ON A NEW CLASS OF SEMI-LINEAR SYSTEMS AND THEIR APPLICATIONS

HEBERTT SIRA-RAMIREZ
Universidad de Los Andes, Mérida-Venezuela

Abstract: A new class of non-linear systems with explicit solutions is introduced. Three applications are presented.

MAIN RESULTS

The non-linear vector differential equation:

$$\frac{d}{dt}\underline{x}(t) = A(t)\underline{x}(t) + h(\underline{x}(t),\underline{u}(t))\,\underline{x}(t) \;;\quad \underline{x}(t_0) = \underline{x}_0 \tag{1}$$

where $h(\,,\,)$ is a scalar function m-homogeneous in the state (i.e. $h(\alpha\underline{x},\underline{u}) = \alpha^m h(\underline{x},\underline{u})$ for any scalar α and m is an integer greater than zero), has a unique solution given explicitely by:

$$\underline{x}(t) = \left(1 - m\int_{t_0}^{t} h(\Phi(\sigma,t_0)\underline{x}_0, \underline{u}(\sigma))\,d\sigma\right)^{-1/m}\Phi(t,t_0)\underline{x}_0 \tag{2}$$

where $\Phi(t,t_0)$ is the state transition matrix associated with $A(t)$. The proof of this fact is easily established through standard Lipschitz conditions type of arguments[1]. A bounded gradient of h over any bounded region of the state space suffices for the r.h.s. of (1) to be Lipschitzian. Straightforward derivation of (2) yields back (1) and the initial conditions are trivially satisfied by the proposed solution. Finite escape time problems are easily circumvented due to the explicit character of the solution formula.

APPLICATIONS

1) Given a linear system $\dot{\underline{x}} = A(t)\underline{x} + B(t)\underline{u}(t)$ with $\underline{u}(t)$ known and initial state unknown but bounded by a convex polyhedron of the form $\{\underline{x} \in R^n : <\underline{x}, \underline{\psi}_{i0}> \leq 1 \;;\; i=1,2,\ldots,N \geq n+1\}$, then the set of possible states at time t is a polyhedron characterized by $\{\underline{x}: <\underline{x}, \underline{\psi}_i(t)> \leq \operatorname{sgn} m_i(t) \;;\; \forall\, i\}$ where $m_i(t) \triangleq 1 + \int_{t_0}^{t} <\underline{\psi}_{i0}, \Phi'(\sigma,t_0)B(\sigma)\underline{u}(\sigma)> d\sigma$ and $\underline{\psi}_i(t)$ satisfies the equation:[2]

$$\frac{d}{dt}\underline{\psi}_i(t) = -A'(t)\underline{\psi}_i(t) - \operatorname{sgn} m_i(t) < \underline{\psi}_i(t), B(t)\underline{u}(t)> \underline{\psi}_i(t) \;;\; \underline{\psi}_i(t_0) = \underline{\psi}_{i0} \tag{3}$$

which is an equation of type (1) with m = 1.

2) State constrained (hemispace restriction) optimal control problems defined on linear systems with initial state restricted to a bounded region of a linear variety characterized by a support vector $\underline{h}_0$ are easily shown to yield differential equations of the form:

$$\frac{d}{dt}\underline{h}(t) = -A'(t)\underline{h}(t) - <\underline{p}(t),\underline{h}(t)><\underline{h}(t),B(t)R^{-1}(t)\underline{h}(t)>\underline{h}(t) \;;\; \underline{h}(t_0) = \underline{h}_0 \tag{4}$$

where $\underline{h}(t)$ is the support vector of the state variety at time t, $\underline{p}(t)$ is a co-state vector arising from the optimization problem defined on the basis of a performance index that penalizes 1) any deviation of $\underline{h}(t)$ from a vector $\underline{q}(t)$ defining the hemispace restriction for the state trajectories and 2) the control power by means of a generalized norm defined by $R(t)=R'(t)>0$. Equation (4) is of form (1) with m=3.

3) The main results are easily extended to matrix differential systems[3] of the form: $d/dt\; \Sigma(t) = A(t)\Sigma(t) + \Sigma(t)A'(t) + h(\Sigma(t),U(t))\Sigma(t)$ with $\Sigma(t_0) = \Sigma_0$.

REFERENCES

(1) Nemytskii, V., and Stepanov, V., Qualitative Theory of Diff. Eq. Princeton Press. 1960.

(2) Sira-Ramirez, H., "Reachable Sets and Set-Theoretc Evolution of the Uncertainty in Linear Systems," II Brazilian Congress of Automatic Control Universidade Federal de Santa Catarina (UFSC), Florianópolis. Sept. 1978

(3) Athans, M., "The Matrix Minimum Principle," Information and Control. Vol. 11 Numbers 5 & 6. November-December 1967.