ref.17)), the response of the average model. This is accomplished by suitably increasing the sampling frequency of the PWM controller. It then follows that if the response of the average model (3.16) is asymptotically stable to the zero error manifold, then the actual PWM controlled system trajectories slide around the average trajectories toward such a manifold. The amplitude of the chattering motion decreases as the sampling frequency increases. Thus, modulo sufficiently large but finite sampling frequency, the following theorem holds true :

Theorem 2 The closed loop PWM controlled system (3.13) is asymptotically stable toward the manifold $e = 0$, if and only if the average PWM system (3.16) is asymptotically stable toward such a manifold.

Proof For simplicity we assume that in (3.13) and (3.16), $y_d(t) = 0$ for all t. The average model of the PWM controller establishes three disctintive regions in the state space of the system. These are : **the saturation regions** $S_{+1} = \{ x : e = -h(x) > 1/\beta \}$ and $S_{-1} = \{ x : e = - h(x) < -1/\beta \}$ and the **linear, or boundary layer region** $S_\beta = \{x : -1/\beta \leq e = -h(x) \leq 1/\beta \}$.

On the saturation regions the actual PWM controlled system (3.13) coincides with the average model (3.16). Hence, under the same initial conditions, the actual PWM system trajectories and those of the average PWM system coincide. By virtue of the continuity of solutions in the initial conditions, it follows that for arbitrarily small perturbations in the initial states, the state trajectories of the actual and the average PWM systems remain arbitrarily close to each other locally in such regions.

Within the boundary layer region the actual PWM system is described as if being governed by a two position switch (i.e., one of the ON-OFF type). Depending on the sign of the error signal, the PWM controlled system is described as follows:

For $e > 0$:

$$dx/dt = f(x) + u\,g(x) \quad ; \quad u = \begin{cases} +1 & \text{for } t_k < t \leq t_k + \tau(e)T \\ 0 & \text{for } t_k\,\tau(e)T < t \leq t_k + T \end{cases} \tag{3.17}$$

for $e < 0$:

$$dx/dt = f(x) + u\,g(x) \quad ; \quad u = \begin{cases} -1 & \text{for } t_k < t \leq t_k + \tau(e)T \\ 0 & \text{for } t_k\,\tau(e)T < t \leq t_k + T \end{cases} \tag{3.18}$$

Model (3.18) can be described in the same form as (3.17), simply by letting $-g_1(x) = g(x)$ and substituting $-u$ by u_1. The above ON-OFF systems were shown to exhibit sliding motions about integral manifolds of the average PWM controlled system (which in fact corresponds to the ideal sliding dynamics of the equivalent sliding motion) in the non-saturation regions of the PWM controller. It follows that, within the boundary layer region, the trajectories of the average PWM controlled system are followed arbitrarily close by the actual PWM controlled responses, in the same manner that a sliding motion follows, on a given switching surface in the state space, the trajectories of the corresponding ideal sliding dynamics model.

It will be now shown, under mild assumptions, that the fundamental qualitative stability characteristics of an actual nonlinear PWM controlled system (3.13) are entirely captured by the average model (3.16). For this, the nature of the error between the open loop state trajectory responses of both systems will be analyzed when starting from arbitrarily close initial states.

Let $e^*(t_k) = x(t_k) - z(t_k)$ be the difference, at the sampling instant t_k, between the state x of the PWM controlled system (3.13) and the state z of the average model (3.16). It will be assumed that the vector field $f(x)$ is globally Lipschitz and that the vector field $g(x)$ is globally bounded on R^n, i.e., there exist constants L_1 and M such that $\|f(x)-f(z)\| \leq L_1 \|x-z\|$ and $\|g(x)\| \leq G$ for all x and z in R^n.

Theorem 3 Under the above assumptions on the vector fields $f(x)$ and $g(x)$, given a small positive constant ε , there exists, for any arbitrary finite time interval, $[0,NT]$, a sampling frequency $F_0 = 1/T_0$ such that, if the initial difference $e^*(t_0) = x(t_0) - z(t_0)$, of the initial states of systems (3.13) and (3.16) is norm bounded by a small positive quantity δ, then $e^*(t_0+NT)$ is norm bounded by $(1+\varepsilon)\,\delta$, for any sampling frequency $F > F_0$

Proof According to (3.13) and (3.16) one has :

$$x(t_k+T) = x(t_k) + \int_{t_k}^{t_k+T} f(x(\sigma))d\sigma + M\int_{t_k}^{t_k+|v(t_k)|\,T} g(x(\sigma))\ \text{sign}\ e_1(t_k)\ d\sigma \tag{3.19}$$

$$z(t_k+T) = z(t_k) + \int_{t_k}^{t_k+T} f(z(\sigma))d\sigma + M\int_{t_k}^{t_k+T} g(z(\sigma))\ \text{sat}[\ e_2(\sigma,\beta)]\ d\sigma \tag{3.20}$$

with $e_1(t_k) = y_d - h(x(t_k))$ and $e_2(t_k) = y_d - h(z(t_k))$, subtracting (3.20) from (3.19), one obtains :

$$e^*(t_k+T) = e^*(t_k) + \int_{t_k}^{t_k+T}\left[f(x(\sigma)) - f(z(\sigma))\right]d\sigma\ +$$
$$M\left[\int_{t_k}^{t_k+|v(t_k)|\,T} g(x(\sigma))\ \text{sign}\ e_1(t_k)\ d\sigma - \int_{t_k}^{t_k+T} g(z(\sigma))\ \text{sat}[\ e_2(\sigma,\beta)]\ d\sigma\right]$$

Hence:

$$\|\ e^*(t_k+T)\ \| \ \le \|\ e^*(t_k)\| + L1\int_{t_k}^{t_k+T}\|\ e^*(\sigma)\ \|\ d\sigma\ +$$
$$M\ \|\ \int_{t_k}^{t_k+\tau|v(t_k)|\,T} g(x(\sigma))\ \text{sign}\ e_1(t_k)\ d\sigma - \int_{t_k}^{t_k+T} g(z(\sigma))\text{sat}[e_2(\sigma),\beta]d\sigma\ \|$$
$$\le \|\ e^*(t_k)\| + L1\int_{t_k}^{t_k+T}\|\ e^*(\sigma)\ \|\ d\sigma\ + M\ (\int_{t_k}^{t_k+T}\|\ g(x(\sigma))\ \|d\sigma + \int_{t_k}^{t_k+T}\|\ g(z(\sigma))\ \|\ d\sigma\ )$$
$$\le\ \|\ e^*(t_k)\| + L1\ \int_{t_k}^{t_k+T}\|\ e^*(\sigma)\ \|\ d\sigma\ +2\ M\ T\ G$$

By the Gronwall-Bellman lemma ((ref. 19), pp. 134, see also (ref. 18) pp. 47) one has :

$$\|e^*(t_k+T)\| \le [\ \|\ e^*(t_k)\ \| + 2\ M\ GT\ ]\ \exp\ (\ L_1 T\ ) \tag{3.21}$$

Using iteratively (3.21) for $k=0,1,\ldots,N-1$, the following loose estimate for the norm of the discrepancy $e^*(t_0+NT)$ is easily obtained,

$$\|\ e^*(t_0+NT)\ \| \le [\ 2MNGT + \|\ e^*(t_0)\ \|\ ]\ \exp\ (\ L_1 NT\ ) \tag{3.22}$$

i.e., at an arbitrary finite time $t_0 + NT$, the error between the actual open loop PWM state response and that of the average system remains bounded by an amount determined by the initial states discrepancy, the system constants L_1 and G and the sampling interval T. The following transcendental equation, which is directly obtained from (3.22)

$$[\,2MNG\,T + \delta] = (1+\varepsilon)\,\delta\,\exp\{-L_1 N\,T\} \tag{3.23}$$

has a unique solution for some $T = T_0 > 0$. This is due to the fact that the left hand side term monotonically increases with T from the value δ at $T = 0$, while the right hand side term monotonically decreases with T from the value $(1+\varepsilon)\delta > \delta$ at $T = 0$. Hence, given an initial error bound, $\| e^*(t_0) \| \leq \delta$, and a small positive constant ε, a sampling frequency $F_0 = 1/T_0$ exists for which a preassigned bound of the form $(1+\varepsilon)\delta$ can be obtained such that $\| e^*(t_0+NT) \| \leq (1+\varepsilon)\delta$. It follows, according to (3.22) that for any sampling frequency $F > F_0$ (i.e., $T < T_0$) the states error $e^*(t_0+NT)$, is strictly bounded by $(1+\varepsilon)\delta$.

Remark Equation (3.23) may be solved iteratively for T (or for NT) once all its parameters have been identified from the system model. Since (3.23) does not depend on β, the task of finding an appropriate sampling frequency is thus independent of the problem of finding a stabilizing parameter b for the PWM controller.

3.2.4 A High-Gain Approach for Average PWM Controller Design

The design tasks are reduced to determining the average PWM operator gain, β, and an appropriate sampling frequency. The gain b is sought which stabilizes, toward the manifold $e = 0$, the response of the average closed loop system. The appropriate sampling frequency must be such that it makes the actual PWM controlled trajectory follow arbitrarily close that of the designed average model. We assume that, within the boundary layer region around $e = y_d - y = 0$, which is uniquely specified by β, a sufficiently large sampling frequency may be used, so as to make the actual PWM response and that of the average system sufficiently close to each other.

It is evident that a Lyapunov-based design approach suffices to determine, in any particular case, the value of β that accomplishes the zeroing of the average model error signal. However, we shall state a high gain result (closely related to the results in Marino (ref. 20)) that allows one to establish a design method for the average PWM controller gain β.

Theorem 4 If the control law $u = M\,\mathrm{sign}(e)$ creates a sliding regime locally around the manifold $e = y_d - y = y_d - h(x) = 0$, then there exists a sufficiently high gain β of the average PWM operator such that the state trajectories of the average PWM system stabilize toward $e = 0$.

Proof Suppose such a sliding regime exists locally on $e = 0$ and, for the sake of clarity, let $y_d = 0$. Then, it follows that, locally,

$$\begin{aligned} &\lim_{e \to +0} de/dt = \lim_{e \to +0} -\langle dh, f+Mg \rangle < 0 \\ &\lim_{e \to -0} de/dt = \lim_{e \to -0} -\langle dh, f-Mg \rangle > 0 \end{aligned} \qquad (3.24)$$

Subtracting these inequalities on $e = 0$ one obtains $\langle dh, g \rangle > 0$ locally on the zero error manifold. It follows from the smoothness assumptions on h and g that there exists a boundary layer of width 2ε around $e = 0$, with ε arbitrarily small, where locally $\langle dh, g \rangle > 0$. Taking $\beta > 1/\varepsilon$ the control law $u = \beta e = -\beta y$ yields a controlled error of the form :

$$de/dt = -\langle dh, f(x) \rangle + \beta y \langle dh, g \rangle \qquad (3.25)$$

It is evident that for sufficiently high β the controlled error dynamics exhibits a time scale separation property. Indeed, dividing by β and letting $\beta \to \infty$ one sees that since $\langle dh, g \rangle > 0$ then $e = -y = 0$ is a **slow manifold** of the controlled system. The corresponding **fast subsystem**, described in the fast time scale $\tau = \beta t$ is given by:

$$de/d\tau = y \langle dh, g \rangle = -e \langle dh, g \rangle \qquad (3.26)$$

which is locally asymptotically stable toward $e = 0$. It follows from Tihonov's theorem (See (ref. 20)) that $e = -y = 0$ is an asymptotically stable manifold for the high gain controlled system.

This result immediately suggests a method for the design of PWM controllers, as far as the parameter β is concerned. One simply verifies the existence of a local sliding regime on the manifold $y = 0$ and then finds a high gain replacement for the discontinuous controller. This entails the finding of an appropriate boundary layer along the sliding manifold. The slope of the linear portion of the high gain controller coincides with the needed stabilizing parameter β of the PWM system.

3.2.5 PWM Controller Design for Linear Plants

For the case of linear controlled plants, a vast amount of classical input-output design methods can be directly used for the design of PWM controllers based on the average PWM model. For ON-OFF-ON controlled switchings, the design problem is reduced to finding the stabilizing gain β corresponding to the linear part of the saturation block in the classical feedback configuration shown in figure 3.3.

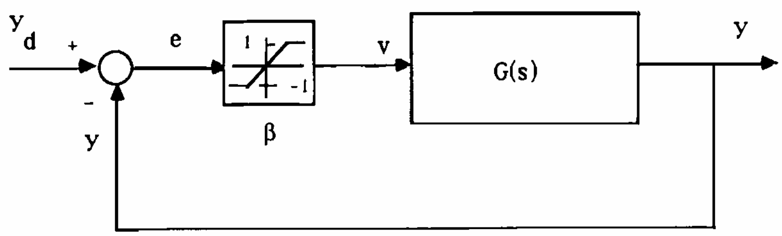

Figure 3.3 An Average PWM Controlled Linear Plant.

Among the many methods available to solve such a design problem one finds : the Small Gain Theorem, the Circle Criterion, Describing Function methods, Nyquist stability criterion, the Popov criterion, etc. All these design techniques, readily found in the literature (See MacFarlane (ref. 21)), are based on well known sufficient conditions for stability and asymptotic stability of linear feedback controlled plants.

3.3. SOME AEROSPACE APPLICATIONS

3.3.1 Example 1.

Consider the kinematic and dynamic model of a single-axis externally controlled spacecraft whose orientation is given in terms of the Cayley-Rodrigues representation of the attitude parameter (Dwyer and Sira-Ramirez, (ref. 22)), denoted by ξ.

$$d\xi/dt = 0.5(1+\xi^2)\omega \quad ; \quad d\omega/dt = \tau / I \tag{3.27}$$

The angular velocity is denoted by ω, while I is the moment of inertia , and τ is the applied external torque, restricted by $\tau \in \{-\tau_{max}, 0, \tau_{max}\}$.

Given arbitrary initial conditions, a slewing maneuver is required which brings the attitude parameter to a final desired value ξ_d and the angular velocity to an equilibrium. For feedback purposes, the following output is made available :

$$y = -\omega + 2\lambda(\xi - \xi_d)/(1+\xi^2) \tag{3.28}$$

with $\lambda < 0$. Notice that if the output function is ideally driven to $y = 0$, in finite time, and the controlled state trajectory is ideally made to stay in such a manifold, then, $\omega = 2\lambda(\xi - \xi_d)/(1+\xi^2)$ and the equation governing the attitude parameter evolution ideally becomes **linear**, $d\xi/dt = \lambda(\xi-\xi_d)$. The controlled trajectories, thus, asymptotically converge to $\xi = \xi_d$ with exponential decay rate set by λ. Simultaneously, ω would tend to zero as desired. Without loss of generality we shall assume that $\xi_d = 0$.

A PWM stabilizing controller is proposed whose average model is represented by a saturation block with limit values $\pm\,\tau_{max}$ and a linear (boundary layer) region characterized by a slope value of $\beta\,\tau_{max}$ (in the notation of equation (3.12) $M(e) = \tau_{max}$ for all e). In order to apply the high gain design method of sections 3.2.3 and 3.24, the following paragraphs consider the existence conditions for a sliding regime on $y = 0$ for the nonlinear discontinuous controlled system (3.27) , (3.28).

The transversality condition (3.A.4) takes, in this case, the simple, global, form: $\langle dy, g \rangle = -1/I < 0$. From (3.27),(3.28) it follows that:

$$dy/dt = -(1/I)u + \lambda\,(1-\xi^2)(1+\xi^2)^{-1}\omega \tag{3.29}$$

The invariance conditions (3.A.5), $y = 0$ and $dy/dt = 0$, yield the equivalent control that ideally keeps the state trajectories constrained to $y = 0$,

$$u^{EQ}(x) = \lambda\, I\,(1-\xi^2)(1+\xi^2)^{-1}\omega \tag{3.30}$$

The region of existence of a sliding regime , on $y = 0$, are easily found from the intermediacy condition (3.A.9) of the equivalent control, which in this particular case takes the form (ref. 18) : $-\tau_{max} < u^{EQ} < \tau_{max}$, i.e., :

$$|\lambda\, I\,(1-\xi^2)(1+\xi^2)^{-1}\omega| < \tau_{max} \tag{3.31}$$

The region of existence of a sliding regime is thus given by:

$$|\omega| < (\tau_{max}/|\lambda\, I|)\,(1+\xi^2)|(1-\xi^2)|^{-1} \tag{3.32}$$

The region defined by (3.32) is depicted in Figure 3.4. Notice that a sufficient condition for avoiding intersection of the sliding manifold, y = 0, with the boundaries of the computed region (3.32), is simply given by $\tau_{max}/|\lambda I| > |\lambda|$. This condition may be strengthened by computing the actual distance between the manifold y = 0 and the corresponding boundary of the region defined by (3.32). To do this, however, a fourth order polynomial equation must be solved.

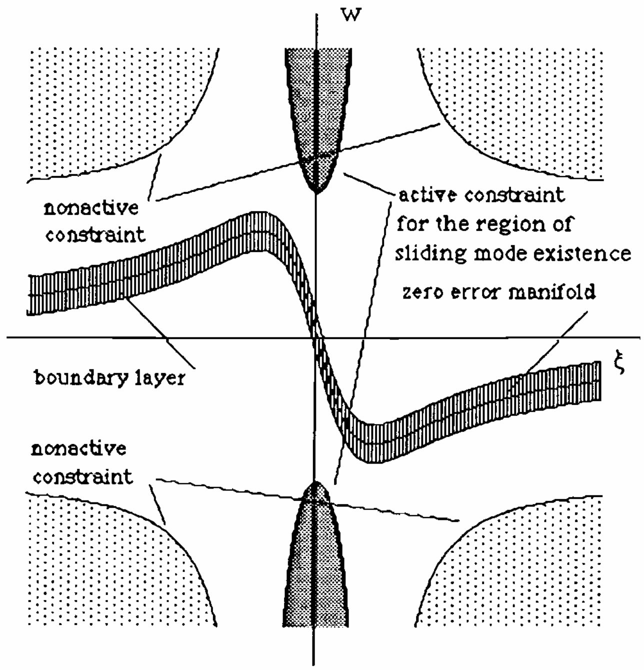

Figure 3.4. Linearizing Manifold and Boundaries of the Region of Existence of a Sliding Mode for the Attitude Control of a Single Axis Spacecraft Model.

Provided that the sliding manifold, y = 0, does not intersect the region defined by (3.32), a sliding regime thus globally exists on such a zero error manifold. It is also easy to see by Lyapunov stability considerations that y = 0 is globally attractive. Theorem 4 can therefore be applied and a high gain parameter β can be prescribed which guarantees asymptotic stability of the average PWM controlled state trajectories toward $y = 0$. For a high gain parameter β the average state response slides on the small boundary layer and quickly adopts $y = 0$ as a sliding surface. On $y = 0$ the controlled motion can be regarded as nearly linear and an estimate of the Lipschitz constant L_1 is simply given by the quantity $|\lambda|$. The rest of the parameters needed to find an estimate of the required sampling frequency, as given by theorem 3, are: $G = 1/I$, $M = \tau_{max}$, and ε and δ are chosen sufficiently small. For instance, they may be chosen as $0.3/\beta$ and $0.5/\beta$, respectively.

A simulated state response of the average and the actual PWM controlled systems is shown in Figure 3.5. For this simulation, the following values were used: $I = 94$ Kg-m^2, $\lambda = -0.11$ s^{-1}, $\tau_{max} = 1.55$ Kg m^2/s^2, $\xi_d = 0$. A sliding regime is guaranteed to exist globally on $y = 0$. The gain parameter was chosen as $\beta = 50$. The sampling frequency of the error signal, in the actual PWM controlled system, was set to 1 sample per second.

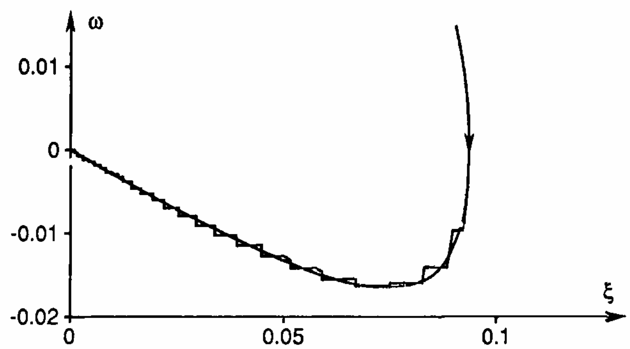

Figure 3.5. Phase trajectory response of Average and Actual PWM Controlled Response of Nonlinear Single Axis Spacecraft Model

The simulations clearly verify that the actual PWM controlled motions slide, very closely, around the shown average model trajectory.

3.3.2 Example 2

Consider a single axis pitch angle attitude control system for a linearized model of a satellite plant (See Howe and Cavanaugh (ref. 23)) shown in Figure 3.6.

The model includes fast rate and position sensors, as well as a first order model preceeded by a small pure time delay to represent the gas jet reaction control system. The pitch angle, θ, and the pitch rate, ω, are used in a PWM feedback control scheme designed to control the satellite orientation to track a pitch command angle θ_d. For simplicity, the commanded pitch angle is taken as zero.

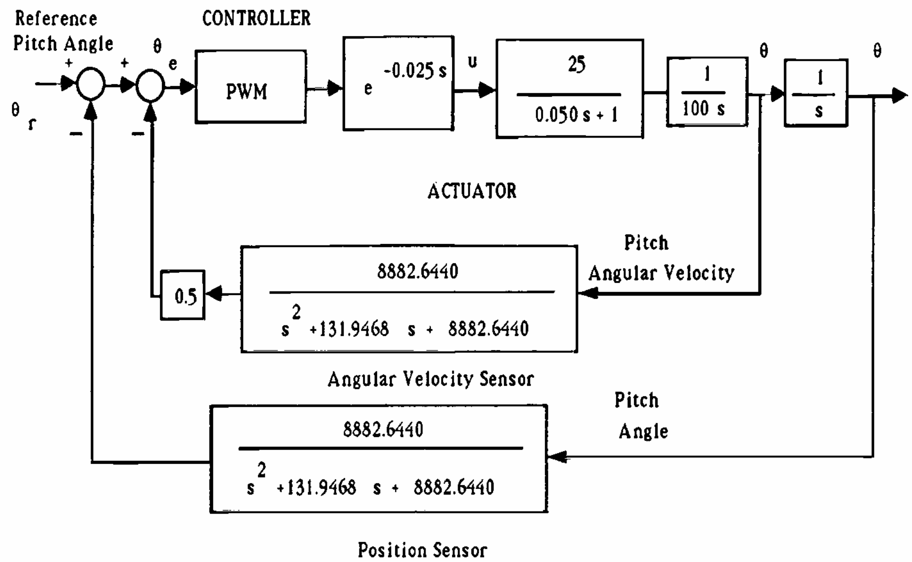

Figure 3.6. A PWM Satellite Attitude Feedback Control System

According to Sandberg's circle criterion for linear time invariant systems (See (ref. 21)), the average PWM closed loop system is guaranteed to be asymptotically stable, if

$$-1/\beta < \inf \operatorname{Re} G(jw) \tag{3.33}$$

where G(jw) represents the complex transfer function of the single-input single-output open loop system (including the actuator and sensor dynamics). Figure 3.7 shows a plot of the real part of the transfer function in terms of the frequency w. The average PWM controlled motion is guaranteed to be stable for any positive value of the controller gain β smaller than 35. However, the sufficiency of the criterion does not conclude instability for larger values of such a design parameter.

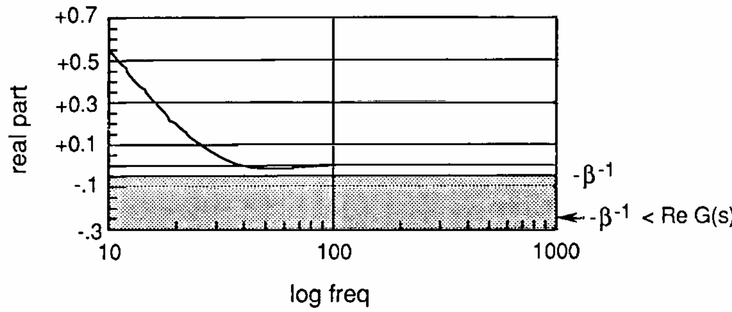

Figure 3.7. PWM gain parameter design using the real part of Nyquist Plot of the Open Loop System Trasfer Function

Simulations of the state trajectories corresponding to the actuator and orientation and velocity variables are shown in figures 3.8 and 3.9. Also, the pure time delay block was simulated using the approximation:

$$e^{-Ts} = \frac{\left[T^2/8\right]s^2 - \left[T/2\right]s + 1}{\left[T^2/8\right]s^2 + \left[T/2\right]s + 1} \tag{3.34}$$

The average and actual closed loop PWM controlled actuator state trajectory responses of the spacecraft model are shown in Figure 3.8. A sliding regime of the actual PWM trajectories about the average PWM response curve is clearly depicted in this figure. Here $\beta = 20$ and the sampling rate was chosen as 8 samples per second.

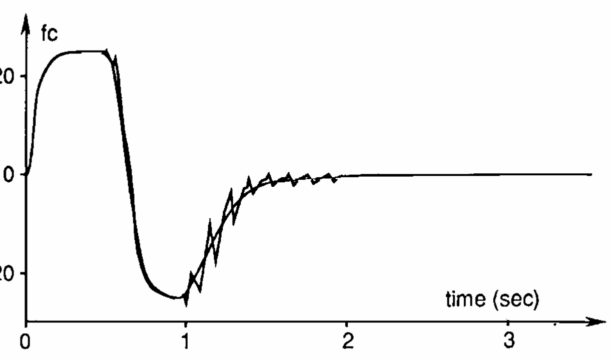

Figure 3.8. Average and actual PWM State Trajectory Response of Controlled Spacecraft Model (actuator state)

Figure 3.9 shows the actual and average responses of the pitch angular position and the pitch rate. The chattering motions are significantly smoothed out due to the several integrations undergone by the PWM block output signal through the actuator and the linear system approximation prescribed for the pure time delay block. The average and PWM controlled trajectories are seen to coincide practically over the entire simulation interval.

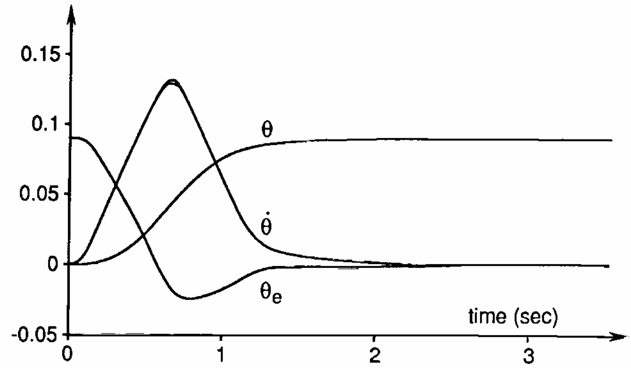

Figure 3.9. Average and Actual PWM Controlled Trajectories Response
(Pitch angular position, Pitch rate and angular position error)

3.4. CONCLUSIONS

In this chapter a design procedure was proposed for the specification of stabilizing feedback PWM controllers. It was shown that, under an infinite sampling frequency assumption, an average model, of the Filippov type, can obtained for PWM controllers. The average model simply replaces the discontinuous PWM switching control function by a duty ratio function. The average model was shown to capture the basic qualitative (i.e. stability) features of the actual PWM controlled system. for sufficiently high sampling frequency. Estimates of such sampling rate are easily found from the Lipschitz constants associated with the state space model of the controlled system. A design based on the average model allows to treat the PWM design problem in a more exact fashion (i.e., without discrete-time approximations) For the design task one may resort to, for example, well known Lyapunov stability theory, or else use a high gain design approach as described in this chapter. It should be noted that dealing with the average model one totally circumvents the technical difficulties associated with the fact that PWM operators are, indeed, unbounded operators on the Banach space of absolutely integrable functions. For the case of a PWM design problem in the particular context of linear controlled plants, classical sufficient conditions, based on frequency domain criteria for asymptotic stability, are readily applicable to the average model. The results were applied, through simple design examples, to PWM controller specification for linear and nonlinear plants.

Acknowledgments. This research was supported by the **Consejo de Desarrollo Cientifico Humanistico y Tecnológico** of the **Universidad de Los Andes** under Research Grant I-280-87. The author is highly indebted to technician **Angel Maestre**, of the Systems Engineering School, for his invaluable assistance in making possible the preparation and installation of the computer packages, used in the simulations presented in this article.

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3.5 APPENDIX

Consider the n-dimensional variable structure system:

$$dx/dt = u\, f_1(x) + (1-u)\, f_2(x) \tag{3.A.1}$$

$$u = \begin{cases} 1 & \text{for } s(x) > 0 \\ 0 & \text{for } s(x) < 0 \end{cases} \tag{3.A.2}$$

with $S = \{ x : s(x) = 0 \}$ being a smooth n-1 dimensional manifold defined in the open set N of R^n, with gradient vector $\partial s/\partial x \neq 0$, for all x in S.

Definition A.1 A **sliding regime** is said to exist locally on the manifold S whenever the following conditions are satisfied (ref. 18) (See also Sira-Ramírez (ref. 24)) :

$$\lim_{s \to +0} ds/dt =: \lim_{s \to +0} \langle ds, f_1(x) \rangle < 0$$

$$\lim_{s \to -0} ds/dt =: \lim_{s \to -0} \langle ds, f_2(x) \rangle > 0 \tag{3.A.3}$$

with $\langle ds, f_i \rangle$ being a shorthand notation for the chain rule $[\partial s/\partial x]^T f_i(x)$.

Proposition A.1 (ref. 24) If a sliding regime exists locally on S then, necessarily, for all $x \in S$ where the sliding regime exists, the following **transversality condition** is satisfied:

$$\langle ds, f_1(x) - f_2(x) \rangle < 0 \tag{3.A.4}$$

Proof Obvious upon subtracting the expressions in (3.A.3) evaluated on S.

Being a necessary condition, (3.A.4) determines the extent of a region which properly contains the region of existence of a sliding regime on the surface S.

If a sliding motion exists locally on S, the state trajectories undergo a chattering motion about the switching (sliding) manifold. An idealized version of such a motion is obtained by assuming that the trajectories smoothly evolve on the sliding manifold. To describe such an **ideal sliding dynamics** two general methods have been proposed: Utkin's method, based on the **Equivalent Control Method** (ref. 18) , and the method of **Filippov's Geometric Averaging** (ref. 25).

The **Equivalent Control method** is based on defining a control function, called the **equivalent control**, and denoted by $u^{EQ}(x)$, locally defined along S, for which the following **invariance conditions** are satisfied:

$$ds/dt = 0 \quad \text{on } s = 0 \tag{3.A.5}$$

Using our shorthand notation, these conditions are expressed as:

$$\langle ds\, ,\, u^{EQ}(x)\, f_1(x) + [1-u^{EQ}(x)]\, f_2(x) \rangle = 0 \quad \text{on} \quad s = 0 \tag{3.A.6}$$

The geometric interpretation of (3.A.6) should be clear: the smooth vector field $u^{EQ}(x)\; f_1(x) + [1-u^{EQ}(x)]\; f_2(x)$ must be locally orthogonal to the surface gradient at every point $x \in S$ located in the region of existence of the sliding mode. From (3.A.6) one finds the **unique** value of the equivalent control, for $x \in S$, as :

$$u^{EQ}(x) = -\langle ds, f_2(x) \rangle / [\langle ds, f_1(x) - f_2(x) \rangle] \tag{3.A.7}$$

To see that $u^{EQ}(x)$ is indeed unique (ref. 24), assume $\mu(x)$ is a different function also satisfying (3.A.6) i.e. $\langle ds, \mu(x)\, f_1(x) + [1-\mu(x)]\, f_2(x) \rangle = 0$. Subtracting from (3.A.6) the obtained expression with $\mu(x)$, one would get : $\langle ds, [u^{EQ}(x)-\mu(x)]\, f_1(x) - [u^{EQ}(x)-\mu(x)]\, f_2(x) \rangle = [u^{EQ}(x)-\mu(x)] \langle ds, f_1(x)-f_2(x) \rangle = 0$. Since necessarily $\langle ds, f_1(x)-f_2(x) \rangle < 0$, it follows that $u^{EQ}(x) = \mu(x)$ which is a contradiction.

When (3.A.7) is formally substituted in place of the discontinuous control u in (3.A.1), the obtained dynamics, constrained to evolve on S, is known as the **ideal sliding mode**. Its explicit expression is readily obtained as follows:

$$dx/dt = [-\langle ds, f_2(x) \rangle f_1(x) + \langle ds, f_1(x) \rangle f_2(x)] / [\langle ds, f_1(x)-f_2(x) \rangle] \quad ; \quad x \in S \tag{3.A.8}$$

Theorem A.1 (ref. 16) Let the transversality condition (3.A.4) be locally satisfied on S. The necessary and sufficient condition for the local existence of a sliding regime of

(3.A.1),(3.A.2) on S, is that the equivalent control $u^{EQ}(x)$ satisfies:

$$0 < u^{EQ}(x) < 1 \qquad \text{on } s(x) = 0 \tag{3.A.9}$$

Proof Suppose (3.A.9) is locally valid on S. Inverting then the expression in (3.A.7) and according to (3.A.9) one obtains :

$$- [\langle ds, f_1(x) - f_2(x) \rangle] / \langle ds, f_2(x) \rangle > 1, \quad x \in S . \tag{3.A.10}$$

i.e.,

$$-\langle ds, f_1(x) \rangle / \langle ds, f_2(x) \rangle > 0 \tag{3.A.11}$$

Hence, $\langle ds, f_1(x) \rangle$ and $\langle ds, f_2(x) \rangle$ have opposite signs on S. According to the validity of the transversality condition (3.A.4), the numerator of expression (3.A.10) is positive, then $\langle ds, f_2(x) \rangle$ is also necessarily positive and hence (3.A.11) implies that $\langle ds, f_1(x) \rangle < 0$, locally on S. It follows that there exists locally an open neighborhood surrounding S where conditions (3.A.3) remain valid. A sliding regime exists locally on S.

To prove necessity, suppose a sliding regime exists locally on S and conditions (3.A.3) are locally valid on S. Then, there exists a positive function $0 < \mu(x) < 1$, such that $\mu(x) \langle ds, f_1(x) \rangle + [1-\mu(x)] \langle ds, f_2(x) \rangle = 0$. Solving for $\mu(x)$ we obtain the same expression as in (3.A.7) for $\mu(x)$. By virtue of the uniqueness of the equivalent control $\mu(x) = u^{EQ}(x)$ and the result follows.

Filippov's Geometric Averaging method (ref. 25), as applied to sliding mode existence conditions, can be phrased as in the following theorem.

Theorem A.2 A sliding regime exists locally on S for system (3.A.1),(3.A.2) if and only if there exists a scalar function $0 < \mu(x) < 1$ defined on S such that S is a local integral manifold for the average dynamics:

$$dx/dt = \mu(x) f_1(x) + [1-\mu(x)] f_2(x) =: f_{av}(x) \tag{3.A.12}$$

One immediately concludes that Filippov's convex combination function $\mu(x)$ is none other that the **equivalent control** and that, therefore, Filippov's Average dynamics coincides with the Ideal Sliding mode.

Remark The above result is not generally true for systems of the form $dx/dt = F(x,u)$, as it was shown in Utkin ((ref. 18), pp. 26-27 and pp. 62). The procedure by which a general VSS of the form $dx/dt = F(x,u)$ is written in the form (3.A.1), (3.A.2) (with $F(x,1) = f_1(x)$ and $F(x,0) = f_2(x)$) is known as "artificial" control-linearization (elsewhere called "pre-linearization"). For the artificially control-linearized system, both Utkin's and Filippov's approaches yield the same results concerning the ideal sliding mode. However, it is clear that if a control $u^{EQ}(x)$ can be found for which $\mu(x)F(x,1) + [1-\mu(x)]\, F(x,0) = F(x,u^{EQ}(x))$, it does not necessarily follow that $u^{EQ}(x) = \mu(x)$.